

# **Dynamic Asset Pricing With Non-Redundant Forwards**

by

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First Draft: February 5, 2000

This draft: May 20, 2001

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## Abstract

We consider an incomplete but frictionless financial market in which non-redundant forward contracts contribute to span the uncertainty present in the economy. When such forward contracts are available for trade, some standard results of portfolio and dynamic asset pricing theory must be amended. When the investment opportunity set is driven by  $K$  state variables, a  $(K+4)$ -mutual fund separation theorem is obtained in lieu of Merton's classic  $(K+2)$ -fund separation. The two additional funds are fully characterized. One fund is a portfolio containing forward contracts only, and the other fund is a portfolio of cash assets and forward contracts that hedges the interest rate risk brought about by the optimal portfolio strategy itself. The latter risk is due to the fact that, when a forward contract is involved, incurred profits or losses that accrue to the investor's wealth at each instant are locked-in in the forward position up to the contract maturity. Thus discounting these gains or losses back at the current date gives rise to an interest rate risk. A second important result is that the mean-variance efficiency of the market portfolio of cash assets is neither a necessary nor a sufficient condition for the linear relationship between expected return and beta to hold. Finally, the pricing equation for a forward contract is shown to contain an extra term relative to that for a cash asset. We name this term a *strategy risk premium*. It compensates the investor for the (systematic) risk that stems from his very *portfolio strategy* when the latter involves a non-redundant forward contract.

JEL Classification: D52, E52, G11, G12.

## I. Introduction

Financial markets and monetary authorities have recently been under pressure to design forward contracts written neither on financial assets nor on commodities but on non-tradable economic variables [see Shiller (1993), Sumner (1995) and Athanasoulis, Shiller and van Wincoop (1999)]. The most obvious example is the discussion surrounding forward contracts written on the Consumer Price Index which are to be launched by various Central Banks as a substitute to inflation targeting [see Cowen (1997), Dowd (1994) and Sumner (1997)]. It has also been suggested that the forward price targeting system be expanded to other macro-economic aggregates, such as GNP and monetary aggregates [Sumner (1995)]. These propositions focus on forward contracts and neglect options and futures, possibly because the non-traded assets or economic aggregates (such as CPI or GDP) under consideration are not sufficiently volatile to warrant the development of option or futures markets.<sup>1</sup>

The most important issue of a price targeting system is the way the forward target price is determined. Proposals in the literature range from setting the forward price equal to the expected value of the underlying at maturity to setting it so as to minimize either the risk premium offered to investors or the Central Bank's loss function. Yet, the economic justification of these recommendations is at best unclear, and the literature offers no convincing model to price such clearly non-redundant forward contracts. Theoretical work on financial innovation has mainly dealt with the welfare effects of adding non-redundant contracts to a set of primitive assets that forms an incomplete market. Another important issue that has been investigated is the endogenous determination of the optimal number of contracts to be created and the nature of their underlying assets.<sup>2</sup> However, little if any attention has been paid to the

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<sup>1</sup> In the US, for instance, a market for futures contracts written on the CPI was organized but failed. This failure was probably due to a lack of volatility of the CPI since, as pointed out by the referee, when the CPI became momentarily more volatile during the oil price crisis, the open interest increased significantly. A different argument, suggested by Horrigan (1987), Dowd (1995), Sumner (1995) and Athanasoulis et al. (1999) among others, is that futures contracts written on non-tradable assets are not viable since (strictly riskless) arbitrage cannot take place.

<sup>2</sup> See, among others, Ohashi (1995, 1997).

portfolio re-allocation effects brought about by the trading of such new instruments. When the latter are options, which are cash assets, or futures, which are similar to cash assets, traditional results are still expected to hold. However, when the added assets are forward contracts (non cash assets), the optimal portfolio allocation and thus the stock market equilibrium are likely to be affected. These effects may be particularly pronounced in a dynamic environment.

Curiously, the issue of the equilibrium pricing of non-redundant forward contracts has not been addressed in the financial literature.<sup>3</sup> Cox, Ingersoll and Ross (1981) provided a comprehensive study of the pricing of *redundant* forward contracts and their futures counterparts. Subsequent research focused on developing pricing models for futures contracts, particularly under stochastic interest rates, while redundant forward contracts continued to be priced according to the cash-and-carry formula. The question of the equilibrium prices of non-redundant forward contracts has been left open although Cox et al. (1981) explicitly showed that the Merton's intertemporal asset pricing model does not hold for forward contracts.

Richard and Sundaresan (1981) derived a general equilibrium pricing of non-redundant forward contracts. They showed in particular that the simple cash-and-carry pricing equation that characterizes redundant forward contracts still holds for non-redundant ones in special cases. However, their work has three limitations. First, they do not provide a CAPM-like equation for the forwards, even when they are redundant,

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<sup>3</sup> Cox et al. (1981) have shown that the standard intertemporal CAPM applies to futures contracts. Breeden (1984) is, to the best of our knowledge, the only paper to have introduced futures in an intertemporal equilibrium framework à la Merton, in addition to a riskless asset and risky cash assets. In his model, however, each and every futures contract is both of instantaneous maturity and perfectly correlated with each and every state variable. This in effect makes the market quasi-complete. Futures are held by Breeden's investors only as perfect hedges against the instantaneous fluctuations of the corresponding state variables. In contrast, our forward contracts are long lived, are not necessarily written on the state variables and the financial market is in general incomplete. The difference will prove crucial in that, for instance, forward contracts will be demanded by our investors for both (imperfect) hedging and speculative purposes. This will give rise to an original separation result and will remove the usual necessary condition of mean-variance efficiency of the market portfolio for the standard ICAPM to hold.

and thus cannot explain the puzzle regarding forwards (redundant or not) raised by CIR. Second, the cash-and-carry relationship does not hold when the underlying asset pays stochastic dividend(s) and/or provides a stochastic convenience yield. To derive their results, they assume these stochastic features away. Third, they consider only the case of forward contracts with linear pay-offs for which, in their infinite horizon setting, the cash-and-carry pricing relationship holds at equilibrium.

When forward contracts are not redundant, the pricing issue is in fact intricate. Moreover, the related and important issue of how forward trading impacts on dynamic optimal asset allocations has been overlooked in the literature. Thus, we do not know how standard results in portfolio theory are affected when non-redundant forward contracts are introduced in the opportunity set. The main objective of this paper is to fill the gap and develop an intertemporal capital asset pricing model (thereafter ICAPM) for such contracts. Along the way, we will provide new mutual-fund separation theorems and amend some standard results in dynamic asset pricing theory. Our main findings can be summarized as follows:

- A  $(K+3)$ -fund and a  $(K+4)$ -fund separation theorems are obtained when the investment opportunity set is driven by  $K$  state variables, in lieu of Merton's classic  $(K+2)$ -fund separation.
- Mean-variance efficiency<sup>4</sup> of the market portfolio of primitive cash assets is neither a necessary nor a sufficient condition for the linear relation between expected return and beta to hold.
- The pricing equation for forward contracts is shown to contain an extra term relative to that for primitive cash assets. This term, which we call a *strategy risk premium*, compensates investors for the (systematic) risk that stems from their very *portfolio strategies* when the latter involve forward contracts.

The unique feature of forward contracts that leads to these results is the additional interest rate risk brought about by any strategy involving them. When a forward

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<sup>4</sup> Unless mentioned otherwise, the phrase “mean-variance efficiency” is meant to characterize portfolios lying on the instantaneous capital market line in the mean-standard deviation plane. Only in the special (standard) case of the static (one-period) model without non-redundant forwards or in the (standard) dynamic model without non-redundant forwards and state variables is the market portfolio the tangent portfolio.

contract is used, incurred profits or losses that accrue to the investor's wealth at each instant are locked-in in the forward position up to the contract maturity. Thus discounting these gains or losses back at the current date gives rise to an additional interest rate risk.

The remainder of the paper is organized as follows. In Section II, we describe the economy and derive an investor's wealth dynamics in a possibly incomplete market. Section III provides the optimal demands for all risky assets and establishes a novel mutual-fund separation theorem. Section IV characterizes the market portfolio and derives the ICAPM that applies to cash assets on the one hand and to non-redundant forward contracts on the other. The last section offers some concluding remarks. Some proofs are gathered in a mathematical appendix.

## II. Economic framework and wealth dynamics

In this section, the dynamics of an investor's wealth is derived when trading in non-redundant forward contracts is allowed.<sup>5</sup> The pure exchange economy under consideration is first described. Then an investor's profit-and-loss process generated by trading forward contracts is obtained. Finally, the investor's wealth dynamics is provided.

### 2.1 The economy

$N$  primitive (cash) risky assets are traded in the economy. The price  $S_i(t)$  of risky asset  $i$  ( $i = 1, \dots, N$ ) evolves through time according to the following stochastic differential equation (SDE):

$$dS_i(t) = S_i(t) \mu_{S_i}(t, Y(t)) dt + S_i(t) \sigma_{S_i}(t, Y(t))' dZ(t) \quad (1)$$

where  $Z(t)$  is an  $((N+K) \times 1)$  - dimensional Wiener process in  $\mathbb{R}^{N+K}$ ,  $Y(t)$  is a  $(K \times 1)$  - dimensional vector of state variables,  $\mu_{S_i}(t, Y(t))$  is a bounded valued function of

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<sup>5</sup> Once the dynamics of the investor's wealth has been derived, the traditional stochastic dynamic programming approach is applied to obtain optimal solutions. The modern martingale approach cannot be directly applied when the set of assets includes forward contracts. No economic insight is lost by adopting the traditional method.

$t$  and  $Y$ ,  $\sigma_{S_i}(t, Y(t))$  is a bounded  $((K+N) \times 1)$  vector valued function of  $t$  and  $Y$ , and the prime ' denotes a transpose. The Wiener process is defined on the usual complete probability space  $(\Omega, \mathbf{F}, \mathbf{P})$  where  $\mathbf{P}$  is the historical probability measure.

The dynamics of the  $K$  state variables is determined by the following system of SDEs:

$$dY(t) = \mu_Y(t, Y(t))dt + \Sigma_Y(t, Y(t))dZ(t) \quad (2)$$

where  $\mu_Y(t, Y(t))$  is a bounded  $(K \times 1)$  vector valued function of  $t$  and  $Y$  and  $\Sigma_Y(t, Y(t))$  is a bounded  $(K \times (N + K))$  matrix valued function of  $t$  and  $Y$ .

For convenience, we will often write the dynamics of the  $N$  asset prices in the following compacted form:

$$dS(t) = I_S \mu_S(t, Y(t))dt + I_S \Sigma_S(t, Y(t))dZ(t) \quad (3)$$

where  $I_S$  is an  $(N \times N)$  diagonal matrix valued function of  $S(t)$  whose  $i^{\text{th}}$  diagonal element is  $S_i(t)$ ,  $\mu_S(t, Y(t))$  is an  $(N \times 1)$ -dimensional vector whose  $i^{\text{th}}$  component is  $\mu_{S_i}(t, Y(t))$  and  $\Sigma_S(t, Y(t))$  is a  $(N \times (N+K))$  matrix valued function whose  $i^{\text{th}}$  element is  $\sigma_{S_i}(t, Y(t))$ .

Non-redundant derivatives are also available for trade in this economy. For simplicity, we assume that all are forward contracts. The net supply of these instruments is of course zero. To simplify notations, we assume that all contracts have the same time to maturity, which, without loss of generality, is supposed to be equal to the investors' (or the economy's) horizon  $(\tau)$ . The  $j^{\text{th}}$  forward price solves the following SDE:

$$dF_j(t) = F_j(t)\mu_{F_j}(t, Y(t))dt + F_j(t)\sigma_{F_j}(t, Y(t))'dZ(t) \quad (4)$$

where  $F_j(t)$  is short for  $F_j(t, \tau)$ ,  $\mu_{F_j}(t, Y(t))$  is a bounded function of  $t$  and  $Y$ , and  $\sigma_{F_j}(t, Y(t))$  is a bounded  $((N+K) \times 1)$  vector valued function of  $t$  and  $Y$ . Note that  $F_j(t)$  denote the forward *price*, not the *market value* of the forward contract, which is, for instance, zero at inception.

We assume that there exist at most  $H \leq K$  forward contracts. Together with the primitive assets, they form the basis of the financial market. Assets in the basis have linearly independent cash flows. Therefore, in the extreme case where  $H$  is equal to  $K$ , the financial market is complete in the sense of Harrison and Kreps (1979). In general, however, the market is incomplete. The vector process of the forward prices writes:

$$dF(t) = I_F \mu_F(t, Y(t))dt + I_F \Sigma_F(t, Y(t))dZ(t) \quad (5)$$

where  $I_F$  is a  $(H \times H)$  diagonal matrix valued function of  $F(t)$  whose  $j^{\text{th}}$  diagonal element is  $F_j(t)$ ,  $\mu_F(t, Y(t))$  is a  $(H \times 1)$ -dimensional vector whose  $j^{\text{th}}$  component is  $\mu_{F_j}(t, Y(t))$  and  $\Sigma_F(t, Y(t))$  is a  $(H \times (N+K))$  matrix valued function whose  $j^{\text{th}}$  element is  $\sigma_{F_j}(t, Y(t))$ .

The variance-covariance matrixes  $\Sigma_S \Sigma_S'$  and  $\Sigma_F \Sigma_F'$  are assumed to be positive definite. The variance-covariance matrix of the percent changes in all asset prices<sup>6</sup>, i.e.  $\Sigma \Sigma'$  where  $\Sigma \equiv \begin{bmatrix} \Sigma_S \\ \Sigma_F \end{bmatrix}$ , is also assumed to be positive definite.

Investors have also access to an instantaneously riskless asset (money market account) yielding  $r(t)$ . The diffusion process followed by  $r(t)$  is completely general and need not be made explicit. It determines the evolution of the whole term structure of interest rates in an endogenous manner. In particular, we assume that one of the  $N$  primitive cash securities is a pure discount bond whose maturity ( $\tau$ ) coincides with that of all forward contracts.<sup>7</sup> Let  $P(t)$ , short for  $P(t, \tau)$ , be its price at time  $t$ . Its dynamics then obeys the following SDE (for  $t$  positive and smaller than or equal to  $\tau$ ):

$$dP(t) = P(t) \mu_p(t, Y(t))dt + P(t) \sigma_p(t, Y(t))' dZ(t) \quad (6)$$

where  $\mu_p(t, Y(t))$  is a bounded function of  $t$  and  $Y$ , and  $\sigma_p(t, Y(t))$  is a bounded  $((N+K) \times 1)$  vector valued function of  $t$  and  $Y$ .

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<sup>6</sup> The phrase « rates of return » is avoided since forward contracts are included.

<sup>7</sup> Alternatively and equivalently, we can assume that this zero-coupon bond is spanned by the existing cash securities.



## 2.2 The value process for the forward position

We now turn to an investor's cumulative gain process generated by his or her trading on forward contracts. Let  $\Theta_j(t)$  be the number of forward contracts *held* (as opposed to *traded*) at time  $t$ . The value of the position on the  $j^{\text{th}}$  contract ( $j = 1, \dots, H$ ) at time  $t$  writes:

$$X_j(t) = P(t) \int_0^t \Theta_j(u) dF_j(u) \quad (7)$$

The RHS of (7) is the current (at date  $t$ ) value of the profits and losses incurred from the forward position. Since these cumulative (algebraic) gains are cashed-in or -out at the contract maturity date only, the discount factor  $P(t)$  is required. Aggregating over all contracts, the total value at time  $t$  of the forward position thus is:

$$X(t) = P(t) \int_0^t \Theta(u)' dF(u) \quad (8)$$

where  $\Theta(t)$  is the  $(H \times 1)$  vector of forward positions held at time  $t$ . We can now turn to the dynamics of the investor's wealth.

## 2.3 Wealth dynamics

The effects of introducing exogenous endowment and/or wage income in the model on the stock market equilibrium are well understood, while far from trivial in general.<sup>8</sup> Therefore, to ease the analysis and the technical derivations without real loss of generality, we follow Merton (1973) and assume that continuous consumption is financed through continuous selling of a fraction of the portfolio.

Let  $\alpha$  be the  $(N \times 1)$  vector of the proportions of wealth invested in the primitive assets,  $\gamma$  the proportion of wealth invested in the riskless asset and  $C$  the instantaneous consumption rate. Dropping the explicit time dependence of all processes to improve readability, using equations (3), (5), (6) and (8) and applying Itô's lemma yields the following wealth dynamics:

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<sup>8</sup> See for instance He and Pagès (1993) and Basak (1999).

$$dW = \left[ \begin{aligned} &W\alpha' \mu_S + \left( P \int_0^t \Theta' dF \right) \mu_P + P\Theta' I_F \mu_F \\ &+ P\Theta' I_F \Sigma_F \sigma_P + W\gamma r - C \end{aligned} \right] dt \\ + \left[ W\alpha' \Sigma_S + \left( P \int_0^t \Theta' dF \right) \sigma_P' + P\Theta' I_F \Sigma_F \right] dZ$$

Using the definition of wealth  $W \equiv W\gamma + W\alpha' \mathbf{1}_N + P \int_0^t \Theta' dF$  to eliminate the term involving the cumbersome integral, the wealth dynamics can be rewritten as:

$$dW = \left[ \begin{aligned} &W\alpha' \mu_S + (W - W\alpha' \mathbf{1}_N - W\gamma) \mu_P + P\Theta' I_F \mu_F \\ &+ P\Theta' I_F \Sigma_F \sigma_P + W\gamma r - C \end{aligned} \right] dt \\ + \left[ W\alpha' \Sigma_S + (W - W\alpha' \mathbf{1}_N - W\gamma) \sigma_P' + P\Theta' I_F \Sigma_F \right] dZ$$

or as:

$$dW = \left[ \begin{aligned} &W\alpha' (\mu_S - \mu_P \mathbf{1}_N) + W\mu_P + P\Theta' I_F (\mu_F + \Sigma_F \sigma_P) + W\gamma(r - \mu_P) - C \end{aligned} \right] dt \\ + \left[ W\alpha' (\Sigma_S - \mathbf{1}_N \sigma_P') + W(1 - \gamma) \sigma_P' + P\Theta' I_F \Sigma_F \right] dZ$$

To further simplify the notations, we denote by  $\theta$  the vector of the ratios of the forward nominal positions (not their values) to the investor's wealth, i.e.  $\theta \equiv \frac{1}{W} P\Theta' I_F$ .

Hence, the investor's wealth dynamics is finally given by:

$$dW = \left[ \begin{aligned} &W\alpha' (\mu_S - \mu_P \mathbf{1}_N) + W\mu_P + W\theta' (\mu_F + \Sigma_F \sigma_P) + W\gamma(r - \mu_P) - C \end{aligned} \right] dt \\ + \left[ W\alpha' (\Sigma_S - \mathbf{1}_N \sigma_P') + W(1 - \gamma) \sigma_P' + W\theta' \Sigma_F \right] dZ \quad (9)$$

We can now turn to the characterization of optimal demands.

### III. Individual demand for risky assets and separation properties

We derive first an investor's optimal demands for risky assets. A novel mutual-fund separation theorem (that can be stated under two alternative forms) then is shown to obtain.

#### 3.1 The optimal demand for risky assets

An investor is endowed with a Von Neuman-Morgenstern utility function and maximizes the expected utility of his or her consumption flow under a budget constraint, i.e. solves:

$$\max_C E \left[ \int_t^\tau U(s, C(s)) ds \right] \quad (10)$$

subject to equation (9)

and to positive consumption  $\mathbf{P}$ -a.s.

$U$  is a well-behaved utility function and  $\tau (\geq t)$  is the investor's horizon. Let  $J(t, W(t), Y(t))$  be the value function. We assume that  $J(\cdot)$  is an increasing and strictly concave function of  $W$ .<sup>9</sup> The obvious notation  $J_i$  (respectively,  $J_{ii}$ ) stands for the first (respectively, second) partial derivative of  $J(\cdot)$  with respect to its argument  $i$ . Our first derivations, shown in the appendix, lead to the following result.

**Proposition 1.** *Under our set of assumptions, an investor's optimal demand for risky assets is equal to:*

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\theta} \end{pmatrix} = (\Sigma \Sigma')^{-1} \begin{pmatrix} \mu_S - \mathbf{1}_N r \\ \mu_F + \Sigma_F \sigma_P \end{pmatrix} \left( -\frac{J_W}{W J_{WW}} \right) + (\Sigma \Sigma')^{-1} \Sigma \Sigma_Y' \left( -\frac{J_{WY}}{W J_{WW}} \right) - (\Sigma \Sigma')^{-1} \Sigma \sigma_P \frac{X}{W} \quad (11)$$

It is important to stress first that in spite of the presence of  $X$  in the last term of the RHS of equation (11), this solution is indeed in closed form. This is because  $X(t)$  depends on the holdings  $\hat{\alpha}(t)$  and  $\hat{\theta}(t)$  of time  $t$ , and is therefore known at date  $t$  when the new holdings are chosen. For instance, at date  $t = 0$ ,  $\hat{\alpha}(0)$  and  $\hat{\theta}(0)$  are selected with  $X(0)$  equal to zero.

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<sup>9</sup> Cox et al. (1985) provide necessary conditions for  $J$  to be an increasing and strictly concave function of  $W$ .

The demand for risky assets thus contains three components. The first term on the RHS of (11) is the usual mean-variance speculative component while the last two are hedging components. We leave the discussion of the former to the next sub-sections and focus here on the last two terms.

The first hedging component is the traditional Merton-Breeden hedge. It is an information-based and dynamic term whose purpose is to hedge wealth against unfavorable shifts of the  $K$  economic state variables that drive the opportunity set. Note that this component is, as usual, preference dependent, and that  $J_{WY} (\equiv \partial J_W / \partial Y)$  is the cross-partial derivative that represents the effect of the state variable  $Y$  on the marginal value of wealth. In the case of logarithmic (Bernoulli) utility, the investor has a myopic behavior and this hedge would disappear since the cross partial derivative  $J_{WY}$  would vanish.

The second hedging component results specifically from trading in forward contracts and would of course vanish in absence of the latter. It is equal to the fraction of wealth  $X/W$  corresponding to the forward position value times a usual covariance/variance ratio. This term however, unlike the preceding one, does not qualify as a Merton-Breeden component for two reasons. First, it is not a hedge against future levels of state variables. Second, it does not depend on the utility coefficient ( $-J_{WY}/J_{WW}$ ), although it is not strictly preference free since optimal  $X$  and  $W$  depend on the investor's utility. Rather, this second hedging component is due to the forward position not being (continuously) marked-to-market, hence bearing an additional interest rate risk on the cumulative algebraic gains that have accrued so far.

Thus, the source of this additional risk is to be found in the *forward trading strategy* itself. Because they anticipate that one period ahead the current value of their forward position will have changed, investors will optimally hedge against the interest rate risk brought about by their strategy. Thus, this hedging component is not due to the presence of an exogenous source of non-diversifiable risk but results from the *endogenous* risk brought about by the particular nature of the strategy involving forward contracts. In addition, as intuition suggests, it depends on the fraction of wealth  $X/W$  that has been generated so far by the forward strategy.

It must be emphasized that this component being independent of  $J_{WY} (\equiv \partial J_W / \partial Y)$ , it will appear in the optimal strategy of even a myopic Bernoulli investor, contrary to

the first hedging component.<sup>10</sup> This result is reminiscent of the strategy of a logarithmic investor endowed with a non-traded position who maximizes his expected utility of wealth by adding to the usual speculative component two hedging terms, one preference dependent and one preference free.<sup>11</sup> In the same way, our investor includes a hedging component in his strategy because forward trading creates a non-traded position in an implicit discount bond.

### 3.2 A separation result

The setting with non-redundant forward contracts that we have chosen generates an interesting separation result. Merton's (1973) traditional separation theorem states that, at equilibrium, any investor divides his investment between the riskless asset, the optimum growth portfolio<sup>12</sup> and the  $K$  funds that are best correlated with the  $K$  state variables. Consequently, a  $(K+2)$  fund separation obtains. Here we can state instead the following proposition.<sup>13</sup>

**Proposition 2.** *Under our set of assumptions, when non-redundant forward contracts are traded, investors are indifferent to the reduction of their investment opportunity set from  $N+1$  primitive cash assets and  $H$  forward contracts to  $(K + 3)$  mutual funds: the riskless asset, a speculative portfolio providing the optimal risk-return trade-off,  $K$  funds hedging against random shifts in the  $K$  state variables, and a hedge portfolio against the interest rate risk generated by the forward position. All these portfolios comprise both risky cash assets and forward contracts. The optimal demand for risky assets is given by:*

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<sup>10</sup> It will also appear in the optimal strategy of a pure hedger, as shown by Lioui and Poncet (2000).

<sup>11</sup> See Adler and Detemple (1988).

<sup>12</sup> This portfolio, also called the logarithmic or the numeraire portfolio, is the one held by the (myopic) Bernoulli investor. It maximizes the expected log of terminal wealth. See Long (1990). It is important to recall that (i) the market portfolio is neither the tangency portfolio nor the numeraire portfolio, and (ii) the weights of the risky assets in the (instantaneous) tangency and numeraire portfolio are homothetic. See Bajeux-Besnainou and Portait (1997).

<sup>13</sup> We are grateful to the referee for having pointed out to us this separation, starting from the  $(K+4)$  separation of Proposition 3 below.

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\theta} \end{pmatrix} = \omega_0 \lambda_0 + \sum_{j=1}^K \omega_j \lambda_j + \omega_{K+1} \lambda_{K+1} \quad (12)$$

with:

$$\begin{aligned} \lambda_0 &= \left( \left( (\Sigma \Sigma')^{-1} \begin{pmatrix} \mu_S - \mathbf{1}_N r \\ \mu_F + \Sigma_F \sigma_P \end{pmatrix} \right)' \begin{pmatrix} \mathbf{1}_N \\ \mathbf{0}_H \end{pmatrix} \right)^{-1} (\Sigma \Sigma')^{-1} \begin{pmatrix} \mu_S - \mathbf{1}_N r \\ \mu_F + \Sigma_F \sigma_P \end{pmatrix} \\ \lambda_j &= \left( \left( (\Sigma \Sigma')^{-1} \Sigma \Sigma_Y' \right)_j' \begin{pmatrix} \mathbf{1}_N \\ \mathbf{0}_H \end{pmatrix} \right)^{-1} \left( (\Sigma \Sigma')^{-1} \Sigma \Sigma_Y' \right)_j \quad j=1, \dots, K \\ \lambda_{K+1} &= \left( \left( (\Sigma \Sigma')^{-1} \Sigma \sigma_P \right)' \begin{pmatrix} \mathbf{1}_N \\ \mathbf{0}_H \end{pmatrix} \right)^{-1} (\Sigma \Sigma')^{-1} \Sigma \sigma_P \end{aligned}$$

where  $\left( (\Sigma \Sigma')^{-1} \Sigma \Sigma_Y' \right)_j$  is the  $j^{th}$   $((N+H) \times 1)$  column vector of the matrix  $(\Sigma \Sigma')^{-1} \Sigma \Sigma_Y'$ , and the  $\omega$ 's are weights defined by:

$$\begin{aligned} \omega_0 &= \left( (\Sigma \Sigma')^{-1} \begin{pmatrix} \mu_S - \mathbf{1}_N r \\ \mu_F + \Sigma_F \sigma_P \end{pmatrix} \right)' \begin{pmatrix} \mathbf{1}_N \\ \mathbf{0}_H \end{pmatrix} \left( -\frac{J_W}{W J_{WW}} \right) \\ \omega_j &= \left( (\Sigma \Sigma')^{-1} \Sigma \Sigma_Y' \right)_j' \begin{pmatrix} \mathbf{1}_N \\ \mathbf{0}_H \end{pmatrix} \left( -\frac{J_{WY_j}}{J_{WW}} \right) \\ \omega_{K+1} &= - \left( (\Sigma \Sigma')^{-1} \Sigma \sigma_P \right)' \begin{pmatrix} \mathbf{1}_N \\ \mathbf{0}_H \end{pmatrix} \frac{X}{W} \end{aligned}$$

The first fund is of course the riskless asset. We recover the  $K$  traditional Merton-Breeden hedging funds ( $\lambda_j$ ,  $j=1, \dots, K$ ) from the second term on the RHS of equation (12). These portfolios hedge against the fluctuations in the opportunity set brought about by the state variables. Fund  $(K+2)$  is a portfolio of risky cash assets and forward contracts constructed as usual to provide optimal diversification. It stems from the first term on the RHS of equation (12), the traditional speculative component of the optimal dynamic strategy of an intertemporal expected utility maximizer. However, while Merton's corresponding fund is a mean-variance efficient fund of cash assets, our  $\lambda_0$  is the growth optimum portfolio generated by risky cash assets and forward contracts. It will be studied in detail in the next subsection.

The first important difference between our separation result and Merton's is that not only the speculative component contains forward contracts, but also the portfolio of risky cash assets that is part of this component need not be mean-variance efficient within the set of portfolios comprising cash assets only. In Merton's separation, the speculative component is homothetic to the portfolio tangent to the instantaneous efficient frontier of risky cash assets. Consequently, the market portfolio in an economy populated by myopic investors turns out to be mean-variance efficient at equilibrium. We examine in the next section the conditions under which these standard results of portfolio choice theory are recovered when forward contracts are included in the opportunity set.

The second, major, difference with Merton's results lies in the presence of an extra fund, number  $(K+3)$ , given by the third term on the RHS of equation (12). As already explained, this fund  $\lambda_X$  allows investors to hedge against the interest rate risk generated by their forward strategies. It contains both forward contracts and primitive assets. Since this component is independent of  $J_{WY} (\equiv \partial J_W / \partial Y)$ , it will be present even in a Bernoulli investor's portfolio, unlike the  $K$  Merton-Breeden hedging portfolios. However, for the same reason, this hedging component will not lead to any additional risk premium at equilibrium since it cancels out by aggregation. Consequently, when the traditional ICAPM will be shown in Section IV not to hold at equilibrium for forward contracts although it does for cash assets, the discrepancy will not be due to the presence of this extra fund.

A comparison of what would occur should futures be substituted for forward contracts is instructive. Lioui and Poncet (2001) have recently obtained an optimal demand equation similar to equation (12) but with an important difference. The optimal demand for risky assets does not contain the last fund that hedges against interest risk. This is because the futures positions are continuously marked-to-market and therefore do not bring about any new source of risk. Our finding (12) thus exhibits an (overlooked) important difference between forward and futures contracts. Since in general financial markets are not complete, this additional interest rate risk cannot be perfectly hedged. Hence the marking-to-market mechanism that characterizes futures contracts and allows for the complete elimination of this risk is valuable to risk-averse agents.

### 3.3 The separation result revisited

Our objective in this subsection is to analyze in more depth the speculative component of equations (11) or (12). The latter can actually be split in two distinct funds, one comprising cash assets only, and the other forward contracts only. We fully characterize these portfolios and provide a new separation result that allows a better understanding of the changes in the standard optimal dynamic strategy brought about by the introduction of forward contracts.

The speculative component in equation (11) can be rewritten as follows:

$$\left(-\frac{J_W}{WJ_{WW}}\right)(\Sigma\Sigma')^{-1}\begin{pmatrix}\mu_S - \mathbf{1}_N r \\ \mu_F + \Sigma_F \sigma_P\end{pmatrix} = \left(-\frac{J_W}{WJ_{WW}}\Lambda\right)\begin{bmatrix}A\lambda_S - (\Sigma_S \Sigma_S')^{-1} \Sigma_S \Sigma_F' B\lambda_F \\ B\lambda_F - (\Sigma_F \Sigma_F')^{-1} \Sigma_F \Sigma_S' A\lambda_S\end{bmatrix} \quad (13)$$

where

$$\lambda_S \equiv \frac{(\Sigma_S \Sigma_S')^{-1}(\mu_S - \mathbf{1}_N r)}{\mathbf{1}_N' (\Sigma_S \Sigma_S')^{-1}(\mu_S - \mathbf{1}_N r)} = \frac{(\Sigma_S \Sigma_S')^{-1}(\mu_S - \mathbf{1}_N r)}{\Lambda} \quad (14)$$

$$\lambda_F \equiv (\Sigma_F \Sigma_F')^{-1}(\mu_F + \Sigma_F \sigma_P) \quad (15)$$

and

$$\Lambda = \mathbf{1}_N' (\Sigma_S \Sigma_S')^{-1}(\mu_S - \mathbf{1}_N r)$$

$$A = \left[ \Sigma_S \Sigma_S' - \Sigma_S \Sigma_F' (\Sigma_F \Sigma_F')^{-1} \Sigma_F \Sigma_S' \right]^{-1} (\Sigma_S \Sigma_S')$$

$$B = \frac{1}{\Lambda} \left[ \Sigma_F \Sigma_F' - \Sigma_F \Sigma_S' (\Sigma_S \Sigma_S')^{-1} \Sigma_S \Sigma_F' \right]^{-1} (\Sigma_F \Sigma_F')$$

Consider the vectors  $\lambda_S$  and  $\lambda_F$ . The former is traditional and concerns risky cash assets *only*. The latter is novel, involves forward contracts *only*, and has itself two components. The first one is a speculative term involving the drifts and diffusion parameters of the forward price processes  $\left( (\Sigma_F \Sigma_F')^{-1} \mu_F \right)$ . The second term,  $(\Sigma_F \Sigma_F')^{-1} \Sigma_F \sigma_P$ , is a minimum variance hedge against the interest rate risk brought about by the forward position and related to the discount bond of same maturity as the



forward contracts.<sup>14</sup> Therefore,  $\lambda_F$  is not mean-variance efficient in the usual sense. Rather, it is efficient within the set of portfolios containing forward contracts only. For convenience, we dub it adjusted-mean-variance efficient. It must also be noted that for a portfolio of cash assets, the sum of the weights must be equal to one, while this restriction is meaningless for a portfolio of forwards. Therefore, the  $(N \times 1)$  vector  $\lambda_S$  is constructed such that it can be interpreted as a portfolio of cash assets whose weights sum to one. This property does not apply to the  $(H \times 1)$  vector  $\lambda_F$ , which in fact is the set of the market prices of risk for the forward contracts.

Now consider the extreme case where the primitive assets and the forward contracts are not correlated.  $A$  and  $\Lambda B$  then are identity matrices so that equation (13) reduces to:

$$\left( -\frac{J_W}{WJ_{WW}} \right) (\Sigma \Sigma')^{-1} \begin{pmatrix} \mu_S - \mathbf{1}_N r \\ \mu_F + \Sigma_F \sigma_P \end{pmatrix} = \begin{pmatrix} -\frac{J_W}{WJ_{WW}} \begin{bmatrix} \Lambda \lambda_S \\ \lambda_F \end{bmatrix} \end{pmatrix} \quad (13')$$

The investor's optimal speculative component then consists of two basic, efficient, portfolios: the first one is the traditional mean-variance efficient portfolio containing primitive assets *only* and the second one is the adjusted-mean-variance efficient portfolio containing forward contracts *only*. Therefore, in this very special case we recover some of the traditional results in portfolio theory. Since the decomposition of the speculative component is preference free, it would lead to a  $(K+4)$ -fund separation result in which the dynamic asset allocation would be generated by trading in the riskless asset, an efficient portfolio of risky cash assets, an efficient portfolio of forward contracts, the  $K$  Merton-Breeden hedging funds, and a fund hedging against interest rate risk.

More generally, however, when some correlation between primitive assets and forward contracts is present, the matrices  $A$  and  $\Lambda B$  are not identity matrices but are completely general. Consequently, portfolios  $A\lambda_S$  and  $B\lambda_F$  are "perturbed" and no longer mean-variance efficient. Furthermore, in the demand for cash assets an extra term is subtracted from  $A\lambda_S$  that depends on  $B\lambda_F$  and thus accounts for cross-hedging

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<sup>14</sup> Note that this term concerns *only* forward contracts while the hedging component found in the demand for risky assets (the third element on the RHS of (11)) combines forward contracts with primitive assets.

(with forwards) effects. A similar structure is found in the demand for forwards. Consequently, the two speculative portfolios are even more perturbed. They both depend on expected returns on all traded assets and their correlations, and both are mean-variance inefficient within their respective mean-variance set. To summarize, each of the two speculative components given by equation (13) is a combination of a (preference free) perturbation of the two basic portfolios adjusted for cross hedging.

Nevertheless, even in this general case, the speculative component can be usefully decomposed into two parts. Define the following two portfolios:

$$\begin{aligned}\lambda_{0S} &= \left( \begin{bmatrix} A\lambda_S - (\Sigma_S \Sigma_S')^{-1} \Sigma_S \Sigma_F' B\lambda_F \\ \mathbf{0}_H \end{bmatrix} \mathbf{1}_{N+H} \right)^{-1} \begin{bmatrix} A\lambda_S - (\Sigma_S \Sigma_S')^{-1} \Sigma_S \Sigma_F' B\lambda_F \\ \mathbf{0}_H \end{bmatrix} \\ \lambda_{0F} &= \begin{bmatrix} \mathbf{0}_N \\ B\lambda_F - (\Sigma_F \Sigma_F')^{-1} \Sigma_F \Sigma_S' A\lambda_S \end{bmatrix}\end{aligned}\quad (16)$$

Then equation (13) can be rewritten as:

$$\left( -\frac{J_W}{WJ_{WW}} \right) (\Sigma \Sigma')^{-1} \begin{pmatrix} \mu_S - \mathbf{1}_N r \\ \mu_F + \Sigma_F \sigma_P \end{pmatrix} = \omega_{0S} \lambda_{0S} + \omega_{0F} \lambda_{0F} \quad (17)$$

where  $\omega_{0S}$  and  $\omega_{0F}$  are scalars defined by:

$$\begin{aligned}\omega_{0S} &= \left( -\frac{J_W}{WJ_{WW}} \Lambda \right) \begin{pmatrix} A\lambda_S - (\Sigma_S \Sigma_S')^{-1} \Sigma_S \Sigma_F' B\lambda_F \\ \mathbf{0}_H \end{pmatrix} \mathbf{1}_{N+H} \\ \omega_{0F} &= -\frac{J_W}{WJ_{WW}} \Lambda\end{aligned}$$

Equation (17) establishes the two-fund separation property of the speculative part of the investor's optimal demands. Consequently, we obtain a (K+4)-fund separation result, stated in the following proposition.

**Proposition 3.** *Under the conditions of Proposition 2, investors are indifferent to the reduction of their investment opportunity set to (K + 4) mutual funds: Merton's K hedging funds, the riskless asset, a hedge portfolio comprising both cash and forward assets, a speculative portfolio of cash assets only, and a speculative portfolio of forward contracts only.*

A remark is in order at this stage. One could argue that the  $(K+1)$  portfolios hedging against the state variables and interest rate risk could, like the speculative portfolio, each be split into a portfolio of cash assets and one of forward contracts. Yet this decomposition is irrelevant for the hedging portfolios, since investors will construct, from both the cash and forward assets, the (unique) portfolio that is best correlated with each and every source of uncertainty to be hedged. It may so happen that, for some given source of risk, the best hedging portfolio contains only some cash assets, only some forwards or a combination of some of them.

Finally, one may again wonder what would occur should futures be substituted for forward contracts. An important difference would be that, since there is no additional risk due to the futures trading strategy, the drift terms  $(\mu_F + \Sigma_F \sigma_P)$  present in the speculative part of the demand (17) would reduce to the drifts  $\mu_F$  of the futures contracts.<sup>15</sup>

## **IV. Financial market equilibrium**

Now, we fully characterize the market portfolio and derive an ICAPM for both primitive cash assets and forward contracts.

### *4.1. Market portfolio characterization*

In the static CAPM, mean-variance efficiency of the market portfolio is a necessary and sufficient condition for an exact linear relationship to exist between the expected return on an asset and its risk measured by its beta. Roll's (1977) critique initiated a flood of research that mainly focused on developing econometric tests of the mean-variance efficiency of the market portfolio. Kandel and Stambaugh (1995) were the first authors to carefully distinguish the two theoretical implications of the static model, i.e. market portfolio efficiency and the linear risk-return relationship. They showed that, if the market portfolio is not exactly efficient, "either (implication) can hold nearly perfectly while the other fails grossly."<sup>16</sup>

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<sup>15</sup> See Lioui and Poncet (2001).

<sup>16</sup> The quote is from their abstract, on p. 157.

In the dynamic ICAPM, some economic state variables are assumed to drive the investment opportunity set. In such a setting, investors' optimal demands for risky assets include hedging components, in addition to the usual mean-variance terms [see equation (11)]. Those hedging components protect investors against unfavorable shifts in their opportunity set brought about by random changes in the state variables. The market portfolio then need not be mean-variance efficient [see, for instance, Fama (1996)]. Yet, a special case is worth mentioning. It arises when investors are endowed with time-additive logarithmic utility functions. It has been thoroughly examined in the literature and is widely used in theoretical finance as it represents the benchmark case for risk aversion.<sup>17</sup> Indeed, logarithmic investors are (uniquely) myopic in the sense that they do not hedge against the random fluctuations of their opportunity set. In that case, like in the static CAPM, the market portfolio must be mean-variance efficient for the ICAPM to hold.

In our setting, this is no longer true. To see this and fully characterize the market portfolio, we restrict ourselves here to the case of logarithmic (Bernoulli) investors. The extension to other types of utility functions is straightforward.

Using the fact that  $J_{WY}$  is equal to zero and the relative risk tolerance coefficient is equal to one for a Bernoulli investor, the demand for risky assets (11) simplifies to:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\theta} \end{pmatrix} = \Lambda \begin{bmatrix} A & -(\Sigma_S \Sigma_S')^{-1} \Sigma_S \Sigma_F' B \\ -(\Sigma_F \Sigma_F')^{-1} \Sigma_F \Sigma_S' A & B \end{bmatrix} \begin{bmatrix} \lambda_S \\ \lambda_F \end{bmatrix} - (\Sigma \Sigma')^{-1} \Sigma \sigma_P \frac{X}{W} \quad (18)$$

Since the aggregate value of  $X$  is zero, summing over all investors yields:

$$\begin{pmatrix} \alpha_M \\ \mathbf{0}_H \end{pmatrix} = \Lambda \begin{bmatrix} A & -(\Sigma_S \Sigma_S')^{-1} \Sigma_S \Sigma_F' B \\ -(\Sigma_F \Sigma_F')^{-1} \Sigma_F \Sigma_S' A & B \end{bmatrix} \begin{bmatrix} \lambda_S \\ \lambda_F \end{bmatrix} \quad (19)$$

where  $\alpha_M$  is the vector of weights in the market portfolio. Eliminating  $B$  by using the second row of (19), the first row becomes:

$$\alpha_M = \Lambda \left( I_N - (\Sigma_S \Sigma_S')^{-1} \Sigma_S \Sigma_F' (\Sigma_F \Sigma_F')^{-1} \Sigma_F \Sigma_S' \right) A \lambda_S \quad (20)$$

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<sup>17</sup> See for instance Merton (1971, 1973), Rubinstein (1976), Breeden (1979), or Cox et al. (1985).

Equation (20) clearly implies that the market portfolio is not proportional to a mean-variance efficient portfolio of primitive assets. It immediately follows that the market portfolio *need not* be mean-variance efficient. This result may have important implications when testing the ICAPM, because the linear relationship between expected return and beta will be shown to hold for primitive assets despite the (possible) inefficiency of the market portfolio.

#### 4.2. Market equilibrium

We now derive the ICAPM for the primitive assets and the non-redundant forward contracts and offer an interpretation and a discussion of the results. We stress that we do not restrict here the class of utility functions to the logarithmic.

To characterize the equilibrium, we now assume, like CIR (1985) among others, that a representative investor exists. Thus, aggregate wealth and individual wealth are equal. We further impose the usual market clearing conditions. Total wealth is invested in the primitive assets ( $\hat{\alpha}' \mathbf{1}_N = 1$ ), net positions in the forward contracts are equal to zero ( $\hat{\theta} = \mathbf{0}_H$ ) as well as the net investment in the riskless asset ( $\hat{\gamma} = 0$ ). Combining these and the optimality conditions leads to the following result.

**Proposition 4.** *Under our set of assumptions, at equilibrium the expected returns on primitive assets satisfy the following relationship:*

$$\mu_{S_i} = r + \beta_{i,M}(\mu_M - r) + \sum_{k=1}^K \beta_{i,Y_k}(\mu_{Y_k} - r) \quad (21)$$

while the expected percent changes in the forward contract prices satisfy:

$$\mu_{F_j} = -\sigma_{F_j}' \sigma_P + \beta_{j,M}(\mu_M - r) + \sum_{k=1}^K \beta_{j,Y_k}(\mu_{Y_k} - r) \quad (22)$$

where  $\beta_{l,U}$  stands for the sensitivity of the return on asset  $l$  to the excess return on the market portfolio ( $U = M$ ) or to the (adjusted for the riskless rate) drift of a state variable process ( $U = Y_k$ ).

The excess returns on the primitive cash assets (21) satisfy the standard ICAPM. This result is interesting to the extent that we have found the market portfolio not to be necessarily efficient. In a dynamic model where the risk emanating from the state

variables is priced by the market, a multi-beta ICAPM thus obtains *regardless of whether the market portfolio is mean-variance efficient or not*. This result is due to the presence of non-redundant forward contracts in the investor's portfolio. Moreover, we have shown our result to hold even for Bernoulli investors who do not price the risks associated with the state variables: the market portfolio may not be mean-variance efficient while the traditional one-beta ICAPM for cash assets still holds for logarithmic investors.

Now, the (multi-beta) ICAPM in its standard form does not hold for forward contracts (see Cox et al. (1981)). Instead, the ICAPM for forward contracts obeys equation (19). It turns out that the expected percent change in the price of a forward contract comprises not only the usual premiums on the market and the state variables but also an additional term.<sup>18</sup> The latter includes the instantaneous covariance between the forward price and the pure discount bond price of identical maturity. The negative sign that affects this component is logical since the correlation between the forward price of an asset and the relevant discount bond *price* is negative. This implies that the expected percent change in the price of a forward contract contains an (additional) element of compensation for the strategy. We name this additional term a *strategy risk premium*. Consequently, we can state that holding a forward contract is rewarded also for the (systematic) risk that results from the covariance between the contract and its associated discount bond. The intuition behind this result is straightforward: when trading on forward contracts the investor is bound to have a non-tradable discount bond position and, being unable to diversify the corresponding systematic risk, she must be compensated for it.

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<sup>18</sup> Since there is no investment in forward contracts (no cost-of-carry), the riskless rate of return  $r$  is absent from the expected return equation.

## V. Concluding remarks

Our findings help clarify the intuition that, when investors trade forward contracts, they bear an additional, interest rate related, risk. More importantly, we show a less intuitive, more fundamental, result regarding the way the presence of this forward trading strategy risk affects the equilibrium expected returns on the cash assets traded in an incomplete economy. It turns out that only the expected percent change in the forward prices are affected, the required returns on the other assets remaining unchanged. Another finding is that the covariance term present (uniquely) in the ICAPM for forwards does not stem from the additional components in the investors' optimal dynamic strategies since the latter cancel out at equilibrium. This implies that the ICAPM (22) for forward contracts holds regardless of whether interest rate risk can be perfectly hedged or not.

We want to stress that knowing the cash-and-carry relationship between the forward and the spot prices and the ICAPM for the underlying cash asset and the relevant discount bond does not make trivial the derivation of the corresponding ICAPM equation for the forward. First, the cash-and-carry relationship in general does not hold for forward contracts, redundant or not. Second, this method would assume away the issue of how the introduction of forwards in an incomplete economy affects the equilibrium prices of existing cash assets. Third, one could not provide a convincing economic interpretation of the result since the precise mechanism that makes the traditional ICAPM invalid for forwards would not be exhibited. By contrast, none of our results is grounded on the cash-and-carry relationship.

We have derived our results within the simple framework of a frictionless, albeit incomplete, financial market. Hopefully, their basic insight will prove useful in more complicated situations. In particular, the equations for the optimal demands and market equilibrium are tractable yet reasonably realistic, so that the model should be applicable to the analysis of a variety of both theoretical and empirical issues. Introducing some market frictions such as transaction costs or portfolio composition restrictions may lead to a further understanding of the particularities of forward contracts. Work is currently in progress in that direction.

## Mathematical appendix

*Proof of Proposition 1.* Let  $J(t, W(t), Y(t))$  be the value function and let  $L(t)$  be the differential generator of  $J(\cdot)$ . We assume that  $J(\cdot)$  is an increasing and strictly concave function of  $W$ .<sup>19</sup> Letting  $\psi \equiv LJ + U$ , the necessary and sufficient conditions for optimality that are derived from the Hamilton-Jacobi-Bellman equation write as follows:

$$\psi_C = U_C - J_W \leq 0 \quad (a)$$

$$\hat{C}\psi_C = 0 \quad (b)$$

$$\begin{aligned} \psi_\alpha = & (\mu_S - \mathbf{1}_N \mu_P) W J_W + (\Sigma_S - \mathbf{1}_N \sigma_P)' \Sigma_Y' W J_{WY} + \\ & \left[ (\Sigma_S - \mathbf{1}_N \sigma_P)' (\Sigma_S - \mathbf{1}_N \sigma_P)' \hat{\alpha} + (1 - \hat{\gamma}) (\Sigma_S - \mathbf{1}_N \sigma_P)' \sigma_P \right. \\ & \left. + (\Sigma_S - \mathbf{1}_N \sigma_P)' \Sigma_F' \hat{\theta} \right] W^2 J_{WW} = 0 \end{aligned} \quad (c)$$

$$\psi_\theta = (\mu_F + \Sigma_F \sigma_P) W J_W + \Sigma_F \Sigma_Y' W J_{WY} + \left[ \Sigma_F (\Sigma_S - \mathbf{1}_N \sigma_P)' \hat{\alpha} + (1 - \hat{\gamma}) \right] W^2 J_{WW} = 0 \quad (d)$$

$$\psi_\gamma = (r - \mu_P) W J_W - \sigma_P' \Sigma_Y' W J_{WY} - \left[ \hat{\alpha}' (\Sigma_S - \mathbf{1}_N \sigma_P)' \sigma_P + (1 - \hat{\gamma}) \sigma_P' \sigma_P \right. \\ \left. + \hat{\theta}' \Sigma_F \sigma_P \right] W^2 J_{WW} = 0 \quad (e)$$

with obvious notations for partial derivatives of the value function with respect to each of its arguments. The hat  $\hat{\cdot}$  above a variable denotes an optimal value. The first two (Kuhn and Tucker) equations concern consumption and follow from the non-negativity constraint imposed on the optimal consumption path.

Combining equations (c) to (e) to eliminate the proportion  $\hat{\gamma}$  of wealth held in the riskless asset leads to the result.

*Proof of Proposition 4.* Using (c), (d) and (e), one easily shows that, at equilibrium,

$$\begin{pmatrix} \mu_S - \mathbf{1}_N r \\ \mu_F + \Sigma_F \sigma_P \end{pmatrix} = V_{SF,W} \left( -\frac{W J_{WW}}{J_W} \right) + V_{SF,Y} \left( -\frac{J_{WY}}{J_W} \right) \quad (g)$$

$$\text{where } V_{SF,W} = \begin{pmatrix} \Sigma_S \Sigma_S' \hat{\alpha} \\ \Sigma_F \Sigma_S' \hat{\alpha} \end{pmatrix} \text{ and } V_{SF,Y} = \begin{pmatrix} \Sigma_S \Sigma_Y' \\ \Sigma_F \Sigma_Y' \end{pmatrix}.$$

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<sup>19</sup> See footnote 8.



Now pre-multiplying (g) by  $\begin{pmatrix} \hat{\alpha}' & \mathbf{0}_H' \end{pmatrix}$  yields the equilibrium risk-return trade-off for total wealth (which is equal to the market portfolio value):

$$\mu_W - r = V_{W,W} \left( -\frac{WJ_{WW}}{J_W} \right) + V_{W,Y}' \left( -\frac{J_{WY}}{J_W} \right) \quad (h)$$

where  $\mu_W$  is the expected return on total wealth,  $V_{W,W}$  is the variance of total wealth and  $V_{W,Y}$  the  $(K \times 1)$  vector of covariances between total wealth and the state variables.

Also, pre-multiplying (g) by  $\Sigma_Y (\Sigma' \Sigma)^{-1} \Sigma'$  gives:

$$\mu_Y - \mathbf{1}_H r \equiv \Sigma_Y (\Sigma' \Sigma)^{-1} \Sigma' \begin{pmatrix} \mu_S - \mathbf{1}_N r \\ \mu_F + \Sigma_F \sigma_P \end{pmatrix} = V_{Y,W} \left( -\frac{WJ_{WW}}{J_W} \right) + V_{Y,Y} \left( -\frac{J_{WY}}{J_W} \right) \quad (i)$$

where  $V_{Y,W} (=V_{W,Y})$  is a  $(K \times 1)$  covariance vector and  $V_{Y,Y}$  is a  $(K \times K)$  variance-covariance matrix.

Combining (i) and (h) yields:

$$\begin{pmatrix} \mu_W - r \\ \mu_Y - \mathbf{1}_H r \end{pmatrix} = \begin{pmatrix} V_{W,W} & V_{W,Y}' \\ V_{Y,W} & V_{Y,Y} \end{pmatrix} \begin{pmatrix} -\frac{WJ_{WW}}{J_W} \\ -\frac{J_{WY}}{J_W} \end{pmatrix}$$

Defining  $\Sigma_{WY} \equiv \begin{pmatrix} V_{W,W} & V_{W,Y}' \\ V_{Y,W} & V_{Y,Y} \end{pmatrix}$ , we get:

$$\begin{pmatrix} -\frac{WJ_{WW}}{J_W} \\ -\frac{J_{WY}}{J_W} \end{pmatrix} = (\Sigma_{WY})^{-1} \begin{pmatrix} \mu_W - r \\ \mu_Y - \mathbf{1}_H r \end{pmatrix} \quad (j)$$

Substituting (j) into (g) and defining  $V_{SF,WY} \equiv (V_{SF,W} \quad V_{SF,Y})$  yields:

$$\begin{pmatrix} \mu_S - \mathbf{1}_N r \\ \mu_F + \Sigma_F \sigma_P \end{pmatrix} = V_{SF,WY} (V_{WY})^{-1} \begin{pmatrix} \mu_W - r \\ \mu_Y - \mathbf{1}_H r \end{pmatrix} \quad (k)$$

Results (18) and (19) follow from (k) since the value of primitive assets represents total wealth at equilibrium so that  $W$  is the value of the market portfolio  $M$ .

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