QUANTUM ADVANTAGE IN BAYESIAN GAMES

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Quantum advantage in Bayesian games, or games with incomplete information, refers to the larger set of correlated equilibrium outcomes that can be obtained by using quantum mechanisms rather than classical ones. Earlier examples of such an advantage go under the title of quantum pseudo-telepathy. By using measurements of entangled particles, the players in the Mermin–Peres magic square game and similar games can obtain a common payoff that is higher than that afforded by any classical mechanism. However, these common-interest games are very special. In general games, where payoffs differ across players and player types, the implementation of specific correlated equilibrium outcomes may require limiting the information that different player types get though the signals or messages they receive from a correlation device or mechanism. Because of the inherently destructive nature of measurements in quantum mechanics, it is well suited for this task. In a *quantum correlated equilibrium*, players choose what part of the information "encoded" in the quantum state to read, and choosing the part meant for their actual type is required to be incentive compatible. This requirement makes the choice of measurement analogous to the choice of report to the mediator in a communication equilibrium, with the measurement value analogous to the massage sent back from the mediator. Each player then makes an incentive-compatible choice of action, which depends on the player's type, chosen measurement and the measurement value.
This paper systematically explores the advantage quantum mechanisms has over comparable classical mechanisms in correlated and communication equilibria, and identifies the specific properties of quantum mechanisms responsible for these advantages. It presents a classification of the equilibrium outcomes (both type-action distributions and equilibrium payoffs) in correlated and communication equilibria according to the kind of (classical or quantum) mechanism employed.

1 Introduction

Quantum advantage refers to the potential use of quantum mechanical phenomena for achieving goals that are not achievable by classical means. In computer science, that goal may be solving a computational problem that no classical algorithm is practically capable of solving. More fundamentally, quantum physics may be used for, in a sense, defying probability theory itself. The mathematical theory of probability is based on the concept of *probability space* (Shiryaev 1996), a main ingredient of which is a *sample space*, which is a set of points Ω. The interpretation is that each point $\omega \in \Omega$ represents a possible "state of the world". Knowing the state amounts to resolving all uncertainty concerning all – past and present – random occurrences. It may be theoretically impossible to ever know the state. For example, it is not possible to learn the outcomes of an infinite sequence of coin tosses. It is only possible to know, after observing the outcomes of finitely many tosses, that the actual state ω belons to a certain subset of Ω, an event. However, a most remarkable aspect of quantum physics is that it denies even the conceptual, theoretical possibility of states of the world. Such states correspond to what are known as "hidden variables". As first shown by Bell (1964), the existence of hidden variables is inconsistent with the actual quantitative predications of quantum mechanics. (For an informal exposition of Bell's findings and their significance, see Albert 1994. See also below.) As these predications were repeatedly verified by experiments,[1] the meaning of Bell's theoretical result is that quantum phenomena involve a kind of uncertainty that is not captured by "classical" probability theory, where the employment of a sample space is standard and is never questioned.

This paper explores the possibilities offered by quantum phenomena when applied to Bayesian, or incomplete information, games. This application is part of what is known as *quantum game theory* (see, for example, Eisert et al. 1999, Guo et al. 2008, Khan et al. 2018). Specifically, the present subject matter is correlated equilibria in Bayesian games (Forges 1993, Lehrer et at. 2013, Milchtaich 2014, Bergemann and Morris 2016). Such equilibria employ a mechanism, or correlation device, that substitutes for direct communication between players. Coordination is achieved by each player observing a (generally, private) signal sent by the mechanism, and these signals are generally correlated. The use of a mechanism prevents the players from sharing any private information they possess. However, with a *quantum* mechanism, the outcome may be as if information was shared, a phenomenon dubbed quantum pseudo-telepathy. In the following subsections, I review two earlier notable applications of this phenomenon to Bayesian games and briefly explain the physics behind them. I then describe what I believe is a significant missing part in this domain of quantum game theory.

1.1 Mermin–Peres magic square game

A beautiful demonstration of how classically impossible equilibrium outcomes in Bayesian games can be achieved by using quantum pseudo-telepathy is provided by the Mermin–Peres magic square game (Mermin 1990a, Peres 1990, Xu et al. 2022). This is a two-player common-interest game where one player, Alice, is randomly allotted one of the three rows of a 3×3 table and the other player, Bob, is allotted one of the columns. All nine possible combinations are equally probable, which in particular means that the players' allotments, or *types*, are independent. Alice, who only knows her own type, has to fill her row by placing either $+1$ or -1 in each of its cells, with the proviso that the product of the three numbers has to be $+1$. Similarly for Bob's column, except that for him the product has to be -1. The players' common payoff is the product of the numbers they place in the cell where their row and column cross.

Alice and Bob can achieve an expected payoff of $(8/9 \times (+1) + 1/9 \times (-1) =)\ 7/9$ by agreeing in advance on the numbers they will put in all cells but one, say the lower-right cell, as in the following example:

$+1$	$+1$	$+1$
-1	-1	$+1$
$+1$	$+1$	± 1

[1] The 2022 Nobel Prize in Physics was awarded to Aspect, Clauser and Zeilinger for their work on "Bell tests" and related experimental measurements that underline the distinction between the quantum and classical worlds (Billings 2022).

(The only case where the payoff is not $+1$ is when the common cell is the corner one, where Alice puts $+1$ and Bob puts -1.) No pair of pure strategies gives a higher expected payoff. The same holds for any correlated strategy, which employs a mechanism that sends a pair of random, correlated signals that the players receive before they fill out the cells. This is because each possible realization of the random signals is interpreted by the players as instructions for playing a particular pair of pure strategies, which gets them at most $7/9$. However, the players can achieve a perfect payoff of 1 *with certainty* by using as signals the measurement values of two pairs of entangled qubits.

A historically important example of a *qubit* is the spin state of a silver atom in the century-old Stern–Gerlach experiment (Peres 2002, Cohen-Tannoudji et al. 2020). All atoms in this experiment initially travel with the same velocity along a line, the y axis. They then pass through an inhomogeneous magnetic field whose mean direction is the positive direction of the z axis. Because of the inhomogeneity of the field, whose z component varies with z, the atoms are deflected. According to classical physics, the angle of deflection along the z axis should be approximately proportional to the magnitude of the z component of each atom's intrinsic magnetic moment, and thus should have some continuous distribution. What is found instead is the result predicted by quantum mechanics, which is that the magnitude of the deflection is the same for all atoms and the only difference between individual atoms is its direction: half the atoms are deflected in the positive direction of the z axis (state $|0\rangle$, "up") and half in the opposite direction (state $|1\rangle$, "down"). Rotating the magnet 90°, so that the spin in the x direction is measured instead, would have given a similar result, with half the atoms deflected in the positive direction (state $|+\rangle$) and half in the negative direction (state $|-\rangle$). However, the first measurement S_z and the second one S_x are incompatible. According to quantum mechanics, measurement of the z component of the spin *puts* the atom in the state corresponding to the outcome ($|0\rangle$ or $|1\rangle$), and thus changes it irreversibly. There is no sense in which the three components of the spin have definite, objective values at any moment in time, as the measurement of any one of them is a destructive process. Correspondingly, alternating between the measurements would yield inconsistent results, with S_x possibly giving a different outcome each time it follows S_z. Only consecutive identical measurements are guaranteed to be consistent.

An example of a pair of *entangled* qubits is two particles that are created as the result of the decay of a spinless particle, for example, the decay of a neutral pion to an electron and a positron (Peres 2002). If the z component of the spin is measured for both particles, the measurements are bound to give opposite results: either the first is up and the second down or vice versa. The same goes for the x and y components, indeed, for measurements of the spin in any common direction.

This prediction of quantum theory is at the heart of the famous Einstein–Podolsky–Rosen paradox (Einstein et al. 1935). The two measurements can be instantaneous or nearly so,[2] and the two particles can be arbitrarily far from one another at the time of measurement, which excludes any transmission of information between them. In their 1935 paper, EPR

[2] According to relativity theory, instantaneity is a relative term. Two events that are simultaneous for one observer are not so for an observer moving in a particular direction relative to the first one. Moreover, a third observer, who moves in the opposite direction, would perceive the *order* of the events to be the opposite of that perceived by the second observer. Such subjectivity is inconsistent with causality (that is, it cannot be that one event caused the other) and occurs whenever the two events are sufficiently close in time or sufficiently distant is space so that light could not have reached one from the other (and *this* criterion is objective, not relative).

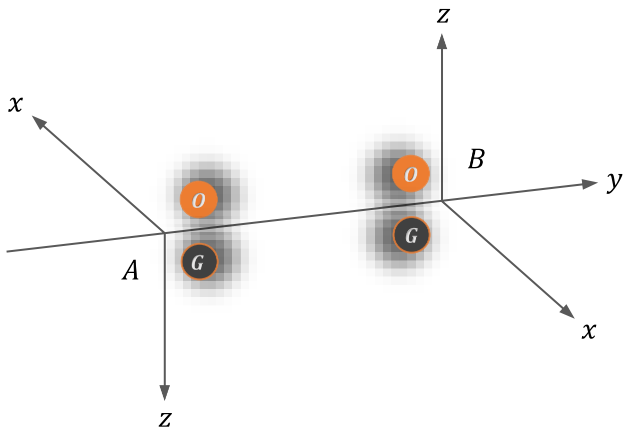

Figure 1 The mechanism used for the Mermin–Peres magic square game. Alice (A) and Bob (B) measure one component of the spin of each of two particles. The spin state of each particle is entangled with that of the similarly colored particle (O or G) of the other player.

argued that, since the measurement of one particle could not have changed in any way the other particle, the latter's three components of the spin must have physical reality. However, as mentioned above, Bell showed in 1964 that this idea of "hidden variables" is in fact unequivocally refuted by quantum mechanics itself (and his proof turns out to be highly relevant for the present work; more on this below).

And so the mechanism used for achieving the perfect outcome in the Mermin–Peres magic square game may take the following form (Figure 1). Pair O and pair G each consists of two spin-entangled particles, as described above. Alice and Bob can measure the z or x component of the spin of a different member of pair O, and similarly for pair G. For convenience, each player's positive directions of the z and x axes are chosen as the other player's negative directions. This choice of signs means that if both players measure the z component of their particle O (measurement S_z^O) or both measure the x component (measurement S_x^O), they are guaranteed to get an equal outcome: either $+1$, corresponding to state $|0\rangle$ or $|+\rangle$ respectively, or -1, corresponding to $|1\rangle$ or $|-\rangle$. The same goes for pair G. The actual measurements that the players perform depend on the cells they have to fill out, as detailed in the following table, which specifies for each cell a specific measurement, the negative of a measurement, or the product of two measurements, which can and should be performed as a single measurement:

S_z^O	S_z^G	$S_z^O S_z^G$
$-S_x^G$	$-S_x^O$	$S_x^G S_x^O$
$S_z^O S_x^G$	$S_z^G S_x^O$	

The players fill out the first two cells in their assigned row (Alice) or column (Bob) according to the outcomes of the measurements specified for these cells. The third cell does not actually require an additional measurement because, to conform with the sign requirements of the game, the number there must be the product (Alice) or the negative of the product (Bob) of the first two numbers. In particular, there is for each player only one legitimate way to complete the last row or column. The number Alice puts in the corner, blank cell must equal the outcome of the measurement $S_z^O S_x^G S_z^G S_x^O$ and for Bob it must be $-S_z^O S_z^G S_x^G S_x^O$. And, magically,

$$S_z^O S_x^G S_z^G S_x^O = -S_z^O S_z^G S_x^G S_x^O, \qquad (1)$$

which means that in this cell, too, the players will put the same number. The identity (1) would not of course hold if the symbols in it stood for numbers. However, the symbols here represent not numbers but measurements, which like all measurements in quantum mechanics are mathematically represented by linear operators in a suitable complex Hilbert space (which in the case of a single spin-$1/2$ particle such as an electron or a positron is a two-dimensional space). The multiplication (composition) of operators, which can be represented as multiplication of (square) matrices, is not generally commutative. In fact, two operators commute if and only if they represent compatible physical quantities: measuring one quantity does not disturb the other. As indicated, this is not the case for the z and x components of the spin of a single particle (in particular, particle G of either player), whose measurements in fact anti-commute: $S_x^G S_z^G = -S_z^G S_x^G$. Hence the identity (1). Moreover, both sides of (1) can be shown to be equal to $S_y^O S_y^G$, the product of the measurements of the y component of the spin of the O and G particles.

1.2 Greenberger–Horne–Zeilinger game

The mechanism described in Figure 1 is more complicated than required for refuting the existence of hidden variables. By only invoking a single pair of entangled particles, Bell proved that quantum mechanics predicts a particular *statistics* of particle measurements that is classically impossible (see Section 2.3). However, as first shown by Greenberger, Horne and Zeilinger (1989), the use of more than two particles enables a more dramatic demonstration of the "spookiness" of quantum mechanics, which concerns a *single* run of an experiment. Such a demonstration does not have to involve four particles, as in Figure 1, but can do with three, for example, three polarization-entangled photons (Pan et al. 2000). Brassard et al. (2005) recast the simplified three-particle account of the GHZ scenario provided by Mermin (1990b) into the framework of pseudo-telepathy in a three-player Bayesian game.

In the GHZ game, each of the three players is assigned a type, which can be I or II, such that the number of I's is odd. The four possible type profiles are equally probable. (Equality is in fact not a crucial assumption. It would suffice to assume that the four probabilities are all positive. Either way, the players' types are *not* independent.) The players, who only know their own type, have to choose between two actions, $+1$ or -1. Their common payoff is determined by the three players' types and the product of their actions. If only one player has type I, then the payoff is 0 or 1 if the product is $+1$ or -1, respectively. If all types are I, then it is the opposite: payoff 1 if the product is $+1$ and 0 if it is -1.

The best the players can do classically is to choose some fixed three actions whose product is -1. This gets them a payoff of 1 with probability 0.75. However, they can get 1 with certainty by using, for example, three spin-entangled particles whose initial state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

Thus, if each player would measure the spin of one particle in the direction of that player's z axis, the measurements would yield equal results – all particles up or all down – with both possibilities equally probable. What the players should measure, however, are not the z but the x or y component of their particle: x if their type is I and y if it is II. Each player then has to choose the action that matches the outcome of the measurement, $+1$ or -1. It can be shown that these strategies guarantee a perfect payoff. For example, if all types are I, then the outcomes of the measurements (the x components of the spin, in this case) will all be

$+1$ or only one of them will be so, and thus the product is guaranteed to be $+1$. This is because the initial state can be presented also as

$$|\psi\rangle = \frac{1}{2}(|+++\rangle + |+--\rangle + |-+-\rangle + |--+\rangle).$$

This representation shows that the only possible triplets of outcomes are those described above, and also shows that the four of them are equally probable.

Note that, for each player, the probability of $|+\rangle$ is $1/2$ (two out of four possibilities). This is true not only when all players measure the x components of the spin of their particle but also when the other players' type is II and they measure the y component. Thus, the outcome of the measurement does not give a player any information about the others' types. This fact is a manifestation of a fundamental no-signaling principle (c.f. footnote 2). A measurement of an entangled particle cannot reveal what the other players are measuring (Albert 1994). It can only provide information about the *outcomes* they (would) get, which can be used for choosing coordinated actions. The quantum mechanism is similar in this respect to a classical mechanism that is unaware of the players' types.

1.3 This work

Common-interest games, like those in the examples above, are somewhat marginal in game theory. This is because they do not give rise to what is arguably the most salient aspect of games: conflict. Game theory is all about (complete or partial) misalignment of interests, which is lacking here. This lack is inconsequential for examples as above because, as Section 3.1 shows, such examples arguably do not genuinely concern *games* at all but only *game forms*, which specify the players' types and possible actions but not the resulting payoffs. Payoff functions only play an incidental role in them. In other words, these examples only show what can and cannot be achieved *mechanically*, when there is no need to take into consideration possible conflicts of interest.

The particular significance of common interests in the context of Bayesian games (which are formally defined in Section 2) is that they entail that the mechanism used in a correlated equilibrium, whose goal is to coordinate the players' choice of actions, can only become more suitable for the job the more information it is able to pass around. There is no reason to hide information. But with non-aligned interests, a mechanism may be required to inform players *selectively*, with each of a player's types only getting to know what it has to know in order to choose the intended action for that type (Milchtaich 2014). Here, too, quantum mechanics may be useful.

An example demonstrating the usefulness of quantum mechanisms for "real" games, where the players' payoffs are not identical, is given in Section 3.2. It concerns a particular correlated equilibrium distribution that cannot be implemented by any classical mechanism that does not know the types of the players. The reason for the impossibility is that one type of player should not be able to infer from the mechanism's message the action that the other type of the same player were supposed to play, because this additional information would also partially betray the *other player's* expected move and reveal that the action indicated by the message is in fact not optimal. A classical type-unaware mechanism cannot tell each type of player only its action. But, as shown in Section 3.3, a quantum mechanism can do that.

Section 4 establish a connection between quantum mechanisms and those used in communication equilibria, where the players self-report their types to the mechanism before it send its messages to them. In a communication equilibrium, truthful reporting is required to be incentive compatible. With a quantum mechanism, the corresponding requirement is incentive compatibility of choosing the measurement assigned to the player's actual type. Section 4.1 gives an example showing that, also in the domain of communication equilibria, the use of quantum mechanisms may extend the set of achievable equilibrium outcomes beyond what is classically possible.

Section 5 presents a general definition of quantum mechanism as well as the definitions of quantum correlated equilibrium (QCE) and equilibrium distribution.

The roots of quantum advantage in correlated and communication equilibria are discussed in Section 6. Theorem 1 shows that, in the case of communication equilibria, the advantage is wholly due to the potential incompatibility of each player's different possible measurements. Allowing only compatible physical measurements would eliminate the difference from (classical) communication equilibria in which the players' type reports do not affect the messages the mechanism sends to the other players. Theorem 2 shows that, in the case of correlated equilibria, the advantage is due to the multiplicity of possible measurements. Quantum mechanisms where each player is only allowed one measurement give rise to the same correlated equilibrium outcomes as classical type-unaware mechanisms. Theorem 3 shows that that a similar result holds for game forms, where there are no incentives to worry about.

Section 7 sheds further light on the connections between QCEs and classical correlated and communication equilibria by classifying the equilibrium outcomes according to the kind of mechanisms employed. This classification is similar to, and in a sense extends, the one previously obtained for classical mechanisms (Milchtaich 2014). The seven categories considered turn out to be nested. The one corresponding to QCEs (quantum mechanisms) is situated somewhere in the middle, and is overshadowed only by classical correlated and communication equilibria where the mechanism's messages to players can depend on the other players' types (which the mechanism either knows or is informed about by the players themselves). It is shown that the same taxonomy applies both to the equilibrium distributions of types and actions (Section 7.1) and to the resulting payoff vectors (Section 7.2).

With the advent of quantum computing, the vision of putting the peculiar properties of entangled particles and similar quantum systems to everyday use rapidly becomes a reality. Soon, the intricate laboratory equipment needed for maintaining and making use of such systems may be replaced by handheld contraptions. This will make quantum mechanisms seem as mundane as stoplights or other classical coordination devices. It is hoped that this paper will contribute to laying the foundations for the use of such devices as part of new solution concepts: finding equilibria where, due to conflicts of interest, none existed before.

2 Bayesian games

A (finite) *Bayesian game*, or game with incomplete information, has a finite set of players $N = \{1,2, \dots, n\}$ and, for each player i, a finite set of types T_i, a finite set of actions A_i and a payoff function $u_i = T \times A \to \mathbb{R}$, where $T = T_1 \times T_2 \times \cdots \times T_n$ and $A = A_1 \times A_2 \times \cdots \times A_n$ are respectively the collections of all type profiles and all action profiles. The actual type profile is random, and can be specified either as a random element $\boldsymbol{t} = (\boldsymbol{t}_1, \boldsymbol{t}_2, \dots, \boldsymbol{t}_n)$ of T (throughout this paper, boldface symbols represent random elements) or as a probability vector $(\lambda_t)_{t \in T}$, the *common prior*. The two specifications are connected by

$$\lambda_t = \Pr(\boldsymbol{t} = t), \qquad t \in T.$$

A *game form* is defined similarly except that no payoff functions are specified.

2.1 Mechanisms and correlated strategies

A *mechanism* for a Bayesian game or game form is an extraneous source of messages. It sends to each player i a message m_i that is an element of some finite set M_i, the player's *message space*. The messages that the players receive may reflect to a lesser or greater extent the type profile. That is, the mechanism may be *type aware*. They may additionally reflect some randomness that is independent of the types. Formally (Milchtaich 2014), a (classical) mechanism is defined as a random element $\boldsymbol{m}$ that is (statistically) independent of $\boldsymbol{t}$, with values that are functions from T to $M = M_1 \times M_2 \times \cdots \times M_n$. Thus, for each type profile $t \in T$, the messages that the players receive when their types are t are given by the corresponding *random message profile* $\boldsymbol{m}(t) = (\boldsymbol{m}_1(t), \boldsymbol{m}_2(t), \dots, \boldsymbol{m}_n(t))$. As the type profile is itself random, the actual messages are given by the random element $\boldsymbol{m}(\boldsymbol{t})$ of M.

There are two senses in which the messages the mechanism sends may *not* reflect the players' types. The difference between them may seem subtle but is in fact highly consequential; each version gives rise to a different set of possible outcomes. With a *type-unaware* mechanism, the (random) message each player receives does not factor in the type profile:

$$\boldsymbol{m}_i(t) = \boldsymbol{m}_i(t'), \qquad i \in N,\ t, t' \in T.$$

This means that the player's own type does not affect the signal,

$$\boldsymbol{m}_i(t) = \boldsymbol{m}_i(t_i', t_{-i}), \qquad i \in N,\ t, t' \in T \tag{S}$$

(where (t_i', t_{-i}) denotes the type profile that takes player i's type from t' and the other players' types from t), and neither do the types of the other players,

$$\boldsymbol{m}_i(t) = \boldsymbol{m}_i(t_i, t_{-i}'), \qquad i \in N,\ t, t' \in T. \tag{O}$$

A corresponding pair of less restrictive conditions is that the message each player i receives does not provide any *information* about that player's type,

$$\boldsymbol{m}_i(t) \overset{d}{=} \boldsymbol{m}_i(t_i', t_{-i}), \qquad i \in N,\ t, t' \in T, \tag{$\tilde{S}$}$$

or about the other players' types,

$$\boldsymbol{m}_i(t) \overset{d}{=} \boldsymbol{m}_i(t_i, t_{-i}'), \qquad i \in N,\ t, t' \in T, \tag{$\tilde{O}$}$$

where $\overset{d}{=}$ denotes equality in distribution. The last property $\tilde{O}$ entails that the message to each player i is conditionally independent of the other players' types, given the player's own type t_i. This property, also referred to as belief invariance (Forges 2006, Bergemann and Morris 2016, Auletta et at. 2021), corresponds to the quantum no-signaling property.

Properties S and $\tilde{S}$ limit a mechanism's ability to provide different cues to different types of player. However, different types may still interpret the same message differently, with each

			Player 2						
			Type I				**Type II**		
		$\lambda_{\text{I,I}}$	$+1$	-1		$\lambda_{\text{I,II}}$	$+1$	-1	
Player 1	**Type I**	$+1$	0	0.5	0.5	$+1$	0.5	0	0.5
		-1	0.5	0	0.5	-1	0	0.5	0.5
			0.5	0.5			0.5	0.5	
		$\lambda_{\text{II,I}}$	$+1$	-1		$\lambda_{\text{II,II}}$	$+1$	-1	
	Type II	$+1$	0.5	0	0.5	$+1$	0.5	0	0.5
		-1	0	0.5	0.5	-1	0	0.5	0.5
			0.5	0.5			0.5	0.5	

TABLE 1 A CORRELATED STRATEGY DISTRIBUTION IN A TWO-PLAYER BAYESIAN GAME. EACH OF THE 2×2 TABLES GIVES THE JOINT DISTRIBUTION (AND THE MARGINALS) OF THE PLAYERS' ACTIONS FOR A SINGLE TYPE PROFILE t, WHOSE PROBABILITY λ_t IS SPECIFIED BY THE COMMON PRIOR. HERE, ALL FOUR PROBABILITIES ARE ASSUMED POSITIVE.

type prompted to choose a different action. Thus, the action of player i is jointly determined by i's type, the message i receives, and the player's *strategy* $\sigma_i: T_i \times M_i \to A_i$, which specifies an action a_i for each type t_i and received message m_i. A pair $(\boldsymbol{m}, \sigma)$ consisting of a mechanism $\boldsymbol{m}$ and a profile of strategies $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is a *correlated strategy*. The corresponding *random action profile* is the random vector $\boldsymbol{a} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)$ where

$$\boldsymbol{a}_i = \sigma_i(\boldsymbol{t}_i, \boldsymbol{m}_i(\boldsymbol{t})), \qquad i \in N. \tag{2}$$

Properties O and $\tilde{O}$ also limit the mechanism, and consequentially restrict the possible random action profiles. These restrictions are not the same: those resulting form O are more severe than for $\tilde{O}$ (Milchtaich 2014). An example showing this is given in the next subsection.

2.2 Correlated strategy distributions

In a Bayesian game or game form, a *correlated strategy distribution* (CSD) is any probability measure on $T \times A$ whose marginal on T coincides with the common prior λ. As the latter specifies the probability of each type profile t, a CSD only adds the joint distribution of the players' action for every t with $\lambda_t > 0$. (For t lying outside the support of the common prior, *any* joint distribution of actions is consistent with any CSD.) The example shown in Table 1 refers to a two-player Bayesian game form where each player has two possible actions ($+1$ and -1) and two types, and the common prior has full support, meaning that all four type profiles are possible.

For every CSD η there exists a mechanism $\boldsymbol{m}$ such that, for some strategy profile σ, η is the correlated strategy distribution *of* the correlated strategy $(\boldsymbol{m}, \sigma)$, in the sense that it is the joint distribution of the random type profile $\boldsymbol{t}$ and the random action profile $\boldsymbol{a}$ given by (2). Any such mechanism is said to *implement* η. A simple mechanism implementing the CSD in Table 1 is one where the message to player 1 is determined by a coin toss between $+1$ and -1 and the message to player 2 is then determined by the type profile: identical to the message to 1 if at least one type is not I and the opposite message if both types are I. The corresponding players' strategies are simply to play according to their signal.

In the coin-toss mechanism just described, each player's message is equally likely to be $+1$ or -1 regardless of the type profile. The message therefore gives no information about either player's type, and so conditions $\tilde{S}$ and $\tilde{O}$ are satisfied. However, the CSD in Table 1 is not implementable by any type-unaware mechanism or even by a mechanism satisfying only

O (Milchtaich 2004, 2014). The reason is that, for a correlated strategy $(\boldsymbol{m}, \sigma)$ in which $\boldsymbol{m}$ has property O, the joint distribution of the players' actions for every type profile $t = (t_1, t_2)$ coincides with the distribution of $(\boldsymbol{a}_1^{t_1}, \boldsymbol{a}_2^{t_2})$, where

$$\boldsymbol{a}_i^j = \sigma_i(j, \boldsymbol{m}_i(j,j)), \qquad i = 1,2, j = \mathrm{I}, \mathrm{II}.$$

The joint distribution of these four elements necessarily satisfies the following pair of *Bell inequalities* (see Brunner and Linden 2013):

$$-2 \leq -\rho_{\mathrm{I,I}} + \rho_{\mathrm{I,II}} + \rho_{\mathrm{II,I}} + \rho_{\mathrm{II,II}} \leq 2, \tag{3}$$

where

$$\rho_{\mathrm{I,II}} = \Pr(\boldsymbol{a}_1^{\mathrm{I}} = \boldsymbol{a}_2^{\mathrm{II}}) - \Pr(\boldsymbol{a}_1^{\mathrm{I}} \neq \boldsymbol{a}_2^{\mathrm{II}}) = \mathbb{E}[\boldsymbol{a}_1^{\mathrm{I}} \boldsymbol{a}_2^{\mathrm{II}}]$$

and similarly for the other ρ's. (When all actions are equally likely to be $+1$ and -1, as in Table 1, the ρ's are also the correlation coefficients between the corresponding actions.) The reason the pair of inequalities (3) must hold is that it is easily seen to be equivalent to the pair

$$\Pr(\boldsymbol{a}_1^{\mathrm{I}} = \boldsymbol{a}_2^{\mathrm{I}}) \leq \Pr(\boldsymbol{a}_1^{\mathrm{I}} = \boldsymbol{a}_2^{\mathrm{II}}) + \Pr(\boldsymbol{a}_1^{\mathrm{II}} = \boldsymbol{a}_2^{\mathrm{I}}) + \Pr(\boldsymbol{a}_1^{\mathrm{II}} = \boldsymbol{a}_2^{\mathrm{II}})$$
$$\Pr(\boldsymbol{a}_1^{\mathrm{I}} \neq \boldsymbol{a}_2^{\mathrm{I}}) \leq \Pr(\boldsymbol{a}_1^{\mathrm{I}} \neq \boldsymbol{a}_2^{\mathrm{II}}) + \Pr(\boldsymbol{a}_1^{\mathrm{II}} \neq \boldsymbol{a}_2^{\mathrm{I}}) + \Pr(\boldsymbol{a}_1^{\mathrm{II}} \neq \boldsymbol{a}_2^{\mathrm{II}}),$$

and these inequalities hold because every realization $(a_1^{\mathrm{I}}, a_1^{\mathrm{II}}, a_2^{\mathrm{I}}, a_2^{\mathrm{II}})$ necessarily satisfies an *even* number of the equalities in the first line. But the CSD described in Table 1 does not satisfy (3), as the expression in the middle equals 4. Therefore, this CSD is not implementable by any mechanism with property O.

Definition 1 A correlated strategy distribution is a *type correlated distribution*, or *type CSD* for short, if it is implementable by some mechanism with property O.

In Definition 1, 'property O' can be replaced with 'properties S and O'. That is, a CSD is a type CSD if and only if it is implementable by a type-unaware mechanism (Milchtaich 2014).

The argument used in the above impossibility result can be generalized into a necessary and sufficient set of conditions for a CSD in a 2×2 Bayesian game with two types for each player to be a type CSD. More precisely, these conditions, which are spelled out below, concern a given quartet of joint distributions of players' actions, one for each type profile t. (As indicated, for t lying outside the support of the common prior, any joint distribution is consistent with any CSD. Therefore, specifying a joint distribution also for such t goes beyond specifying the CSD.)

As explained, every correlated strategy where the mechanism has property O induces a distribution over the 16 possible values of the four-tuple $(a_1^{\mathrm{I}}, a_1^{\mathrm{II}}, a_2^{\mathrm{I}}, a_2^{\mathrm{II}})$. This distribution can be viewed as a probability vector lying in the unit simplex in $\mathbb{R}^{16}$. In the other direction, every such probability vector corresponds to some correlated strategy where the mechanism has both properties S and O, a type-unaware mechanism. An example of such a correlated strategy is the one where a four-tuple is randomly drawn according to the probability vector, each player i is told $(a_i^{\mathrm{I}}, a_i^{\mathrm{II}})$, and the players then choose the first or second action depending on their actual types.

The joint distribution of actions for each type profile is a marginal of the distribution over action four-tuples. Specifically, each entry in the 2×2 table describing it (as in Table 1) is the sum of four elements of the probability vector described above. For example, when player 1 has type I and player 2 has type II, the probability that the former plays -1 and the latter plays $+1$ is given by

$$\Pr(-1,+1,+1,+1)+\Pr(-1,+1,-1,+1)+\Pr(-1,-1,+1,+1)+\Pr(-1,-1,-1,+1).$$

The collection of all possible 16 numbers in these four 2×2 tables is therefore a linear transform of the unit simplex and is thus also a polytope in $\mathbb{R}^{16}$. As such, it can be described as the intersection of a finite number of half-spaces, in other words, as the collection of all points satisfying a particular set of linear equalities and (weak) inequalities. Some of these are obvious. The four numbers in each 2×2 table must be nonnegative and sum up to 1, and the marginals on the actions of each player type must be the same for both types of the other player: a "no-signaling" condition for actions. The non-obvious inequalities are called the Bell inequalities. They are obtained from (3) by moving the single minus sign in the middle expression to each of the four ρ's in turn. No-signaling and the eight Bell inequalities are therefore necessary and sufficient conditions for the existence of a distribution over action four-tuples such that the marginals on action pairs coincide with the given four joint distributions of players' actions.[3] As shown, this "marginality" condition is necessary for the actions to be determined by a mechanism with property O and is sufficient for them to be determined by a mechanism satisfying both S and O. These logical relations mean that the three conditions are equivalent.

Observe that being a type CSD is also equivalent to implementability by a mechanism with property O that *directly* tells the players how to play; the corresponding players' strategies are to do as told. This is because, for any mechanism $\boldsymbol{m}$ with property O and any corresponding strategy profile, $\boldsymbol{m}$ can be modified by adding to it an element that "translates" each message according to the receiving player's strategy and directly instructs the player how to act. The modified mechanism obviously also has property O.

2.3 Quantum advantage in CSDs

A different correlated strategy distribution in the game form considered in Table 1 is shown in Table 2. Bell inequalities still do not hold, as

$$-\rho_{\mathrm{I,I}}+\rho_{\mathrm{I,II}}+\rho_{\mathrm{II,I}}+\rho_{\mathrm{II,II}}=-\left(-1/\sqrt{2}\right)+1/\sqrt{2}+1/\sqrt{2}+1/\sqrt{2}=2\sqrt{2}>2.$$

Therefore, this CSD is not a type CSD; it cannot be obtained by using any classical mechanism that does not know the players' types. However, the CSD can be obtained by using a quantum mechanism. A suitable quantum mechanism consists of a single pair of entangled particles, like either of the pairs in Figure 1, with each player measuring the spin of a different particle. Unlike in Figure 1, Alice's x and z directions are rotated only 135° with respect to Bob's, not 180°, so that her x direction forms this angle with both his x and z directions and her z direction is 135° from his z and 45° from his x. Suppose that Alice measures the z component of the spin of her particle if her type is I and the x component if it is II, while Bob does the opposite: x for type I and z for II. The angle between the two directions of measurement is then 45° if both types are I and 135° otherwise. In the former case, the correlation coefficient between the outcomes is $-\cos 45°=-1/\sqrt{2}$, and in the

[3] The necessity of the Bell inequalities is proved by the argument presenting above (which concerns the specific pair (3)). Their sufficiency was first established by Fine (1982), whose proof uses an alternative form of the Bell inequalities, sometimes called the CH inequalities (after Clauser and Horne 1974), which is equivalent to the form presented above when no-signaling holds.

		Player 2								
			Type I				Type II		
		$\lambda_{\mathrm{I,I}}$	$+1$	-1		$\lambda_{\mathrm{I,II}}$	$+1$	-1	
Player 1	Type I	$+1$	$(1-\rho)/4$	$(1+\rho)/4$	0.5	$+1$	$(1+\rho)/4$	$(1-\rho)/4$	0.5
		-1	$(1+\rho)/4$	$(1-\rho)/4$	0.5	-1	$(1-\rho)/4$	$(1+\rho)/4$	0.5
			0.5	0.5			0.5	0.5	
		$\lambda_{\mathrm{II,I}}$	$+1$	-1		$\lambda_{\mathrm{II,II}}$	$+1$	-1	
	Type II	$+1$	$(1+\rho)/4$	$(1-\rho)/4$	0.5	$+1$	$(1+\rho)/4$	$(1-\rho)/4$	0.5
		-1	$(1-\rho)/4$	$(1+\rho)/4$	0.5	-1	$(1-\rho)/4$	$(1+\rho)/4$	0.5
			0.5	0.5			0.5	0.5	

TABLE 2 QUANTUM ADVANTAGE IN CORRELATED STRATEGY DISTRIBUTIONS. THIS CSD, WHERE $\rho = 1/\sqrt{2}$, IS NOT IMPLEMENTABLE BY ANY CLASSICAL TYPE-UNAWARE MECHANISM BUT IS IMPLEMENTABLE BY A QUANTUM MECHANISM.

latter, it is $-\cos 135° = 1/\sqrt{2}$ (Greenberger et al. 1990). These correlation coefficients and the fifty-fifty marginals mean that, if Alice and Bob always choose the action matching the outcome of their measurement, then (regardless of who is player 1 and who is 2) the joint distribution of their actions for every type profile is as in Table 2.

Significantly, even a quantum mechanism cannot implement the CSD in Table 1. This is because a value of 4 for the expression in the middle of (3) cannot be reached. In fact, the largest possible deviation from Bell inequalities is the last example's value of $2\sqrt{2}$ (Tsirelson 1980).

3 Correlated equilibria

In the previous section, none of the subsections refers to the players' payoff functions. The discussion only concerns the game form. This makes it only a first step in the analysis of a Bayesian game, albeit a useful one, particularly in regard to impossibilities. For if a counterexample shows that attaining a certain outcome with a certain kind of mechanism is *mechanically* impossible, then it also proves impossibility in the more complicated case where the players cannot be assigned arbitrary actions because their incentives need to be taken into account. The incentive-compatibility constraint is that, given the type $\boldsymbol{t}_i$ of a player i and the message $\boldsymbol{m}_i(\boldsymbol{t})$ the player received, the conditional expectation of i's payoff cannot be increased by replacing the action $\boldsymbol{a}_i$ specified by the player's strategy σ_i (Eq. (2)) with any action a_i' .

Definition 2 In a Bayesian game, a correlated strategy $(\boldsymbol{m}, \sigma)$ is a *correlated equilibrium* if the corresponding random action profile $\boldsymbol{a}$ is such that, for every player i and action a_i' for that player,

$$\mathbb{E}[u_i(\boldsymbol{t}, \boldsymbol{a}) - u_i(\boldsymbol{t}, (a_i', \boldsymbol{a}_{-i})) \mid \boldsymbol{t}_i, \boldsymbol{m}_i(\boldsymbol{t})] \geq 0.$$

The correlated strategy distribution of a correlated equilibrium is a *correlated equilibrium distribution* (CED), and the n-tuple $(\mathbb{E}[u_i(\boldsymbol{t}, \boldsymbol{a})])_{i=1}^{n}$ of the players' expected payoffs is a *correlated equilibrium payoff vector* (CEP). In the special cases where the mechanism $\boldsymbol{m}$ satisfies O or satisfies both S and O, the correlated equilibrium is referred to as a *type* or *strategy* correlated equilibrium, respectively, and the same qualifiers are attached to the CED and CEP.

The qualifier 'type' (Cotter 1994) refers to the fact that the mechanism is allowed to send different messages to different types of the same player i, while 'strategy' (Cotter 1991) means that it is left for the player to interpret the mechanism's message as a map from T_i to A_i, in other words, as a strategy in the Bayesian game.[4]

Example 1 The CHSH game (Clauser et al. 1969) is the common-interest 2×2 Bayesian game obtained by completing the specification of the game form considered in Section 2.2 with $\lambda_t = 0.25$ for all t and defining the payoff functions in the following manner: If both players have type I, their payoffs are both -1 if they choose the same action and 1 if they choose different actions. For the other three type profiles, the payoffs are 1 in the first case and -1 in the second case. Obviously, the CSD in Table 1 is a CED in this game, with the correlated equilibrium payoff 1 for both players. However, these are not type or strategy correlated equilibrium distribution and payoff. This is because, with any mechanism that has property O, the expected payoff for each player, which is easily seen to be given by one-quarter the expression in the middle of the Bell inequalities (3), can be at most $1/2$.[5]

As shown in Section 2.3, the quantum mechanism consisting of a pair of spin-entangled particles, and the players' strategies of measuring the spin in the directions specified in that section and playing accordingly, achieve the (quantum) highest possible expected payoff of $1/4 \times 2\sqrt{2} = 1/\sqrt{2}$. They therefore constitute a *quantum correlated equilibrium* in the CHSH game, with the CED in Table 2. Viewed in this light, Table 2 may be interpreted as demonstrating quantum advantage also in CEDs or, perhaps more significantly, in correlated equilibrium payoffs. Indeed, this is how this example is usually presented (Brunner and Linden 2013). Similarly, the examples of the Mermin–Peres magic square game and the GHZ game (Sections 1.1 and 1.2) also revolve around classically unachievable payoffs. However, as shown below, all these examples can also be viewed in a different light, which demotes payoff functions to a supporting role.

3.1 Separating hyperplanes

The correlated strategy distribution of a correlated strategy $(\boldsymbol{m}, \sigma)$ is, by definition, a probability measure η on the (finite) product space $T \times A$, assigning a probability $\eta(\{(t, a)\})$ to every pair (t, a) consisting of a type profile and an action profile. It may therefore be viewed as a point in $\mathbb{R}^{|T \times A|}$. Of particular interest here is the subset of all points that correspond to some type-unaware mechanism. The next lemma shows that this set is a polytope.

Lemma 1 In every Bayesian game form, the collection of all type CSDs is a polytope.

Proof. The assertion follows immediately from Proposition 2 in Milchtaich (2014), which shows that the collection in question is a linear transform of a polytope. That polytope consists of all probability distributions on $A_1^{T_1} \times A_2^{T_2} \times \cdots \times A_n^{T_n}$, the set of strategy profiles in the game. (In the special case of a 2×2 game form with two types for each player, the linear transformation is explicitly given towards the end of Section 2.2.) ■

[4] Among the many alternative names used in the literature are "agent normal form correlated equilibrium" (Forges 1993, 2006; Lehrer et al. 2010) for type correlated equilibrium, and "strategic (normal) form correlated equilibrium" (Forges 1993, 2006; Lehrer et al. 2010), or just "correlated equilibrium" (Auletta et al. 2021), for strategy correlated equilibrium.

[5] The maximum of $1/2$ is achieved by the trivial strategy profile where both players always play $+1$. Therefore, this strategy profile and any type-unaware mechanism together constitute a strategy correlated equilibrium in the CHSH game.

							Player 2						
			Type I						Type II				
		0.25	$+1$	-1	$+1$	-1		**0.25**	$+1$	-1	$+1$	-1	
	Type I	$+1$	0,0	4,0	9/68	25/68	0.5	$+1$	15,0	0,0	32/68	2/68	0.5
		-1	4,0	0,0	25/68	9/68	0.5	-1	0,0	0,0	2/68	32/68	0.5
Player 1					0.5	0.5					0.5	0.5	
		0.25	$+1$	-1	$+1$	-1		**0.25**	$+1$	-1	$+1$	-1	
	Type II	$+1$	15,0	0,0	32/68	2/68	0.5	$+1$	0,0	4,0	9/68	25/68	0.5
		-1	0,0	0,0	2/68	32/68	0.5	-1	4,0	0,0	25/68	9/68	0.5
					0.5	0.5					0.5	0.5	

TABLE 3 QUANTUM ADVANTAGE IN CORRELATED EQUILIBRIUM DISTRIBUTIONS. FOR EACH TYPE PROFILE, THE LEFT 2×2 TABLE IS THE CORRESPONDING PAYOFF MATRIX AND THE RIGHT TABLE SPECIFIES THE JOINT DISTRIBUTION OF THE PLAYERS' ACTIONS. SEE EXAMPLE 2 FOR MORE DETAILS.

It follows from the lemma, by a standard separation theorem, that a CSD is not a type CSD if and only if there is a hyperplane separating it from all type CDSs. In other words, there is, in this case, a linear functional f on $\mathbb{R}^{|T\times A|}$ whose value at that CSD is greater than the maximum of f in the polytope identified by Lemma 1. Now, the linearly of f means that the value it returns at every CSD η can be written as

$$f(\eta) = \sum_{(t,a)\in T\times A} b_{(t,a)}\, \eta(\{(t,a)\}),$$

where the b's are constant coefficients. This expression can be naturally interpreted as an expected payoff. Specifically, it is the expected payoff for any correlated strategy having the CSD η when the players share the common payoff function u specified by the coefficients,

$$u(t,a) = b_{(t,a)}, \qquad t \in T, a \in A.$$

It is thus a corollary of Lemma 1 that, for every Bayesian game form, a correlated strategy $(\boldsymbol{m}, \sigma)$ produces (via (2)) a joint distribution of type and action profiles that is different from that produced by any correlated strategy $(\boldsymbol{m}', \sigma')$ with type-unaware mechanism if and only if there is some common payoff function u for which $(\boldsymbol{m}, \sigma)$ gives a higher expected payoff than every such $(\boldsymbol{m}', \sigma')$ does. This corollary means that, as indicated, all the examples above, in which the players' payoffs are always equal, can be viewed as concerning game forms rather than games, and correlated strategy distributions rather than correlated equilibria.

3.2 Games with non-identical payoffs

A major difference between correlated equilibria in common-interest games as above and those in general Bayesian games is that the latter may require a mechanism that sends different messages to different types of some player i, which means that condition S does not hold. Specifically, the cue provided to some type t_i needs to be such that it only indicates the action assigned to that type and not those of the other types of player i. The reason the actions assigned to the other types should not be disclosed is that they may tell t_i too much about the actions of the *other* players (which may be correlated with i's action), so that it would sometimes be possible for t_i to realize that the assigned action is in fact not optimal: some alternative action can be expected to yield a higher payoff. The next example illustrates this possibility and demonstrates its significance.

Example 2 In the two-player Bayesian game presented in Table 3, the specified correlated strategy distribution is a correlated equilibrium distribution. This is because, with a mechanism that simply randomizes according to the specified joint distribution of actions for each type profile and tells each player what action to take, heeding the mechanism always guarantees player 1 the highest expected payoff for the player's type, and the same is trivially true for player 2, whose payoff function is constant. For example, if player 1 has type I and the mechanism's recommendation is to play -1, doing so yields the player an expected payoff of $25/68 \times 4$, whereas playing $+1$ would yield the lower payoff $9/68 \times 4 + 2/68 \times 15$. The correlated equilibrium payoff for player 1 is 5.

The mechanism just described has properties $\tilde{S}$ and $\tilde{O}$. The message it sends to each player does not provide any information about either player's type, as it is equally likely to be $+1$ or -1 regardless of the type profile. But the mechanism does base its recommendations on the types and it is thus not type-unaware (properties S and O). However, as the correlated strategy distribution satisfies the Bell inequalities (in particular, the middle expression in (3) is $120/68 < 2$), it is also implementable by a type-unaware mechanism. An example of such a mechanism is the following one. Messages to player i are of the form $m_i = (a_i^{\mathrm{I}}, a_i^{\mathrm{II}})$, where the first and second components are the actions recommended for the player's types I and II, respectively. Pairs of messages (m_1, m_2), which are strategy profiles in the Bayesian game, can therefore be written as $(a_1^{\mathrm{I}}, a_1^{\mathrm{II}}, a_2^{\mathrm{I}}, a_2^{\mathrm{II}})$. This four-tuple has the value $(-1,-1,-1,-1)$, $(-1,+1,+1,-1)$ or $(-1,-1,+1,+1)$ with probability $9/68$, $23/68$ and $2/68$, respectively, and the same probabilities apply to the "opposite" three quartets, where all the signs are inverted. It is easy to check that the correlated strategy consisting of this mechanism and the players' strategies of following the recommendations has the above correlated strategy distribution. However, although, as shown, this CSD is a correlated equilibrium distribution, the correlated strategy just described is not a correlated equilibrium. If player 1 receives the message $m_1 = (-1,-1)$, he can conclude that m_2 is either $(-1,-1)$ or $(+1,+1)$. In the former case, both types of player 1 would increase their expected payoff from 0 to $1/2 \times 4$ by playing $+1$ rather than the assigned action -1, and in the latter case, they would increase it from $1/2 \times 4$ to $1/2 \times 15$. Thus, for both types of player 1, playing according to the message is guaranteed not to be optimal.

The incentive-incompatibility of the last mechanism's recommendations points to the problem with trying to implement the CED in Table 3 with a type-unaware mechanism. However, it is not a definite proof that the problem cannot be overcome, in other words, that the CED in question is not a strategy CED. What would prove this is a demonstration that the indicated deviation is beneficial for some type of player 1 under *any* distribution of the four-tuples $(a_1^{\mathrm{I}}, a_1^{\mathrm{II}}, a_2^{\mathrm{I}}, a_2^{\mathrm{II}})$ that gives the four marginals on action pairs specified in Table 3. (There are infinitely many such distributions.) This goal can be formulated as a linear programming problem and tackled by a standard mathematical software package. I used Wolfram Mathematica for this and found that the above deviation is indeed always profitable, so that no type-unaware mechanism is fit for the job. It turns out, however, that a quantum mechanism is so.

3.3 Quantum advantage in CEDs

A quantum correlated equilibrium with the distribution in Table 3 can be constructed by again using a mechanism employing one of the two pairs of spin-entangled particles in Figure 1. A strategy for each player specifies two directions. The first direction indicates the spin component of the player's particle measured by type I and the second direction indicates this for type II. The player then chooses the action ($+1$ or -1) coinciding with the outcome of the measurement. For player 2, the two directions will be Bob's x and z

directions, respectively, in Figure 1. As player 2 is indifferent about the outcome, from that player's perspective this choice is as good as any other. Player 1's first direction can be specified by the angle θ it forms with Bob's x direction; the angle with Bob's z direction is $90° - \theta$.[6] It remains to find the optimal θ, that is, the best response (with respect to the payoffs in Table 3) of type I of player 1 to player 2's strategy. By symmetry, player 1's optimal second direction, which is that employed by type II, is at the same angle of θ with Bob's z direction; the angle with Bob's x direction is $90° - \theta$.

For every $-180° < \theta \leq 180°$, the correlation coefficient between the outcome of player 1's measurement and that of the same type of player 2 is $-\cos\theta$ (Greenberger et al. 1990). As both measurement values are equally likely to be $+1$ or -1, their joint distribution is

	$+1$	-1	
$+1$	$(1-\cos\theta)/4$	$(1+\cos\theta)/4$	0.5
-1	$(1+\cos\theta)/4$	$(1-\cos\theta)/4$	0.5
	0.5	0.5	

For the other type of player 2, θ in the table is replaced with $90° - \theta$, equivalently, $\cos$ is replaced with $\sin$. Using these two conditional distributions and the fact that player 2's two types are equally probable, player 1 can compute, upon getting a particular measurement value, the conditional expectation of the payoff for each of the two actions, $+1$ and -1, and choose the action giving the higher payoff. The player's expected payoff is the average of these maximal payoffs for the two possible outcomes of measurement, an expression which simplifies to

$$\frac{23}{8} + \max\left\{\frac{15}{8}, \left|\frac{15}{8}\sin\theta - \cos\theta\right|\right\}.$$

The optimal θ is therefore determined by the first-order condition $(15/8)\cos\theta + \sin\theta = 0$. It follows that a best-response strategy for player 1 is to measure in the direction $\theta = \arctan(-15/8) \approx -62°$ and to play according to the outcome. Plugging this θ into the above tables gives that the joint distributions of actions are as in Table 3. The conclusion proves that these quantum mechanism and strategies constitute a quantum correlated equilibrium having the CED in the table. This contrasts with the classical case, where, as shown, no type-unaware mechanism can give such a correlated equilibrium.

4 Communication equilibria

A communication equilibrium differs from correlated equilibrium is that the message exchange between the mechanism and the players is two-way. First, each player sends a message to the mechanism, which without loss of generality may be assumed to be a type report.[7] Then, based on the profile of reported types, messages are sent in the opposite direction. The mechanism in a communication equilibrium thus serves as a mediator, or is just a (possibly, noisy) communication protocol.

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[6] This description assumes that player 1's two directions also lay in Bob's xz plane. For simplicity, this could be made an explicit restriction on the choice of strategies. However, such a restriction is in fact unnecessary, because no direction outside the plane outperforms all those in it. The underlying reason is that it cannot be more informative than them.

[7] Since the messages players send can only depend on their types, each player could in principle use a gadget that takes type as input and outputs any required message. Such gadgets can be viewed as part of the mechanism.

The formal structure of a communication equilibrium (Milchtaich 2014) is similar to that of a correlated equilibrium. Namely, it is a correlated strategy $(\boldsymbol{m}, \sigma)$ (see Section 2.1) satisfying a particular incentive-compatibility condition. However, here, the type profile fed into the mechanism $\boldsymbol{m}$ is not that of the true types but the reported ones. Correspondingly, the incentive-compatibility condition for communication equilibrium differs from that for correlated equilibrium in also requiring truthful reporting to be optimal. For every player i, reporting the type truthfully and then choosing the action according to the player's strategy σ_i should give the same or higher expected payoff than that obtained by (i) misreporting any of the player's types as another type t_i', and/or (ii) switching to another strategy σ_i'.

Definition 3 In a Bayesian game, a correlated strategy $(\boldsymbol{m}, \sigma)$ is a *communication equilibrium* if for every player i, type t_i' of that player and strategy $\sigma_i' \colon T_i \times M_i \to A_i$,

$$\mathbb{E}[u_i(\boldsymbol{t}, \boldsymbol{a}) - u_i(\boldsymbol{t}, \boldsymbol{a}') \mid \boldsymbol{t}_i] \geq 0, \tag{4}$$

where $\boldsymbol{a}$ is the random action profile defined by (2) and $\boldsymbol{a}' = (\boldsymbol{a}_1', \boldsymbol{a}_2', \dots, \boldsymbol{a}_n')$ is given by

$$\begin{aligned} \boldsymbol{a}_i' &= \sigma_i'(\boldsymbol{t}_i, \boldsymbol{m}_i(t_i', \boldsymbol{t}_{-i})), \\ \boldsymbol{a}_j' &= \sigma_j(\boldsymbol{t}_j, \boldsymbol{m}_j(t_i', \boldsymbol{t}_{-i})), \qquad j \neq i. \end{aligned}$$

The correlated strategy distribution of a communication equilibrium is a *communication equilibrium distribution* (MED), and the n-tuple of the players' expected payoffs is a *communication equilibrium payoff vector* (MEP). In the special case where the mechanism $\boldsymbol{m}$ satisfies O, the distribution and payoff vector are referred to as *type* MED and MEP.

The significance of the requirement that truthful type reports be incentive compatible is demonstrated by the following simple example (Milchtaich 2014).

Example 3 *A correlated equilibrium distribution that is not a communication equilibrium distribution.* Two players play a 2×2 coordination game: their common payoff is 1 if they choose the same action and -1 otherwise. There are two types, I and II. Player 1 is always type I but 2 is equality likely to be either type. The following CSD is a CED but is not a MED:

			Player 2						
			Type I				**Type II**		
		0.5	$+1$	-1		**0.5**	$+1$	-1	
Player 1	**Type I**	$+1$	0.5	0	0.5	$+1$	0.25	0.25	0.5
		-1	0	0.5	0.5	-1	0.25	0.25	0.5
			0.5	0.5			0.5	0.5	

A correlated equilibrium with this distribution is one where two fair coins are flipped, a player of type I observes the outcome for the first coin and type II observes the second coin, and the players play $+1$ if and only if they observe heads. The reason there is no communication equilibrium with this distribution is that type II of player 2 would always be able to increase his expected payoff from 0 to 0.5 by mimicking the action of type I. That is, for $i = 2$, inequality (4) does not hold when $\boldsymbol{t}_2 = \mathrm{II}$ if $t_2' = \mathrm{I}$ and σ_2' is a strategy such that $\sigma_2'(\mathrm{II}, \cdot) = \sigma_2(\mathrm{I}, \cdot)$.

As the proof of the following proposition shows, Definition 2 is equivalent to Definition 3 with the truthfulness of the reports taken for granted.

Proposition 1 Every communication equilibrium is a correlated equilibrium, and therefore every MED is a CED and every MEP is a CEP.

Proof With truthful reports (that is, t_i' chosen to match the true type $\boldsymbol{t}_i$), inequality (4) becomes

$$\mathbb{E}[u_i(\boldsymbol{t},\boldsymbol{a}) - u_i(\boldsymbol{t},(\sigma_i'(\boldsymbol{t}_i,\boldsymbol{m}_i(\boldsymbol{t})),\boldsymbol{a}_{-i})) \mid \boldsymbol{t}_i] \geq 0. \tag{5}$$

Therefore, a necessary condition for communication equilibrium is that (5) holds for all i and σ_i'. For a strategy of the form

$$\sigma_i'(t_i,m_i) = \begin{cases} a_i', & t_i = t_i^*, m_i = m_i^* \\ \sigma_i(t_i,m_i), & \text{otherwise} \end{cases},$$

where t_i^* and m_i^* are such that $\Pr(\boldsymbol{t}_i = t_i^*, \boldsymbol{m}_i(\boldsymbol{t}) = m_i^*) > 0$ and a_i' is any action, this requirement is equivalent to

$$\mathbb{E}[u_i(\boldsymbol{t},\boldsymbol{a}) - u_i(\boldsymbol{t},(a_i',\boldsymbol{a}_{-i})) \mid \boldsymbol{t}_i = t_i^*, \boldsymbol{m}_i(\boldsymbol{t}) = m_i^*] \geq 0.$$

The latter is the requirement in Definition 2. ■

4.1 Quantum advantage in MEDs

In the context of correlated equilibria, quantum advantage (Section 3.3) is examined with respect to classical type-unaware mechanisms, those satisfying S and O. But in the context of communication equilibria, these properties are too strong, as they mean that the mechanism totally ignores the players' type reports. (This is why there is no "strategy MED" in Definition 3; it would be the same thing as strategy CED.) The appropriate framework in this context for the consideration of quantum advantage is that of type MEDs, which are implementable by classical mechanisms with property O. The message such a mechanism sends to each player is not affected by the other players' type reports. It is allowed to be affected by the player's own report, but in practice, there may be no actual reports at all. For example, each player i may be required to choose among several sealed envelopes, each marked with a different player type. Choosing an enveloped is interpreted as reporting to be the corresponding type. If the (possibly, random) messages inside i's envelopes are not always identical, then the mechanism does not have property S. However, O still holds, as the other players' reported types (that is, the envelopes they choose to open) have no effect on the content of i's envelopes.

With a quantum mechanism, too, players need to make a choice: among measurements. (The necessity may reflect the destructive nature of measurements in quantum mechanics; see Section 1.) What may reasonably be interpreted as quantum advantage in MEDs is the fact that there are type CEDs that are attainable by quantum correlated equilibria but are not type MEDs. A type CED that is not a type MED is implementable by a classical mechanism with property O when the signal each player receives is dictated by the player's type (correlated equilibrium) but not when the signal may effectively be chosen by the player through a type report (communication equilibrium). Attainability by a quantum correlated equilibrium means that implementability can be restored by using a quantum mechanism, in which players choose among the *measurements* delivering the messages (which are the measurement values).

Example 4 *A type CED that is attained in a quantum correlated equilibrium but is not a type MED.* In a 2×2 coordination game as in Example 3, and with the common prior λ shown in

					Player 2				
			Type I				**Type II**		
		$\mathbf{0}$	$+1$	-1		$\mathbf{0.3}$	$+1$	-1	
	Type I	$+1$				$+1$	$(1+\rho_{\mathrm{I,II}})/4$	$(1-\rho_{\mathrm{I,II}})/4$	0.5
		-1				-1	$(1-\rho_{\mathrm{I,II}})/4$	$(1+\rho_{\mathrm{I,II}})/4$	0.5
							0.5	0.5	
Player 1		$\mathbf{0.3}$	$+1$	-1		$\mathbf{0.4}$	$+1$	-1	
	Type II	$+1$	$(1+\rho_{\mathrm{II,I}})/4$	$(1-\rho_{\mathrm{II,I}})/4$	0.5	$+1$	$(1+\rho_{\mathrm{II,II}})/4$	$(1-\rho_{\mathrm{II,II}})/4$	0.5
		-1	$(1-\rho_{\mathrm{II,I}})/4$	$(1+\rho_{\mathrm{II,I}})/4$	0.5	-1	$(1-\rho_{\mathrm{II,II}})/4$	$(1+\rho_{\mathrm{II,II}})/4$	0.5
			0.5	0.5			0.5	0.5	

TABLE 4 QUANTUM ADVANTAGE IN COMMUNICATION EQUILIBRIUM DISTRIBUTIONS. THE 2×2 TABLES, WHERE $\rho_{\mathrm{I,II}} = \rho_{\mathrm{II,I}} = -\cos 150° = \sqrt{3}/2$ AND $\rho_{\mathrm{II,II}} = -\cos 60° = -1/2$, SPECIFY THE JOINT DISTRIBUTION OF ACTIONS FOR TYPE PROFILES t WITH $\lambda_t > 0$. SEE EXAMPLE 4 FOR MORE DETAILS.

Table 4, the CSD in that table is not a type MED. This fact hinges on the nonoccurrence of the type profile (I, I), which means that if both players report their type as I, at least one report if false. To deter such misreporting, the probability of miscoordination of actions (hence, payoff -1) when both types are reported as I should be high, equivalently, the correlation coefficient $\rho_{\mathrm{I,I}}$ should be low. Specifically, by misreporting his type as I, a player of type II changes the expected payoff from $0.3 \times \sqrt{3}/2 + 0.4 \times (-1/2)$ to $0.3 \times \rho_{\mathrm{I,I}} + 0.4 \times \sqrt{3}/2$, and so making this unprofitable requires $\rho_{\mathrm{I,I}} \leq \sqrt{3}/6 + 2/3 = -0.955$. This requirement makes it impossible to satisfy the Bell inequalities (3), according to which $\rho_{\mathrm{I,I}} \geq \rho_{\mathrm{I,II}} + \rho_{\mathrm{II,I}} + \rho_{\mathrm{II,II}} - 2 = \sqrt{3} - 5/2 = -0.768$. As shown in Section 2.2, the impossibility of satisfying the Bell inequalities means that incentive compatibility cannot hold with a classical mechanism satisfying O. Thus, the above CSD is not a type MED.

The CSD is, however, a type CED. This is because, with $\rho_{\mathrm{I,I}} = \sqrt{3} - 5/2 = -0.768$ for example, all eight Bell equalities hold. As remarked at the end of Section 2.2, this means that there is a classical mechanism with property O that tells the players how to act, which together with the players' strategies of doing as told gives the joint distributions of actions in Table 4 (as well as the above $\rho_{\mathrm{I,I}}$). These mechanism and strategies constitute a correlated equilibrium, as it is easy to check that a player heeding the mechanism always gets positive expected payoff while not doing so would result in negative expected payoff. (Note that the correlation coefficient $\rho_{\mathrm{I,I}}$ does not actually matter, and it does not materialize, as there is no type reporting – or misreporting – in a correlated equilibrium. The mechanism is assumed capable of matching the message with the player's true type.)

The joint distributions of actions in Table 4 are also attained in a quantum correlated equilibrium. A suitable quantum mechanism consists of a pair of spin-entangled particles whose spin both players can measure either in the z direction, which they do when their type is I, or in an angle of $150°$ with respect to the z direction, which they do when their type is II, such that the angle between the two players' second directions is $(360° - 2 \times 150° =)\ 60°$. The players then play according to the measurement value. The specified angles yield the required values for the correlation coefficients $\rho_{\mathrm{I,II}}$, $\rho_{\mathrm{II,I}}$ and $\rho_{\mathrm{II,II}}$. They also give $\rho_{\mathrm{I,I}} = -1$, which means that if both players "report" their type as I by measuring in the z direction, they are punished by getting -1 for sure. As indicated, this is sufficient for deterring the players from taking the wrong measurement for their type, so that the above quantum mechanism and strategies constitute a quantum correlated equilibrium.

5 General quantum correlated equilibria

A general *quantum state* is a unit vector $|\psi\rangle$ in a finite-dimensional complex Hilbert space $\mathcal{H}$, the *state space*. It may be specified either abstractly or as a concrete state of a physical system.[8] An *observable* is a Hermitian, or self-adjoint, linear operator on $\mathcal{H}$. In the physical case, it represents a measurable physical quantity. An n-player *quantum mechanism* is a quantum state together with a nonempty set of admissible observables for each player i such that each of i's observables is compatible with every observable of every other player j. Mathematically, compatibility means that the two observables are commuting operators. Physically, it means that the players' measurements do not interfere with one another, so that the outcome for player i is statistically the same for all choices of measurements by the other players and there is thus no way for i to know what the others are measuring. The set of possible measurement values for an operator is its spectrum, the set of all eigenvalues. The probability of obtaining each outcome is determined by projecting $|\psi\rangle$ on the corresponding eigenspace; the probability is the square of the length of the projection (Cohen-Tannoudji et al. 2020). In the following, the set M_i of all possible measurement values for a player i (which is the union over all the player's observables, a potentially infinite set) is assumed finite. For example, the only possible values may be $+1$ and -1.

For an n-player Bayesian game (or game form) Γ, an n-player quantum mechanism determines a new game (respectively, a new game form), where a *strategy* for a player i has two components. One component specifies the (admissible) observable each type t_i chooses. The other is a "classical" strategy as in Section 2, that is, a mapping $\sigma_i: T_i \times M_i \longrightarrow A_i$ that specifies, for each type t_i and measurement value m_i, the action $a_i = \sigma_i(t_i, m_i)$ that player i takes. A quantum mechanism and a corresponding profile of strategies together constitute a *quantum correlated strategy* (QCS) in Γ.[9] A *quantum CSD* is any correlated strategy distribution (see Section 2.2) that is the joint distribution of the players' type and action profiles in some QCS. Given the payoff function u_i of a player i, a quantum CSD determines the distribution of the player's payoff, and thus also the expected payoff.

Definition 4 In a Bayesian game, a quantum correlated strategy where the expected payoff for every player i cannot be increased by changing only i's strategy is a *quantum correlated equilibrium* (QCE). The corresponding correlated strategy distribution is a *quantum CED* and the n-tuple of the players' expected payoffs is a *quantum CEP*.

A quantum CED or CEP is also a CED or CEP, respectively, according to Definition 2. That is, it coincides with the correlated strategy distribution or payoff vector of some classical correlated equilibrium as in Section 3. In fact, a stronger result holds (cf. Proposition 1).

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[8] A possible extension is to allow also *mixed*, or "random", quantum states, which are probability distributions over (pure) states. However, such a formal extension would not translate to a substantial one. This is because a mixed state can always by purified by considering a higher-dimensional state space (see Hughston et al. 1993). In other words, probabilistic uncertainly about the quantum state of a system can always be viewed as reflecting the quantum uncertainly embodied in the (pure) quantum state of a larger system, of which the one under consideration is a part.

[9] Auletta et al.'s (2021) notion of "quantum solution" is similar to QCS. The main difference is the implicit assumption in these authors' setting that each player i can choose *any* observable on some Hilbert space $\mathcal{H}_i$ such that the state space is the tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ (and so the players' observables are automatically compatible). This assumption describes a subset of the mechanisms considered here. The definition of quantum correlated equilibrium in Auletta et al. (2021) is also similar to that in Definition 4 below, except for the above limitation on mechanisms.

Proposition 2 Every quantum CED is a MED, and therefore every quantum CEP is a MEP.

This connection between quantum correlated equilibria and communication equilibria is somewhat paradoxical. As discussed in the Introduction, a fundamental property of quantum mechanisms is that they *exclude* any kind of communication between players. The connection is based on the correspondence, already suggested Section 4.1, between the truthfulness requirement of communication equilibrium and the requirement in a quantum correlated equilibrium that no player type has an incentive to choose an observable different from that assigned to it by the QCE.[10]

To prove Proposition 2, consider, for a given QCS, a classical mechanism $\boldsymbol{m}$ as follows. For every type profile $t = (t_1, t_2, \dots, t_n)$, the distribution of $\boldsymbol{m}(t) = (\boldsymbol{m}_1(t), \boldsymbol{m}_2(t), \dots, \boldsymbol{m}_n(t))$ coincides with that of the (random) profile of measurement values that results when each player i chooses the observable specified by the QCS for player type t_i. The mechanism $\boldsymbol{m}$ and the profile $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ of classical strategies specified by the QCS together constitute a classical correlated strategy $(\boldsymbol{m}, \sigma)$ whose CSD is easily seen to coincide with the quantum CSD. If the QCS is a QCE, then in particular no player i can increase his expected payoff by (i) replacing the observable specified for one of i's types with that specified for another type, and/or (ii) switching to another classical strategy. This means that $(\boldsymbol{m}, \sigma)$ must satisfy condition (4) for all i, t_i' and σ_i'; it is a communication equilibrium. This necessary condition for a QCS to be a QCE proves Proposition 2.

Observe that the last condition is also sufficient if the following "tightness" condition holds: every observable admissible to a player i is assigned by the QCS to at least one of the player's types. Thus, a QCS for which the tightness condition holds is a QCE if and only if condition (4), which is formulated entirely in terms of a corresponding classical correlated strategy $(\boldsymbol{m}, \sigma)$, is satisfied.

The tightness condition may hold only if the cardinality of the set of a player's admissible observables is no greater than the number of types. This is not the case for the example in Section 3.3, where the set has the cardinality of the continuum. However, restricting attention to QCSs for which the tightness condition holds would not affect the sets of quantum CSDs and CEDs. This is because eliminating possibilities (that is, observables) that are left unused can obviously only reinforce the optimality of those that are being used.

6 The roots of quantum advantage

The three theorems in this section identify the specific features of quantum mechanisms responsible for their advantage over comparable classical mechanisms. Theorem 1 concerns quantum advantage in communication equilibrium distributions, Theorem 2 concerns the advantage in correlated equilibrium distributions, and Theorem 3 concerns correlated strategy distributions.

[10] The same observation underlies proposition 21 in Auletta et at. (2021), which states that the condition for a quantum solution (see footnote 9) to be a QCE coincides with the condition for communication equilibrium. However, in the communication equilibrium this result refers to, the "reports" that the players send to the mediator are actually their chosen observables. Here, by contrast, the perspective is that players report their *types* by their choice of observables, and so their only alternatives are the observables assigned to some other types. This difference is technically significant because, in Auletta et al.'s setting, a player has finitely many types but uncountably many admissible observables.

As shown in Section 4.1, quantum correlated equilibria may allow for outcomes that cannot be attained in any classical communication equilibrium in which the players' reported types do not influence the signals sent to other players (property O). The following theorem identifies as responsible for this advantage the uniquely quantum phenomenon of incompatible observables. However, somewhat unintuitively, it is the incompatibility of a player's *own* alternative observables that is responsible for this. The observables of different players are always compatible by definition of quantum mechanism.

Theorem 1 A quantum CED is a type MED if and only if it is implementable by a quantum mechanism where all the observables (and not only those of different players) are mutually compatible.

Proof To prove that the compatibility condition is sufficient, it suffices to show that for every quantum correlated equilibrium in which the quantum mechanism has compatible observables, there is a corresponding classical mechanism $\boldsymbol{m}$ (see Section 5) with property O. The compatibility of the observables means that the operators commute, and it is therefore possible to express the quantum state $|\psi\rangle$ specified by the quantum mechanism in an orthonormal basis $\{|\psi_l\rangle\}_{l=1}^{L}$ whose elements are eigenvectors common to all the operators. Thus,

$$|\psi\rangle = \sum_{l=1}^{L} \langle\psi_l|\psi\rangle|\psi_l\rangle.$$

Each basis element $|\psi_l\rangle$ defines a mapping from T to M. It is given by $(t_1, t_2, \dots, t_n) \mapsto (m_1, m_2, \dots, m_n)$, where each m_i is the eigenvalue corresponding to the eigenvector $|\psi_l\rangle$ of the observable specified by the QCE for player type t_i. (Hence, m_i is also the measurement value obtained in state $|\psi_l\rangle$ when player i chooses that observable.) Assigning to each $|\psi_l\rangle$ the probability $|\langle\psi_l|\psi\rangle|^2$ makes this a *random* mapping $\boldsymbol{m}$. The latter is easily seen to be a classical mechanism corresponding to the quantum one. The mechanism $\boldsymbol{m}$ has property O because a player's chosen observable uniquely determines the outcome (i.e., the corresponding eigenvalue) for every $|\psi_l\rangle$; the other players' observables have no effect.

To prove necessity, consider any communication equilibrium $(\boldsymbol{m}, \sigma)$ such that $\boldsymbol{m}$ has property O. This property implies that for arbitrary, fixed type profile t^*,

$$\boldsymbol{m}(t) = (\boldsymbol{m}_1(t_1, t^*_{-1}), \boldsymbol{m}_2(t_2, t^*_{-2}), \dots, \boldsymbol{m}_n(t_n, t^*_{-n})), \qquad t \in T.$$

Suppose, without loss of generality, that the messages to each player i are binary representations of nonnegative integers, specifically, $M_i = \{0,1\}^{K_i}$ for some $K_i \geq 1$. Construct a quantum mechanism employing $\sum_{i=1}^{n} |T_i| K_i$ numbered entangled qubits in a state that emulates $\boldsymbol{m}$, as follows. For each player i and type t_i, a K_i-tuple of consecutive qubits gives the message to that player type. Only player i is allowed to read this message, by measuring the state of each of the K_i bits, which is either $|0\rangle$ or $|1\rangle$. A player's observables thus correspond one-to-one with the player's types, and the observables are compatible because they address disjoint subsets of qubits. Any quantum state of the entire system can be written as

$$|\psi\rangle = \sum_{m_1^1=0}^{2^{K_1}} \sum_{m_1^2=0}^{2^{K_2}} \cdots \sum_{m_i^{j_i}=0}^{2^{K_i}} \cdots \sum_{m_n^{|T_n|}=0}^{2^{K_n}} a_{11\cdots i\cdots n}^{12\cdots j_i\cdots |T_n|} |m_1^1 m_1^2 \cdots m_i^{j_i} \cdots m_n^{|T_n|}\rangle,$$

where $|m_i^{j_i}\rangle$ is the state of the K_i-tuple corresponding to the j_ith type of player i. The emulation of $\boldsymbol{m}$ is achieved by choosing the coefficients according to

$$\left|a_{11\cdots\ i\ \cdots\ n}^{12\cdots j_i\cdots|T_n|}\right|^2 = \\ = \Pr\Big((\boldsymbol{m}_1(1, t_{-1}^*), \boldsymbol{m}_1(2, t_{-1}^*), \dots, \boldsymbol{m}_i(j_i, t_{-i}^*), \dots, \boldsymbol{m}_n(|T_n|, t_{-n}^*)) = (m_1^1, m_1^2, \dots, m_i^{j_i}, \dots, m_n^{|T_n|})\Big). \tag{6}$$

Consider the quantum correlated strategy with the quantum mechanism constructed above where the players' strategies are to choose the observables corresponding to their true types and then to act according to σ. This QCS emulates the communication equilibrium $(\boldsymbol{m}, \sigma)$ in that $\boldsymbol{m}$ is a corresponding classical mechanism and σ is the profile of classical strategies specified by the QCS. As shown in Section 5, the communication equilibrium condition (4) (together with the tightness condition) entails that this QCS is actually a QCE. Thus, the correlated strategy distribution of $(\boldsymbol{m}, \sigma)$ is a quantum CED that is implementable by a quantum mechanism with compatible observables. ∎

As shown in Section 3.3, quantum correlated equilibria have an advantage over strategy correlated equilibria, which employ classical type-unaware mechanisms. The following theorem identifies as responsible for this advantage the greater generality afforded by the quantum mechanisms. Specifically, it shows that type-unaware classical mechanisms correspond in their effect to a special kind of quantum mechanisms, namely, those where players have no choice of observables.

Theorem 2 A quantum CED is a strategy CED if and only if it is implementable by a quantum mechanism where every player has only one observable.

Proof Sufficiency follows from the observation that, if all types of each player must use the same observable, then the classical mechanism $\boldsymbol{m}$ constructed in the first part of the proof of Theorem 1 has property S as well as O. Necessity follows from the observation that, if the classical mechanism $\boldsymbol{m}$ in the second part of that proof has both properties, then the probability in (6) is different from zero only if $m_i^1 = m_i^1 = \cdots = m_i^{|T_i|}$ for all i. The conclusion means that it is possible to modify the construction in the proof so that the quantum mechanism allocates a common observable to all types of player i – a single K_i-tuple of consecutive qubits. ∎

Section 2.3 demonstrates an advantage of quantum correlated strategies over correlated strategies with classical type-unaware mechanisms. The next theorem, which easily follows from the last one, elucidates the root of this advantage.

Theorem 3 A quantum CSD is a type CSD if and only if it is implementable by a quantum mechanism where every player has only one observable.

Proof As the theorem concerns only correlated strategy distributions, the players' payoff functions u_i are irrelevant, and can be assumed constant. As indicated in Section 2.2, a CSD is a type CSD if and only if it is implementable by a type-unaware mechanism. Because of this fact, there is no difference in a Bayesian game with constant payoff functions between type CSD and strategy CED, and similarly, quantum CSD is the same thing as quantum CED. The equivalence in the theorem therefore follows from Theorem 2. ∎

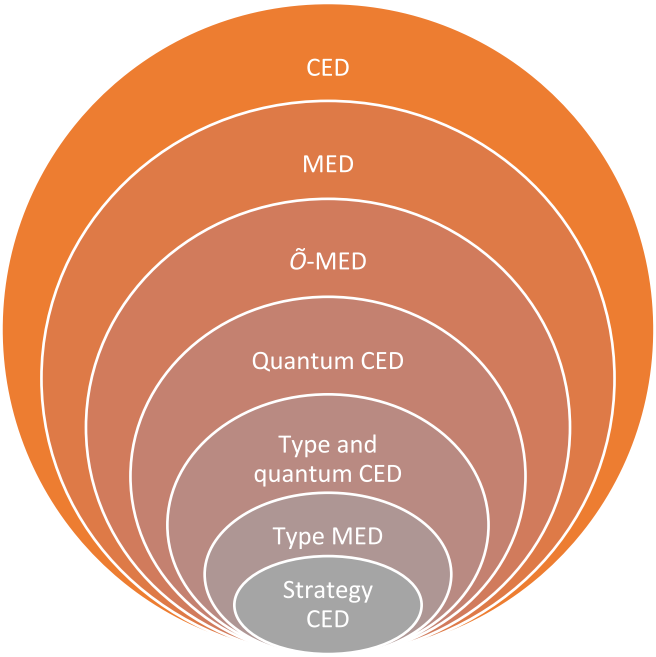

FIGURE 2 CLASSIFICATION OF CEDS IN BAYESIAN GAMES. A CED OR MED IS THE JOINT DISTRIBUTION OF THE PLAYERS' TYPES AND ACTIONS IN SOME CORRELATED OR COMMUNICATION EQUILIBRIUM, RESPECTIVELY. THE PROPERTIES OF THE (CLASSICAL OR QUANTUM) MECHANISM EMPLOYED IN SOME SUCH EQUILIBRIUM DEFINE EACH OF THE SEVEN ATTRIBUTES SHOWN.

7 Classifications

CSDs, CEDs and MEDs can be classified, and their different categories partially ordered, according to the properties of the mechanisms capable of implementing them. With classical mechanisms, these classifications take the form of two-dimensional lattices (Milchtaich 2014, Fig. 1, 2 and 3). This section tackles the question of where do quantum CEDs fit in.

For a finite list $\mathcal{P}$ of properties of classical mechanisms, a CSD, CED or MED is *$\mathcal{P}$-implementable* (a $\mathcal{P}$-CSD, $\mathcal{P}$-CED or $\mathcal{P}$-MED, for short) if it is the CSD of some correlated strategy, correlated equilibrium or communication equilibrium, respectively, with a mechanism that has all the properties in $\mathcal{P}$. Allowing for quantum mechanisms extends the notion of implementability to also cover quantum implementability. This extension yields the concepts of quantum CSD and quantum CED introduced in this paper. Quantum or any of the classical kinds of implementability is an *attribute* of CSDs, CEDs or MEDs. For each of the three kinds of distributions, the various attributes are partially ordered according to the logical relations between them, that is, whether a particular attribute implies another. If the implication holds but the reverse implication does not hold (that is, the two attributes are not equivalent), then the second attribute is *weaker* than the first. In every Bayesian game, the collection of all CSDs, CEDs or MEDs having the weaker attribute includes all those having the other, stronger attribute, while the reverse inclusion does not hold in general.

A simplified, unified classification of CEDs and MEDs (with the unification justified by Proposition 1) is presented in Figure 2. Only the seven attributes deemed most relevant for this paper are considered, and of the properties of classical mechanisms defined in Milchtaich (2014), only $\tilde{O}$, O and S (see Section 2.1) are used. As seen in Figure 2, the classification is one-dimensional. That is, the seven attributes are all comparable. The weakest attribute of CEDs is simply being one and the strongest is being a strategy CED.

These logical relations are proved in the following subsection. The subsequent subsection shows that the same classification that applies to CEDs or MEDs (that is, to type-action distributions) also applies to the corresponding payoff vectors.

7.1 Classification of distributions

It is a simple observation that a MED can be a quantum CED only if it is $\tilde{O}$-implementable. This is because the compatibility of different players' observables in a quantum mechanism entails that players do not get from their measurements any information about the others' types, which for a corresponding classical mechanism $\boldsymbol{m}$ spells the no-signaling property $\tilde{O}$. Thus, the following stronger form of Proposition 2 holds.

Proposition 3 Every quantum CED is an $\tilde{O}$-MED.

The converse is false; not every $\tilde{O}$-MED is a quantum CED. For example, the correlated strategy described in Section 2.2, which uses a coin-toss mechanism that has property $\tilde{O}$, is obviously a communication equilibrium in the common-interest CHSH game (Example 1). But the MED, which is shown in Table 1, is not a quantum CED because it violates Tsirelson's bound (see Section 2.3).

The next, one-step more exclusive, category in the classifications of classical CEDs and MEDs (Milchtaich 2014) is the O-CEDs and O-MEDs, in other words, the type CEDs and MEDs. Property O goes beyond $\tilde{O}$ by requiring the messages to the players to be completely unaffected by the (true or reported) types of the other players. The next result illustrates the significance of this difference.

Proposition 4 In a Bayesian 2×2 game with two types for every player and a common prior with full support, an $\tilde{O}$-implementable CSD, CED or MED is O-implementable if and only if it satisfies the Bell inequalities.

Proof Consider any $\tilde{O}$-CSD, $\tilde{O}$-CED or $\tilde{O}$-MED η. As shown in Milchtaich (2014, Sections 4.4 and 4.5), by a version of the revelation principle, there exists, respectively, a correlated strategy, correlated equilibrium or communication equilibrium $(\boldsymbol{m}, \sigma)$ with the distribution η such that (i) the players' message spaces coincide with their action spaces, and (ii) the strategies are to act according to the received message, that is,

$$\sigma_i(j, m_i) = m_i, \qquad i = 1,2, j = \mathrm{I}, \mathrm{II}.$$

With such strategies, the joint distribution of actions for each type profile t coincides with the distribution of $\boldsymbol{m}(t)$. Now, inspection of the correlated and communication equilibrium conditions shows that they are left unaffected by a replacement of $\boldsymbol{m}$ with any other mechanism $\tilde{\boldsymbol{m}}$ that has the same message distributions, that is,

$$\tilde{\boldsymbol{m}}(t) \overset{d}{=} \boldsymbol{m}(t), \qquad t \in T.$$

What remains to be determined is whether some such $\tilde{\boldsymbol{m}}$ has property O. As shown in Section 2.2, this is so if and only if η satisfies the Bell inequalities and the no-signaling condition. The latter condition is automatically implied by the assumed $\tilde{O}$-implementability of η. This leaves the Bell inequalities as a necessary and sufficient condition for O-implementability. ■

A type CED is not necessarily a quantum CED and vice versa. For example, as shown in Section 3, the CSD in Table 2 is a quantum CED in the CHSH game, but it follows from

Proposition 4 that it is not even a type CSD. The CED in Example 3 is a type CED as the indicated two-coin mechanism has property O, but it is not even a MED. These examples point to the existence of an additional, distinct category: type CEDs that are also quantum CEDs. Example 4, which presents a type CED that is also a quantum CED (hence, a MED) but is not a type MED, shows that this category does not coincide with the type MEDs. In fact, it (properly) includes the latter. The inclusion follows from Proposition 1, which shows that every type MED is a type CED, and from the next proposition, which shows that it is also a quantum CED.

Proposition 5 Every type MED ($= O$-MED) is a quantum CED.

Proof This result is established by the second part of the proof of Theorem 1, which in fact proves more than the necessity of the condition in that theorem. It shows that *every* classical mechanism with property O can be emulated by a quantum mechanism with mutually compatible observables. It follows that the collection of quantum CEDs implementable by a quantum mechanism in which all the observables are compatible coincides with that of *all* type MEDs. ■

The next rung in the classifications of CEDs and MEDs are the S, O-CEDs and S, O-MEDs, that is, those implementable by type-unaware mechanisms. As already indicated, these two are actually the same category: the strategy CEDs. This is because, with a mechanism that totally ignores the players' types, there is no difference between communication and correlated equilibrium. The distinction of this category from O-CEDs and O-MEDs is proved by the next example (Milchtaich 2014).

Example 5 *A type MED (and type CED) that is not a strategy CED ($= S, O$-CED).* Consider the common-interest Bayesian game with the payoffs specified by the left 2×2 table for each type profile.

					Player 2								
			Type I						**Type II**				
		0.25	$+1$	-1	$+1$	-1		**0.25**	$+1$	-1	$+1$	-1	
Player 1	**Type I**	$+1$	2	0	0.5	0	0.5	$+1$	3	0	0.25	0.25	0.5
		-1	0	2	0	0.5	0.5	-1	0	0	0.25	0.25	0.5
					0.5	0.5					0.5	0.5	
		0.25	$+1$	-1	$+1$	-1		**0.25**	$+1$	-1	$+1$	-1	
	Type II	$+1$	3	0	0.25	0.25	0.5	$+1$	2	0	0.5	0	0.5
		-1	0	0	0.25	0.25	0.5	-1	0	2	0	0.5	0.5
					0.5	0.5					0.5	0.5	

The CSD specified by the right 2×2 tables is a type MED (hence, a type CED) in this game. It is implementable by the two-coin mechanism described in Example 3, which has property O. Together with the strategies of playing $+1$ if and only if heads is observed, this mechanism constitutes a communication equilibrium, as it is easy to check that misreporting the type would reduce a player's payoff from $(1/2 \times 2 + 1/8 \times 3 =)\ 1.375$ to $(1/4 \times 2 + 1/4 \times 3 =)\ 1.25$.

This type MED is not a strategy CED; no implementing type-unaware mechanism exists. The reason is that, since the players' actions are perfectly correlated whenever they have the same type, any message m_1 that a type-unaware mechanism sends to player 1, thereby causing the player's two types to choose particular actions, must always be accompanied by a message to player 2 that elicit the same pair of actions from that player's types. Either the two types' actions are different or they are the same. The first possibility is excluded by the equilibrium assumption, because the type of player 1 prompted by m_1 to play -1 will get $1/2 \times 2$ from doing so but $1/2 \times 3$ from choosing $+1$, a profitable deviation. This leaves only the second possibility, in which the two *players* always choose the same action. However, the conclusion contradicts the fact that there is probability 0.5 that the players' actions differ when their types differ.

7.2 Classification of payoff vectors

There exists a parallel classification of CEPs and MEPs, which pertains to the n-tuple of the players' expected payoffs rather than the joint distribution of their types and actions. In every Bayesian game, the joint distribution of types and actions uniquely determines the expected payoffs. However, this mapping if obviously not one-to-one, which is significant for the question of implementability. For example, the expected payoff of both players in the CED in Example 3 is 0.25. And although this CED is not a MED, hence not a quantum CED, the CEP *is* a MEP, a quantum CEP and even a strategy CEP. (A type-unaware classical mechanism implementing the 0.25 payoffs is one where the players receive binary signals with this correlation coefficient.) If the same were true for every CEP, then the move from distributions to payoff vectors would result in a merger of the seven categories in Figure 2. But in fact, as the following theorem shows, no two of these merge; the classification remains untucked.

Theorem 4 For every two attributes of correlated (or communication) equilibrium distributions in Figure 2, the corresponding collections of correlated equilibrium payoff vectors are generally not identical. Thus, the classification of CEPs and MEPs mirrors that of CEDs and MEDs.

Proof The argument is similar to that in the proof of Theorem 3 in Milchtaich (2014). It shows that, if a CED has one attribute $\mathcal{A}$ of those in Figure 2 (say, it is an $\tilde{O}$-MED) but not another attribute $\mathcal{B}$ (say, it is not both a type CED and a quantum CED), then there is in some *other game* a CED with attribute $\mathcal{A}$ whose payoff vector differs from that of every CED with attribute $\mathcal{B}$.

That other Bayesian game is the extension of the original game obtained by adding as "dummy players" all the elements of $T \times A$. Each such player (t, a) has only one possible type and a single action and thus cannot affect the true, original players but is only affected by them. Specifically, the dummy player's payoff function $u_{(t,a)}$ is defined as the indicator function $1_{(t,a)}$, which means that the expected payoff is the probability that the true players' type profile is t and their action profile is a. Incorporation of the unique types and actions of the dummy players extends every CSD η in the original game in a trivial manner to a CSD $\hat{\eta}$ in the extended game, and this extension is clearly a bijective mapping. The payoff vector of $\hat{\eta}$ extends that of η by adding the expected payoffs of the dummy players, which (being the probabilities of type-action pairs) coincide with η. It follows that no other CSD in the extended game has the same payoff vector as $\hat{\eta}$. To complete the proof, it remains to show that a CED η in the original game and its extension $\hat{\eta}$ have the same set of attributes (of those in Figure 2).

First, every attribute $\mathcal{A}$ of η is also shared by $\hat{\eta}$. This is because every correlated strategy $(\boldsymbol{m}, \sigma)$ whose CSD is η can be extended to a correlated strategy $(\widehat{\boldsymbol{m}}, \hat{\sigma})$ with the CSD $\hat{\eta}$ by sending to the dummy players some constant messages and assigning them their single strategies. The two correlated strategies – one in the original game and the other in the extended game – clearly share the same properties in $\{S, \tilde{S}, O, \tilde{O}\}$, and if one of them is a correlated or a communication equilibrium, then so is the other. For quantum correlated strategies, an extension with a similar function is obtained by allowing every dummy player only the trivial observable, i.e., the identity operator on the state space. These players' "measurements" then have no effect on the true players.

Second, every attribute $\mathcal{B}$ of $\hat{\eta}$ is also shared by η. The argument is similar to that in the previous paragraph, except for the possible complication arising from interactions between the mechanism and the dummy players, which need to be nullified first. For a given classical correlated strategy $(\widehat{\boldsymbol{m}}, \hat{\sigma})$ whose CSD is $\hat{\eta}$, this means that any non-constant signals that the mechanism $\widehat{\boldsymbol{m}}$ sends to dummy players need to be replaced with constant signals. This replacement is clearly inconsequential, and it turns the correlated strategy into one that is the extension, and thus shares all the properties, of a correlated strategy in the original game. For a given quantum correlated strategy whose CSD is $\hat{\eta}$, the possible complication is that the quantum mechanism may allow some dummy players non-trivial observables. Depending on the outcomes of these players' measurements, the true players may be left with the system in several possible quantum states – a mixed state. However, as indicated in footnote 8, measurements performed on a system in a mixed state are functionally the same as (that is, their outcomes are statistically indistinguishable from) local measurements in some larger system that is in a pure state. That state and the possible measurements indicated by the true players' observables constitute a quantum mechanism, which together with these players' given strategies constitute a QCS in the original game. It is easy to see that the latter is a QCE if and only if the given QCS in the extended game is so. ■

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