Charges and Bets: A General Characterisation of Common Priors

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Abstract

The seminal no betting theorem on the equivalence of common priors and absence of agreeable bets obtains only over compact state spaces. We show here that this equivalence can be generalised to any infinite space if we expand the set of priors to include probability charges as priors. Going beyond the strict prior/no common prior dichotomy, we further uncover a fine-grained decomposition of the space of type spaces into continuum many subclasses in each of which an epistemic condition approximating common priors is equivalent to a behavioural condition limiting acceptable bets. Several additional concepts relating to approximations of common priors and type spaces admitting common priors are studied, elucidating more aspects of the structure of the class of type spaces.

Keywords: Common prior, no betting, probability charge

1 Introduction

The elucidation of the interplay and interaction between interpersonal beliefs and economic behaviours and choices has been a hallmark of game

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theory research, and more broadly economic research, for decades. One of the clearest articulations of this is expressed in the Aumann No Disagreement, or No Betting Theorem, which has justifiably been impactful in many fields of inquiry. In its brief and concise statement, the theorem establishes a sharp and dichotomous connection between interpersonal epistemology and behaviour: players can find a bet that is agreeable between them if and only if their beliefs cannot possibly be derived from a common prior. This means that a modeller studying speculative trade can assume common priors and conclude that such trade is not possible or conversely suppose that the beliefs of the players cannot have been derived from a common prior for a different conclusion.

This seminal result of 'common priors if and only if there is no bet', however, was shown to hold only when the space of states is compact (Feinberg (2000)): otherwise, while a common prior still implies no betting, the converse may not obtain, with all the attendant modelling implications. Apart from the fact that assuming compactness involves injecting an extraneous topological condition to the subject, this left open the question of whether it is possible to express an analogous characterisation relating epistemic and behavioural properties in general over all infinite state spaces.

We show here that when we expand the focus more broadly to include probability charges as priors, we attain such a generalisation and more. The clear and direct equivalence of 'common priors if and only if there is no bet' is restored. Beyond the stark common priors/no common prior dichotomy, we uncover a fine-grained decomposition of the class of types spaces over a state space Ω into a continuum of subclasses, each of which in itself expresses an equivalence between beliefs and economic behaviour. We further use the tools developed to elucidate aspects of the structure of the space of type spaces using approximations to common priors and consistent type spaces.

1.1 Sigma Additive Priors and No Betting

Relative to a state space Ω and set of players N, the literature considers type spaces, using concepts pioneered by Harsányi (1967-68) and Aumann (1976). Intuitively, the interpretation is that there is a true state $\omega \in \Omega$. A player i receives an informative signal at ω from which he or she deduces a probability distribution $t_i(\omega)$ – his or her 'belief' – over the states of Ω. Furthermore, in a common interpretation, player i is supposed to have a prior distribution (a σ-additive probability measure) $p \in \Delta(\Omega)$ at an *ex ante* stage from which the beliefs expressed by t_i, upon receipt of a signal, are derived by updating. Then if two (or more) players start from the same prior p and update to respective type functions they have a common prior.

Formally, however, there is no need for the interpretive baggage of presuming an *ex ante* stage or even a single shared prior. For one thing, a modeller or analyst may not have access to the history of updating from a prior, even if we assume such history exists; we may have only the current beliefs of the players with which to work.

More deeply, for the purposes of studying disagreements the actual history may be misleading, in the sense that two players can begin with two distinct priors from which they respectively update to a type space, yet for our purposes here we would still say that they have common priors. In fact, for a given type function t_i there may be a *set* of elements $\pi_i \subset \Delta(\Omega)$ all of which can serve as priors for player I, with no reason to prioritise one prior above the others. A type space admits a common prior if $\bigcap_i \pi_i \neq \emptyset$; all the information needed for ascertaining whether there is a common prior is in principle available from the data of the type space without presuming updating from an *ex ante* stage.

Denote the collection of type spaces over Ω by $\mathfrak{T}$. Following Harsányi, a type space that admits a common prior is called a consistent type space. Consistency partitions the class of type spaces in two: denote the collection of consistent spaces by $\mathfrak{C}$ and its complement by $\mathfrak{I}$.

Up to here, only epistemic elements of interactive belief have been considered. For behavioural aspects, one adds a new element exogenous to the type spaces: bets. A bet is a collection of bounded state-dependent payoff functions, one for each player, which together are zero sum; that is, after the true state is ascertained the wager is cleared, with the losers paying the payoff of the winners. A bet is agreeable to the players if at each state each player believes he or she has expectation of positive gain bounded away from zero (despite the fact that the bet is by definition zero sum). The concept naturally extends to broader concepts in the study of speculative trade in Arrow–Debreu securities, contract theory, interactions between Bayesian agents, Bayesian persuasion, and many other fields.

Denote the collection of type spaces admitting an agreeable bet by $\mathfrak{B}$ and its complement by $\mathfrak{D}$. The No Disagreements Theorem states that when Ω is a compact topological space this does not actually define another partition of $\mathfrak{T}$: namely, $\mathfrak{C} = \mathfrak{D}$ and equivalently $\mathfrak{I} = \mathfrak{B}$ (Figure 1(a)), i.e., the players can find an agreeable bet if and only if there is no common prior (Feinberg (2000)).

The power of the No Disagreements Theorem lies in the sharp dichotomy

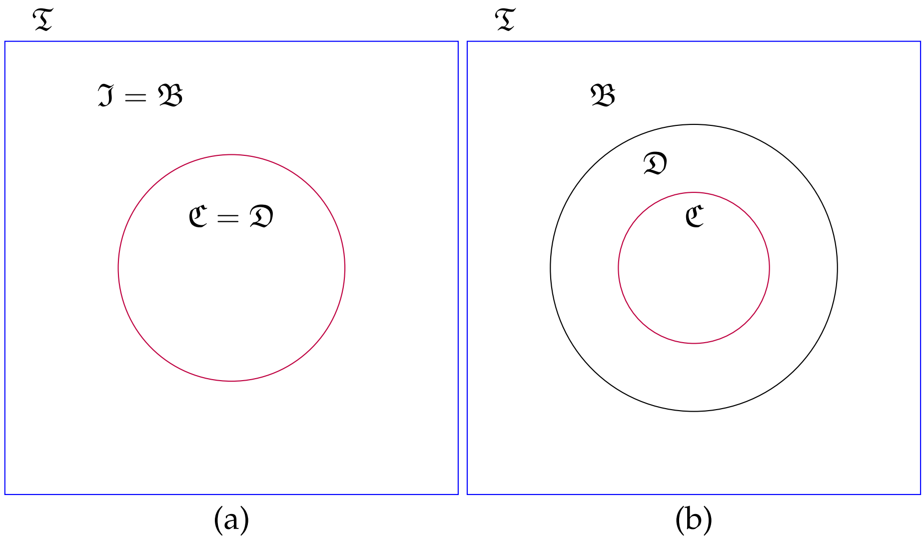

Figure 1: A schematic representation of the relations between the collections of type spaces admitting σ-common priors ($\mathfrak{C}$) and those not admitting agreeable bets ($\mathfrak{D}$) in (a) the case of compact state spaces and (b) non-compact state spaces.

it establishes and its exact coupling of epistemic and behavioural properties that *a priori* are not related. Unfortunately, the statement of the theorem does not extend as-is to non-compact spaces: Feinberg (2000) includes an example of a countable space that does not admit a σ-additive common prior yet also has no agreeable bet. In such cases the most that can be said is that $\mathfrak{C} \subsetneq \mathfrak{D}$ (Figure 1(b)).

Lehrer and Samet (2014), in the context of countable state spaces, later provided a more complex three-levelled classification of belief consistency consisting of strong consistency, consistency, and weak consistency, with a parallel and equivalent classification of behaviour related to bets. This still, however, left open the question of finding a simple characterisation of common priors in more general non-compact spaces in behavioural terms.

1.2 Finitely Additive Priors and No Betting

To begin considering how to contend with the general infinite and non-compact case, consider first a quick sketch of the two-player proof in the case of a compact state space Ω, as in Heifetz (2006). In this context, the sets of priors (π_1, π_2) are a pair of closed and compact subsets of the set of regular Borel probability measures over Ω. If there is no common prior this pair can be strongly separated by a continuous linear functional F. By an appeal to the Riesz–Markov–Kakutani theorem, F is associated to a continuous function $f \in C(\Omega)$, which with some minor manipulation becomes the agreeable bet that is sought.

Dually, instead of starting from probability distributions one can start instead with the set of payoff functions over Ω. Stating that there are no agreeable bets becomes equivalent to the separation of a certain subset of payoff functions from the negative orthant. The resulting separating functional is equivalent to a common prior

When Ω is not compact, the duality inherent in the subject matter at hand may not be available. This is underscored in Lehrer and Samet (2014): in the context of a countable state space, beliefs are elements of the normed space $\ell_1(\Omega)$ of σ-summable functions over Ω, and bets are elements of $\ell_\infty(\Omega)$, the normed space of bounded functions. The space $\ell_\infty(\Omega)$ is the dual of $\ell_1(\Omega)$, but reflexivity does not hold here: taking the dual of $\ell_\infty(\Omega)$ does not return one to $\ell_1(\Omega)$, because there are continuous functionals on $\ell_\infty(\Omega)$ that are finitely additive probability charges that are not σ-additive.

That last sentence contains the grain of inspiration for the approach of this paper. Extend the set of priors associated with the type function t_i of a player i to include now probability charges; denote this by Π_i. What heretofore was termed the set of priors, π_i, which contains only σ-additive probability measures, now becomes a subset of Π_i, and we now call the elements of $\pi_i \subseteq \Pi_i$ the set of σ-priors of player i.

We can similarly extend the space of type spaces over Ω to contain type functions whose range is the collection of probability charges over Ω; denote this extended space of type spaces $\mathcal{T}$. Further generalising, let the set of players N possibly be infinite; we then define a bet to be a state dependent transfer of money amongst a finite subset of players in N. In this new context, denote the set of common priors $\mathcal{C}$, the complement of $\mathcal{C}$ by $\mathcal{I}$, the set of agreeable bets by $\mathcal{B}$ and its complement $\mathcal{D}$.

Putting it all together, Theorem 15 here restores the 'common priors iff no agreeable bets' characterisation in the broadest generality: $\mathcal{C} = \mathcal{D}$ (Figure 2(a)). As Π_i is entirely determined by the type function t_i (as is π_i), we again have epistemics determining behaviour, and the converse.

This result is expressed in terms of probability charges, not probability measures, and indeed it may be necessary to work with priors that are probability charges. However, we stress that this need not require supposing that the player are actually updating in the *ex ante* stage from a prior that is a probability charge but not a probability measure (which may not

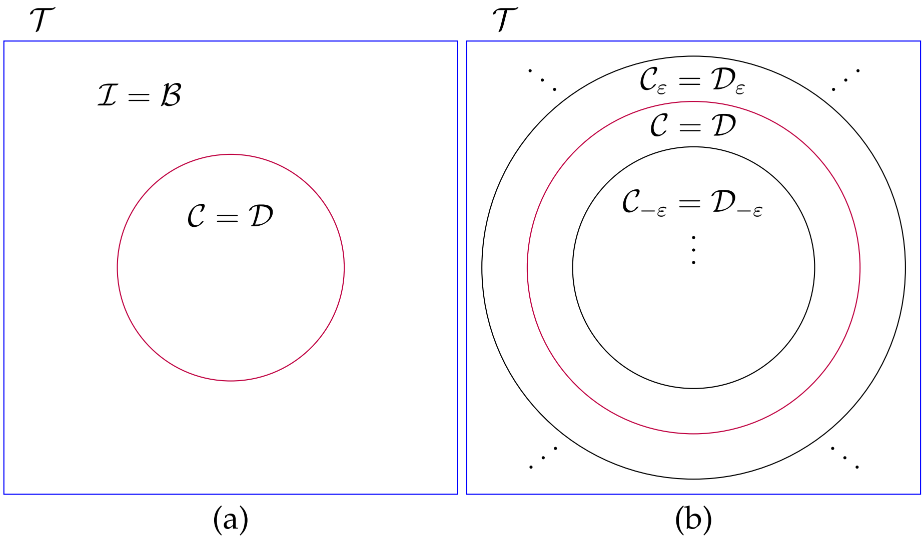

Figure 2: Schematic representations of (a) the relations between the collections of type spaces admitting common priors ($\mathcal{C}$) and those not admitting agreeable bets ($\mathcal{D}$) when probability charges are taken into account; (b) a fine grained decompositoin of the space of types spaces in terms of ε-common priors and ε-agreeable bets.

be possible in a practical way). Indeed, as pointed out above, we do not actually need an ex ante stage; it suffices to have an epistemic condition dependent solely on the beliefs as expressed in the type functions to determine whether or not behavioural consequences such as agreeable betting obtain. This is attained in Theorem 15.

1.3 Fine Grained Decomposition of the Class of Type Spaces

With the tools developed here, we can also go beyond the binary dichotomy of consistent/inconsistent type spaces to study a more fine grained decomposition of the class of type spaces over state space Ω. Agreeable bets are defined above as bets in which each player believes at each state that there is positive expectation of payoff gain bounded away from zero. One can imagine the players demanding more stringent conditions for accepting a bet, such as an expectation of gaining at least $\varepsilon > 0$ at each state. In the other direction, we needn't require only positive gains: a player could conceivably agree to a bet with losses as long as losses are limited to being no greater than ε.

Since bets can be scaled up or down, it makes more sense in this context for the expectations to be proportional to the stake a player has in the bet. For $\varepsilon \in [-1, 1]$ define an ε-agreeable bet to be a bet such that the expectation of player i for all i at each state is greater than $\varepsilon \|f_i\|_{\sup}$, using the sup-norm. Denote the collection of type spaces admitting an ε-agreeable bet by $\mathcal{B}_\varepsilon$, and its complement by $\mathcal{D}_\varepsilon$.

In parallel, on the epistemic side, for $\varepsilon > 0$ define Π_i^ε intuitively as Π_i 'thickened' by expanding Π_i by adding to it probability charges outside of Π_i that are ε distant from the boundary of Π_i by the total variation norm (see Section 3.2 for the formal definition). For $\varepsilon < 0$ define Π_i^ε intuitively by removing from Π_i probability charges that are ε distant from the boundary of Π_i; hence in this case Π_i^ε is 'thinner' than Π_i.

For $-1 \leq \varepsilon \leq 1$, an element of Π_i^ε is an ε-prior of player i. If $\cap_i \Pi_i^\varepsilon \neq \emptyset$ the type space has an ε-common prior. Denote the collection of spaces attaining an ε-common prior by $\mathcal{C}_\varepsilon$; what was previously written as $\mathcal{C}$ is $\mathcal{C}_0$ in this notation. This defines a nested collection of subsets that covers all of $\mathcal{T}$.

The epistemic and behavioural concepts developed here for ε are related: we show that $\mathcal{C}_\varepsilon = \mathcal{D}_\varepsilon$ for all $\varepsilon \in [-1, 1]$. In words, a type space attains an ε-common prior if and only if the players have no ε-agreeable bet between them, with the No Disagreements Theorem a special case of this broader theorem (Figure 2(b)).

We see that the space of types spaces can be decomposed into continuum-many subsets, each of which relates epistemic inter-relations between the players to behavioural constraints.

We also study here further concepts of approximation within the class of type spaces. Common priors in general, which are probability charges and may have infinite support, can be complicated objects with which to work. σ-priors with finite support are simpler. In Section 5 we show that any common prior of a type space can be approximated, in an appropriately defined way, by σ-priors with finite support. In Section 6, we approximate the type spaces themselves, showing that if a type space T is finitely approximable (as defined in that section) by approximating finite type spaces that all attain approximating common priors, then T itself attains a common prior. The converse, however, does not hold: a type space approximated by finite type spaces with common priors may not itself attain a common prior, indicating a 'discontinuity' in the space of type spaces with regards to the attainment of common priors.

2 Preliminaries

2.1 Mathematical Preliminaries

Given a set X, a field[1] $\mathcal{A}$ is a collection of subsets of X that is closed under complements and under finite intersections (hence also under finite unions) of sets, and contains X itself.

If $\mathcal{A}$ is a field on X then the pair $(X, \mathcal{A})$ is called a *chargeable space*. A set function $\mu\colon \mathcal{A} \to \mathbb{R}$ on a chargeable space is a *charge* on $(X, \mathcal{A})$ if μ is additive; it is a *probability charge* if in addition it is non-negative and $\mu(X) = 1$. If $\mathcal{A}$ is a σ-field the pair $(X, \mathcal{A})$ is called a *measurable space*. A charge μ over $\mathcal{A}$ that is σ-additive is a *measure*; it is a *probability measure* if it is non-negative and $\mu(X) = 1$.

The distinction between charges and (σ-additive) measures will be crucial throughout this paper. Note in particular that each measure is a charge, but not the converse.

Let $\mathrm{pba}(X, \mathcal{A})$, $\Delta(X, \mathcal{A})$, and $D(X, \mathcal{A})$ denote, respectively the set of probability charges, the set of probability measures, and the set of probability measures with finite support on $(X, \mathcal{A})$. Moreover, let $\mathrm{ba}(X, \mathcal{A})$, $\mathrm{ca}(X, \mathcal{A})$ and $\mathrm{da}(X, \mathcal{A})$ denote, respectively, the set of bounded finitely additive set functions, the set of bounded measures, and the set of measures with finite support on $(X, \mathcal{A})$.

Let $B(X, \mathcal{A})$ denote the collection of all uniform limits of finite linear combinations of characteristic functions of sets in $\mathcal{A}$. We will call the elements of $B(X, \mathcal{A})$ *$\mathcal{A}$-integrable functions*. This is a very non-standard designation: functions are usually termed μ-integrable for μ a measure; here we are speaking of $\mathcal{A}$-integrable functions where $\mathcal{A}$ is a field. We justify this by the fact that the class of $\mathcal{A}$-integrable functions satisfies the property that it is the class of functions that are integrable over $\mathcal{A}$ with respect to any bounded charge (Aliprantis and Border (2006), Chapter 11).

We will make use of the sup-norm on $B(X, \mathcal{A})$, defined by $\|f\|_{\sup} = \sup_{x \in X} |f(x)|$, $f \in B(X, \mathcal{A})$. Denote by $B^*(X, \mathcal{A})$ the dual of (the set of linear functionals on) $B(X, \mathcal{A})$, noting that $\mathrm{ba}(X, \mathcal{A}) = B^*(X, \mathcal{A})$. Let $\mathrm{ba}(X, \mathcal{A})$ be equipped with the weak* topology by $B(X, \mathcal{A})$. For any $\mu \in \mathrm{ba}(X, \mathcal{A})$, $\varepsilon > 0$, and $F \subseteq B(X, \mathcal{A})$ such that $|F| < \infty$, denote by

(1) $$\mathcal{O}^*(\mu, F, \varepsilon) = \left\{ \nu \in \mathrm{ba}(X, \mathcal{A})\colon \left| \int f \ \mathrm{d}\,\nu - \int f \ \mathrm{d}\,\mu \right| < \varepsilon, \ \forall f \in F \right\},$$

[1] The mathematical literature uses two synonyms for the same concept: field and algebra. We elected in this paper to use the term field for this concept.

a neighbourhood of μ. The collection of such neighbourhoods forms a subbase for the weak* topology of $\mathrm{ba}(X, \mathcal{A})$. $\overline{A}^*$ will denote the weak* closure of a set $A \subseteq \mathrm{ba}(X, \mathcal{A})$.

It is well-known that $\mathrm{da}(X, \mathcal{A})$, the set of measures with finite support on $(X, \mathcal{A})$, is weak* dense in $\mathrm{ba}(X, \mathcal{A})$. As a consequence, since $\mathrm{da}(X, \mathcal{A}) \subseteq \mathrm{ca}(X, \mathcal{A})$, the set of bounded measures, $\mathrm{ca}(X, \mathcal{A})$, is also weak* dense in $\mathrm{ba}(X, \mathcal{A})$.

In addition to the weak* topology, we will also make use of a different topology on $\mathrm{ba}(X, \mathcal{A})$, the one generated by the total variation norm. The total variation norm is defined as follows. For any $\mu \in \mathrm{ba}(X, \mathcal{A})$ let

$$\|\mu\|_{TV} = \sup_{P \in \Pi(X, \mathcal{A})} \sum_{A \in P} |\mu(A)|,$$

where $\Pi(X, \mathcal{A})$ denotes the class of $\mathcal{A}$-measurable finite partitions of X. Then the collection of sets

$$\mathcal{O}_{TV}(\mu, \varepsilon) = \{\nu \in \mathrm{ba}(X, \mathcal{A}) \colon ||\mu - \nu||_{TV} < \varepsilon\},$$

$\mu \in \mathrm{ba}(X, \mathcal{A})$, $\varepsilon > 0$, forms a subbase of the topology by the total variation norm.

For pairs of sets $A, B \subset \mathrm{ba}(X, \mathcal{A})$, we will make use of the concept of the Minkowski sum of A and B: $A + B = \{a + b : a \in A, b \in B\}$.

2.2 Review of Previous Results in the Literature

Mainly following Heifetz (2006), let I be a finite set of players and let Ω be a Hausdorff topological space. Denote by Σ the Borel σ-field generated by the topology of Ω and by $\Delta(\Omega)$ the associated space of regular Borel probability measures, endowed with the weak* topology. A σ-type function for player i is a continuous mapping $t_i : \Omega \to \Delta(\Omega)$. A σ-type space is a tuple consisting of a σ-type function for each player i; denote by $\mathfrak{T}$ the collection of all σ-type spaces over Ω.

Let $T \in \mathfrak{T}$ be a σ-type space. A probability measure $P_i \in \Delta(\Omega)$ is a σ-prior for i if for every event $E \in \Sigma$,

$$P_i(E) = \int_\Omega t_i(\cdot)(E) \, \mathrm{d}P_i.$$

A probability measure $P \in \Delta(\Omega)$ is a common σ-prior (for T) if it is a prior for each $i \in I$. Call a σ-type space that admits a common σ-prior

σ-consistent. Denote the collection of all σ-consistent type spaces over Ω by $\mathfrak{C} \subset \mathfrak{T}$, and the inconsistent type spaces by $\mathfrak{I} := \mathfrak{C}^C = \mathfrak{T} \setminus \mathfrak{C}$.

An agreeable bet, with respect to a σ-type space T, is a set of continuous random variables $f_i : \Omega \to \mathbb{R}$, one for each $i \in I$, satisfying $\sum_i f_i = 0$ and $\int_\Omega f_i(\cdot)\, \mathrm{d}t_i(\omega) > 0$, for all $\omega \in \Omega$. Denote the collection of all σ-type spaces admitting an agreeable bet by $\mathfrak{B} \subset \mathfrak{T}$, and denote $\mathfrak{D} = \mathfrak{B}^C$.

In these terms, the no betting theorem can be stated succinctly as: $\mathfrak{C} = \mathfrak{D}$, (equivalently $\mathfrak{B} = \mathfrak{I}$) when Ω is compact (Feinberg (2000); Heifetz (2006)).

When Ω is not compact, the above does not necessarily hold. Feinberg (2000) provides an example of a type space over $\mathbb{N}$ for which $\mathfrak{C} \subsetneq \mathfrak{D}$.

In the case of a non-compact countable state space, Lehrer and Samet (2014) introduced a concept that they term strong trade consistency, related to bets, and used that to characterise the existence of common priors in type spaces over countable state spaces, i.e., such a type space admits a common prior if and only if it is strong trade consistent. More broadly, Lehrer and Samet (2014) present a three-levelled epistemic classification – weakly belief consistent, belief consistent, strongly belief consistent – which are respectively shown to be equivalent to three behavioural properties that they label weak trade consistency, trade consistency, and strong trade consistency.

3 Type Spaces and Priors

3.1 Type space

We suppose throughout the existence of a a chargeable space $(\Omega, \mathcal{M})$. The elements of the field $\mathcal{M}$, the *general epistemic field*, serve as the *events* of interest in our model.

The model also includes a set of players N (which is not necessarily assumed to be a finite set). Each player $i \in N$ is associated with a field $\mathcal{M}_i \subset \mathcal{M}$ (i's *private epistemic field*), which together with Ω forms a chargeable space $(\Omega, \mathcal{M}_i)$.

Definition 1. *A* type function *is a mapping $t_i : \Omega \times \mathcal{M} \to [0, 1]$ satisfying*

1. *$t_i(\omega, \cdot)$ is a probability charge on $\mathcal{M}$ for all $\omega \in \Omega$,*

2. *$t_i(\cdot, E)$ is $\mathcal{M}_i$-integrable for each event E in the field $\mathcal{M}$.*

A type function t_i for a player i is very similar to what is called a kernel in the probability theory literature; the difference is that where a standard kernel Property 2 above would state that $t_i(\cdot, E)$ is $\mathcal{M}_i$-measurable for each event E, we require it to be $\mathcal{M}_i$-integrable. We may regard type functions defined here as 'generalised kernals', because in the context of a σ-field if $t_i(\cdot, E)$ is $\mathcal{M}_i$-integrable then it is $\mathcal{M}_i$-measurable.

By Property 1 of Definition 1, at any fixed state ω a type function yields a probability charge by $t_i(\omega, \cdot)$; this captures the intuitive idea that when ω is the 'true' state, player i, who has incomplete information, holds a particular belief, expressed by $t_i(\omega, \cdot)$, regarding which states are possibly true. The real value $t_i(\omega, E)$ represents the probability that player i assigns to event E when the true state is ω.

Property 2 of Definition 1 requires that player i know his or her own belief, meaning if player i has two different beliefs in two states then he or she can distinguish between these states This property will also be important for the definition of a prior distribution (Definition 4), which requires integrability with respect to a range of probability charges.

Definition 2. *A tuple $((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i\in N}, \{t_i\}_{i\in N})$ is called a* type space *if for each player $i \in N$, player i's type function t_i satisfies the property that for each $E \in \mathcal{M}_i$, for each $\omega \in E$, $t_i(\omega, E) = 1$.*

Note that Definition 2 excludes models in which a player 'does not know what he believes'.

For a fixed state space Ω, denote the collection of all type spaces by $\mathcal{T}$.

Definition 3. *A σ-type space $((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i\in N}, \{t_i\}_{i\in N})$ is a special case of a type space as defined in Definition 2 in the obvious manner: $\mathcal{M}$ and $\mathcal{M}_i$, for $i \in N$, are σ-fields, and for each player i the type function t_i satisfies the conditions that $t_i(\omega, \cdot)$ is a probability measure on $\mathcal{M}$ for each ω and $t_i(\cdot, E)$ is $\mathcal{M}_i$-measurable for each event E.*

Every σ-type space is a type space, but not vice versa.

3.2 Priors

Definition 4. *Let $T = ((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i\in N}, \{t_i\}_{i\in N})$ be a type space, and let $i \in N$ be a player. Then a probability charge $P_i \in \mathrm{pba}(\Omega, \mathcal{M})$ is a* prior *of player i (relative to T) if for each $A \in \mathcal{M}$ and $B \in \mathcal{M}_i$*

(2)
$$P_i(A \cap B) = \int_B t_i(\cdot, A)\ \mathrm{d}\, P_i.$$

Π_i will be used to denote the set of player i's priors (with T understood from context).

Remark 5. Note that 'prior', as defined here in Definition 4 and used throughout this paper, is a probability charge. The word prior is usually used to mean a probability measure; we reserve the term σ-prior for that. Since probability measures are special cases of probability charges, a σ-prior is a (charge) prior.

Similarly, type spaces as defined in Definition 2 involve charges, whereas the term type spaces in the literature usually refers to what we term here σ-type spaces, which are restricted to using only probability measures. $\mathcal{T}$, the collection of type spaces over a state space Ω in this paper is broader than $\mathfrak{T}$, the collection of σ-type spaces, i.e. $\mathfrak{T} \subset \mathcal{T}$. ♦

Definition 6. *Using the notation of the Minkowski sum, and with $\mathcal{O}_{TV}(0, \varepsilon)$ denoting the ε-neighbourhood of the origin in $\mathrm{ba}(\Omega, \mathcal{M})$ by the total variation norm, define for each player i and each $\varepsilon \in [-1, 1]$ the set*

$$\Pi_i^\varepsilon = \begin{cases} \overline{(\Pi_i + \mathcal{O}_{TV}(0,\varepsilon))}^* \cap \mathrm{pba}(\Omega, \mathcal{M}) & \text{if } \varepsilon > 0, \\ \Pi_i & \text{if } \varepsilon = 0, \\ \overline{\{\mu \in \mathrm{pba}(\Omega, \mathcal{M}) \colon \mathcal{O}_{TV}(\mu, \varepsilon) \cap \mathrm{pba}(\Omega, \mathcal{M})) \subseteq \Pi_i\}}^* & \text{if } \varepsilon < 0, \end{cases}$$

*A charge $P \in \mathrm{pba}(\Omega, \mathcal{M})$ is an ε-*common prior, *for $\varepsilon \in [-1, 1]$, if*

$$P \in \bigcap_{i \in N} \Pi_i^\varepsilon.$$

A 0-common prior will be termed simply a common prior.

Denote the subcollection of $\mathcal{T}$ of type spaces admitting an ε-common prior by $\mathcal{C}_\varepsilon$. The special case of $\mathcal{C}_0$ will be denoted simply as $\mathcal{C}$. In this notation, $\mathcal{C}_\varepsilon \subset \mathcal{C}_\delta$ if $\varepsilon < \delta$, i.e., these are nested collections.

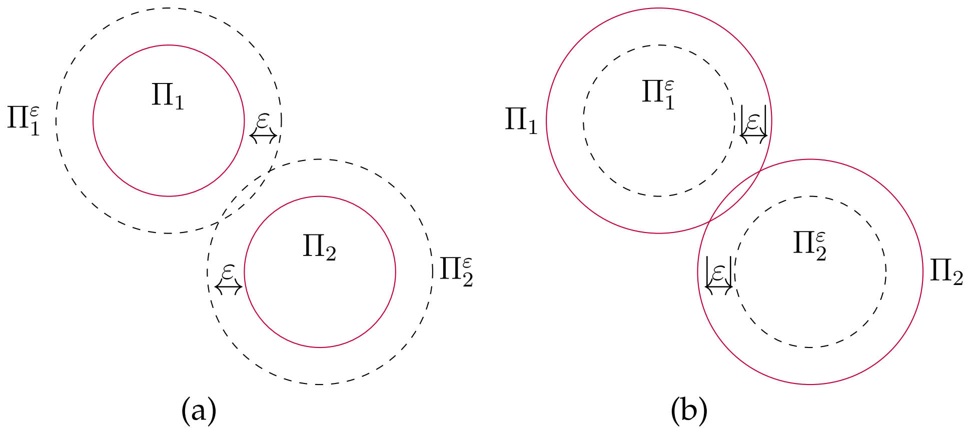

Figure 3: A schematic illustration of deriving Π_i^ε from Π_i when (a) $\varepsilon > 0$, in which case adding to Π_i probability charges that are 'ε distant' from the boundary of Π_i yields Π_i^ε and (b) $\varepsilon < 0$, in which case removing from Π_i probability charges that are '$|\varepsilon|$ distant' from the boundary of Π_i yields Π_i^ε.

Note that ε in Definition 6 may be positive or negative. When $\varepsilon < 0$, the set of ε-common priors is contained in the set of common priors; this is a 'strong' notion of common prior, robust against ε-variations of the types. When $\varepsilon > 0$, the set of ε-common priors is a super-set of the set of common priors, and hence may contain charges that are not common priors but are 'almost common priors' in being 'near' the set of common priors in a sense (this is an infinite version of a similar concept in Hellman (2013)).

An intuitive explanation of the construction of Π_i^ε, for $\varepsilon > 0$, is illustrated in Figure 3(a). Letting Π_1 denote the set of priors of player 1, imagine 'thickening' Π_1 into Π_1^ε by adding to Π_1 probability charges that are 'ε distant' (by the total variation norm) from the boundary of Π_1. Π_2 is 'thickened' to Π_2^ε with a similar construction. Even if Π_1 and Π_2 are disjoint, hence the type space is inconsistent, it is possible that $\Pi_1^\varepsilon \cap \Pi_2^\varepsilon \neq \emptyset$ for sufficiently large ε, in which case there is an ε-common prior.

An illustration for $\varepsilon < 0$ is in Figure 3(b). In this case, imagine 'thinning' Π_1 into Π_1^ε by removing from Π_1 probability charges that are '$|\varepsilon|$ distant' from the boundary of Π_1, and similarly obtaining Π_2^ε from Π_2. Here, even if Π_1 and Π_2 share a common prior, it is possible for Π_1^ε and Π_1^ε to be disjoint for sufficiently large $|\varepsilon|$.

Remark 7. In the literature (in the context of compact spaces and countable spaces, cf. Samet (1998); Heifetz (2006); Lehrer and Samet (2014)) the set of

σ-priors Π_i of a player i is obtained as the closed convex hull of the σ-types of i, and a common prior exists if and only if $\cap_{i \in N} \Pi_i \neq \emptyset$. More specifically, in Heifetz (2006) the set of priors is the weak* closure of the convex hull of the types, and in Lehrer and Samet (2014) it is the the total variation closure (strong closure) of the convex hull of the types.

Although it may not be immediately clear from Definition 6, when $\varepsilon = 0$ our definition of common prior recapitulates these approaches. This is because our sets of priors are weak* closed, as given by the result of Lemma 8 below. ♦

Lemma 8. *Let* $((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$ *be a type space and let* $i \in N$ *be a player. Then* Π_i^ε, $\varepsilon \in [-1, 1]$, *is a weak* compact, convex set; in particular,* Π_i, *the set of player* i*'s priors, is the weak* closure of the convex hull of* i*'s types, that is,*

$$\Pi_i = \overline{\mathrm{conv}(\{t_i(\omega, \cdot) \colon \omega \in \Omega\})}^{*}.$$

Lemma 9. $T \in \mathcal{C}_\varepsilon \iff T \in \mathcal{C}_\delta$, *for all* $\delta > \varepsilon$.

In particular, a type space T *attains a common prior if and only if it attains an* ε*-common prior for every* $\varepsilon > 0$.

Remark 10. Call a pairing (P_i, t_i), where t_i is type function and $P_i \in \Pi_i$ is a prior for player i, a prior-posterior pair. The definition of prior in Definition 4 can be termed as representing a 'posterior first' approach, in the sense that the type function t_i is given first and a prior is then defined by Equation (2) with respect to t_i.

In much of the literature, a 'prior first' approach is taken, with a prior probability considered the given and the posteriors of the type function subsequently defined from the prior by application of Bayes's rule. This accords with a picture of epistemic fields arising from signals inducing partitions of the state space: in the *ex ante* stage before a signal arrives a player has a prior distribution over the entire state space. In the interim stage after a signal has been received, the player 'learns' his or her true type (i.e., a partition element) and updates the probability of an event E accordingly.

Defining a prior with respect to a given type function reverses this picture. Definition 4 is also a more stringent definition of a prior, in the sense that fewer prior-posterior pairs are admissible in a posterior first approach than in a prior first approach. A more detailed discussion of this point appears in Section 8 below. ♦

3.3 Bets

Definition 11. *Let $T = ((\Omega, \mathcal{M}), \{\Omega, \mathcal{M}_i\}_{i \in N}, (t_i)_{i \in N})$ be a type space. A set of functions $\overline{f} = \{f_{i_1}, \dots, f_{i_n}\} \in B(\Omega, \mathcal{M})$, for a finite index set $i_1, \dots, i_n \in N$, is a* bet *if $\sum_{m=1}^{n} f_{i_m} = 0$.*

*A bet is an ε-*agreeable bet *(relative to T) for $\varepsilon \in [-1, 1]$ if there exists $\alpha \in \mathbb{R}$ such that*

$$\int f_{i_m} \, \mathrm{d}\, t_{i_m}(\omega, \cdot) \geq \alpha > \varepsilon \|f_{i_m}\|_{\sup}, \tag{3}$$

for every state $\omega \in \Omega$ and every player i_m with $m \in \{1, \dots, n\}$.

Denote the collection of type spaces admitting an ε-agreeable bet by $\mathcal{B}_\varepsilon$ (for 'bet'), and let $\mathcal{D}_\varepsilon$ (for 'do not bet') denote $\mathcal{T} \setminus \mathcal{B}_\varepsilon$.

Note that in Definition 11 the set of functions $f_{i_1}, \dots, f_{i_n}$ comprising a bet is finite, even though the player set N is not restricted to being finite.

If $\varepsilon > 0$, then $\overline{f}$ is an ε-agreeable bet if each player i not only believes that at each state f_i grants him positive expectation, he believes more strongly that in expectation he can gain more than an amount bounded away from $\varepsilon \|f_{i_m}\|_{\sup}$ (this is an infinite version of a similar concept in Hellman (2013)).

If $\varepsilon < 0$, then in particular every $T \in \mathcal{C}_\varepsilon$ admits a common prior, hence there can be no agreeable bet as it is commonly understood in the literature. But it still might be possible for the players to agree to bet under such circumstance: under an ε-agreeable bet each participant in the bet might have no choice but to accept a loss at some types (i.e., it may be possible that $\int f_{i_m} \, \mathrm{d}\, t_{i_m}(\omega, \cdot) < 0$) but that loss is bounded away from $\varepsilon \|f\|_{\sup}$ (a similar idea appears in Lehrer and Samet (2014)).

When $\varepsilon = 0$ we get back the ordinary notion of agreeable bet in the literature, i.e., $\overline{f} = \{f_{i_1}, \dots, f_{i_n}\} \in B(\Omega, \mathcal{M})$ is a 0-agreeable bet if each player i believes that at each state f_i grants him positive payoff expectation. Note, however, that in this case the payoff expectation of each player must be bounded away from zero, that is, $\alpha > 0$ in Equation (3. We show by example in Appendix B why this must hold.

4 Main result

We begin with preliminary results, towards the main theorem.

Theorem 12. *Let $K_1, \ldots, K_n$ be non-empty, weak* compact, convex sets in a locally convex topological vector space X such that $0 \notin K_m$, $m = 1, \ldots, n$. Then $\cap_{m=1}^{n} \operatorname{cone}(K_m) = \{0\}$, where $\operatorname{cone}(B) = \{\alpha x \colon \alpha \geq 0, \text{ and } x \in B\}$, if and only if there exist continuous linear functionals $f_1, \ldots, f_n$ over X, and $\alpha > 0$, such that $f_m(x) \geq \alpha$ for each $m = 1, \ldots, n$ and for all $x \in K_m$, and in addition $\sum f_m = 0$.*

Lemma 13. *Let $A \subseteq \operatorname{ba}(\Omega, \mathcal{M})$, $\varepsilon, \alpha \in \mathbb{R}$, and $f \in B(\Omega, \mathcal{M})$. Then*

$$f(x) \geq \alpha + \varepsilon ||f||_{\sup} \ \ \forall x \in A \iff f(x) \geq \alpha \ \ \forall x \in \overline{A + \mathcal{O}_{TV}(0, \varepsilon)}^{*},$$

and equivalently

$$f(x) \geq \alpha \ \ \forall x \in A \iff f(x) \geq \alpha - \varepsilon ||f||_{\sup} \ \ \forall x \in \overline{A + \mathcal{O}_{TV}(0, \varepsilon)}^{*}.$$

Lemma 14. *For any $\varepsilon > 0$*

$$\bigcap_{i \in N} \Pi_i^{\varepsilon} = \emptyset \iff \bigcap_{i \in N} \overline{\Pi_i + \mathcal{O}_{TV}(0, \varepsilon)}^{*} = \emptyset.$$

Our main result is as follows:

Theorem 15. *Let $T = ((\Omega, \mathcal{M}), \{\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$ be a type space and $\varepsilon \in [-1, 1]$. Then only one of the following two cases is possible:*

- *T admits an ε-common prior.*
- *There exists an ε-agreeable bet.*

That is, $\mathcal{C}_\varepsilon = \mathcal{D}_\varepsilon$.

Proof. By Lemma 8, for each $i \in N$ and $\varepsilon \in \mathbb{R}$ the set Π_i^ε is a weak* closed subset of $\mathrm{pba}(\Omega, \mathcal{M})$, which is itself weak* compact, hence $\cap_{i \in N} \Pi_i^\varepsilon = \emptyset$, i.e., there is no ε-common prior, if and only if there exists a finite set of indices $i_1, \ldots, i_n \in N$ such that $\cap_{m=1}^n \Pi_{i_m}^\varepsilon = \emptyset$.

If $\varepsilon = 0$ then let $K_m = \Pi_{i_m}$; if $1 > \varepsilon > 0$ then let $K_m = \overline{\Pi_{i_m} + \mathcal{O}_{TV}(0, \varepsilon)}^*$ (it is clear that there always exists 1-common prior but there cannot exist 1-agreeable bet); and if $\varepsilon < 0$ then let $K_m = \Pi_{i_m}^\varepsilon$, $m = 1, \ldots, n$, as in the statement of Theorem 12. By Lemma 8 the sets K_m, $m = 1, \ldots, n$, are convex, weak* compact subsets of $\mathrm{ba}(\Omega, \mathcal{M})$, $0 \notin K_m$, $m = 1, \ldots, n$, and $\cap_{m=1}^n \mathrm{cone}(K_m) = \{0\}$. This last property holds if and only if $\cap_{m=1}^n K_m = \emptyset$. We can therefore apply Theorem 12 to deduce $\cap_{m=1}^n K_m = \emptyset$ if and only if there exist continuous linear functionals $f_1, \ldots, f_n$ satisfying $\sum f_m = 0$ as in the statement of that theorem.

Appeals to Lemmata 13 and 14 then complete the argument leading to our desired result: if there is no ε-common prior, then there exists an ε-agreeable bet, and if there exists an ε-agreeable bet, then there is no ε-common prior. □

Corollary 16. *A type space admits a common prior if and only if it admits no agreeable bet.*

The characterisation in Corollary 16 fills a lacuna in the literature. The parallel characterisation for σ-additive common priors, which famously holds for finite state spaces, fails in certain cases (such as non-compact spaces). In contrast, as Corollary 16 shows that when one permits common priors to be probability charges, rather than restricted to probability measures then the full characterisation is restored and holds for all type spaces. A more detailed discussion of this point appears in Section 7 below.

Corollary 16 generalizes the Corollary on p. 174 in Samet (1998) to the infinite state space case. Notice that every tight probability charge is a tight probability measure (and vice versa), moreover, in Feinberg (2000) and in Heifetz (2006) the beliefs of the players are tight probability measures, hence Corollary 16 also generalises Theorem 5 on p. 150 in Feinberg (2000) and Proposition 1 on p. 109 in Heifetz (2006). A more detailed discussion of this point appears in Section 7 below. Furthermore, in the case $\varepsilon > 0$, Theorem 15 generalises Theorem 1 on p. 406 in Hellman (2013) to the infinite state space case.

As a corollary of Lemma 9 and Theorem 15 we obtain the following result:

Theorem 17. *Let $T = ((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$ be a type space. Then only one of the following two cases is possible:*

- *T admits an ε-common prior for every $\varepsilon > 0$.*
- *There exists an agreeable bet.*

We can also derive a corollary relating to σ-type spaces and σ-priors, showing that the fine-grained decomposition of the space of type spaces of Theorem 15 is equally applicable to the standard σ-additive setting of most of the literature, if compactness is assumed.

Theorem 18. *Let $T = ((\Omega, \mathcal{M}), \{\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$ be a σ-type space sastisfying the property that Ω is a compact Hausdorff topological space and that $t_i(\omega, \cdot)$ is a tight probability measure for all $\omega \in \Omega$ and $i \in N$. Let $\varepsilon \in [-1, 1]$. Then only one of the following two cases is possible:*

- *T admits an ε-common σ-prior.*
- *There exists an ε-agreeable bet that is continuous with respect to the topology on Ω.*

It might seem at first glance as if Theorem 18 follows almost trivially from Theorem 15, since a σ-type space is a special case of a type space under Definitions 1 and 2. But Theorem 18 is saying more than just a direct application of Theorem 15 to a special case, which would be a statement of a dichotomy in the case of σ-type spaces between ε-agreeable bets and ε-common priors; Theorem 18 instead refers specifically to ε-common σ-priors. For proving Theorem 18, one shows that Theorem 12 and Lemmata 13 and 14 apply as-is to type spaces satisfying the statement of Theorem 18, and then completes the proof of the theorem in a manner similar to the proof of Theorem 15, adapted to this setting. We omit the full details.

5 Finite Support Common Prior Measures

The results in Section 4 relate to priors over infinite state spaces that, like the beliefs by the type functions, may be of infinite support. Here we ask whether the existence of a common prior, and by extension no betting, can be ascertained solely by looking at probability measures with finite support, which in a sense are easier to work with. The answer is affirmative, if we consider finite support measures that are 'nearby' to the full priors. We call these 'approximate common σ-priors with finite support'.

Intuitively, a type space admits approximate common σ-priors with finite support if for any finite set of players, any finite collection of functions, and any $\varepsilon > 0$ there exists a probability measure P in $D(\Omega, \mathcal{M})$ (recall that this is the set of probability measures with finite support) such that the weak* ε-neighbourhood around P intersects the sets of priors of all the players.

Definition 19. *A type space $((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i\in N}, \{t_i\}_{i\in N})$ admits* approximate common σ-priors with finite support *if for any $\varepsilon > 0$, and any $F \subset B(\Omega, \mathcal{M})$ such that $|F| < \infty$, there exists $P \in D(\Omega, \mathcal{M})$ such that for all players $i \in N$*

$$\mathcal{O}^*(P, F, \varepsilon) \cap \Pi_i \neq \emptyset.$$

Lemma 20. *A type space $((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i\in N}, \{t_i\}_{i\in N})$ admits approximate common σ-priors with finite support if and only if it admits a common prior.*

Lemma 20 relates common priors – which are probability charges – to finitely supported probability measures. It essentially states that $\bigcap_i \Pi_i \neq \emptyset$ if and only if for each $\varepsilon > 0$ there is a finitely supported probability measure P that is weak* ε-close to each Π_i. Moreover, by Lemmata 9 and 20 we obtain the following corollary:

Corollary 21. *A type space $((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i\in N}, \{t_i\}_{i\in N})$ admits approximate common σ-priors with finite support if and only if it admits an ε-common prior for every $\varepsilon > 0$.*

As a direct corollary of Theorem 15 and Lemma 20 we obtain:

Theorem 22. *Let $T = ((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i\in N}, \{t_i\}_{i\in N})$ be a type space. Then only one of the following two cases is possible:*

- *T admits approximate common σ-priors with finite support.*

- *There exists an agreeable bet.*

Theorem 22 is a generalisation of Theorem 2 p. 170 in Lehrer and Samet (2014) (further details on this appear in Section 7.1).

6 Approximation by Finite Type Spaces

In Section 5 we considered infinite type spaces and approximate common σ-priors with finite support. In this section we continue with the project of considering only finite constructions by approximating the type spaces themselves by finite type spaces (comprised of finite state spaces and a finite number of players). More specifically, we wish to study whether it is possible to approximate, in an appropriately defined sense, infinite type spaces with common priors by 'entirely finite constructions', namely finite type spaces with approximate common priors.

Definition 23. *Let $T = ((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$ be a type space. Another type space $T' = ((\Omega', \mathcal{M}'), \{(\Omega', \mathcal{M}'_i)\}_{i \in N'}, \{t'_i\}_{i \in N'})$ is a finite (N', F, ε)-approximation of T if*

- $0 < |\Omega'| < \infty$,

- $N' \subseteq N$, $|N'| < \infty$,

- $F \subseteq B(\Omega, \mathcal{M})$, $|F| < \infty$,

- $\varepsilon > 0$,

- *there exists $\psi \colon \Omega' \to \Omega$ such that*
  - $\psi^{-1}(A) \in \mathcal{M}'$ *for each* $A \in \mathcal{M}$,
  - $\psi^{-1}(A) \in \mathcal{M}'_i$, *for each* $A \in \mathcal{M}_i$ *and each* $i \in N'$,

- *for all $\omega \in \Omega'$, $i \in N'$, and $f \in F$,*

$$\left| \int f \, \mathrm{d}t_i(\psi(\omega), \cdot) - \int f \circ \psi \, \mathrm{d}t'_i(\omega, \cdot) \right| < \varepsilon.$$

The definition above says that given the means of approximation (F and ε) a finite type space approximates a type space if the beliefs of the related (by ψ) type pairs are close to each other.

In the finite setting both of our approximation concepts, ε-common priors and approximate σ-common priors (Definitions 6 and 19) lead to the following notion:

Definition 24. *A finite type space* $((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$ *achieves an* ε-common prior, *for* $\varepsilon > 0$*, if there exist priors* $P_i \in \Pi_i$*,* $i \in N$*, such that for each pair of players* $i, j \in N$

$$\left| \int f \, \mathrm{d}P_i - \int f \, \mathrm{d}P_j \right| < \varepsilon, \; f \in B(\Omega, \mathcal{M}), \; -1 \leq f \leq 1.$$

Incorporating together concepts from Definitions 23 and 24 gives us our next definition:

Definition 25. *A type space* $T = ((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$ *is* finitely approximable *if there exists a finite* (N', F, ε)*-approximation of* T *admitting a* δ*-common prior for any* $N' \subseteq N$*,* $|N'| < \infty$*,* $\varepsilon, \delta > 0$ *and* $F \in B(\Omega, \mathcal{M})$ *such that* $|F| < \infty$*.*

Theorem 26 is our main approximation result in this section. It may be regarded as a form of 'continuity' statement, in the sense that if a type space has approximations with a δ-common prior, for any $\delta > 0$, then in the limit the existence of these approximations guarantees the existence of a common prior for the type space itself.

Theorem 26. *If a type space* T *is finitely approximable then it admits a common prior.*

Note that the converse of Theorem 26 does not hold. This may happen for two reasons: 1) Definition 23 may fail in the sense that for some ε there may not exist a desired finite (N', F, ε)-approximation of T or 2) Definition 24 may fail in the sense that T may have many perfectly good approximations but they do not admit ε-common priors for some ε. The next two examples exhibit, respectively, these possible ways of failure.

Example 27. Consider the following type space taken from Section 7 of Feinberg (2000):

- Anne and Bob are the two players,
- $\Omega = \{1, 2, \ldots\} = \mathbb{N}$,
- $\mathcal{M} = \mathcal{P}(\Omega)$,
- $\mathcal{M}_{\text{Anne}}$ is the field generated by the sets $\{1\}$, $\{n, n+1\}$, $n \in \mathbb{N}$, $n \geq 2$, and $\mathcal{M}_{\text{Bob}}$ is the field generated by the sets $\{n, n+1\}$, $n \in \mathbb{N}$, $n \geq 1$.

$$t_{\text{Anne}}(n, \{m\}) = \begin{cases} 1 & \text{if } n = m = 1, \\ 2/3 & \text{if } n = m = 2(1+k+k^2),\ k = 0, 1, 2, \ldots, \\ 2/3 & \text{if } n = 2(1+k+k^2) \text{ and } m = 2(1+k+k^2),\ k = 0, 1, 2, \ldots, \\ 1/3 & \text{if } n = m = 2(1+k+k^2)+1,\ k = 0, 1, 2, \ldots, \\ 1/3 & \text{if } n = 2(1+k+k^2) \text{ and } m = 2(1+k+k^2),\ k = 0, 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{and } t_{\text{Bob}}(n, \{m\}) = \begin{cases} 1/2 & \text{if } n = m = 2k+1,\ k = 0, 1, 2, \ldots, \\ 1/2 & \text{if } n = m = 2(k+1),\ k = 0, 1, 2, \ldots, \\ 1/2 & \text{if } n = 2k+1 \text{ and } m = 2(k+1),\ k = 0, 1, 2, \ldots, \\ 1/2 & \text{if } n = 2(k+1) \text{ and } m = 2k+1,\ k = 0, 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases}$$

1. As Feinberg (2000) demonstrated there does not exist an agreeable for this type space, hence by Theorem 15 it admits a common prior.

2. Let

$$f(\omega) = \begin{cases} 1 & \text{if } \omega \text{ is odd,} \\ 0 & \text{if } \omega \text{ is even,} \end{cases}$$

   and $\varepsilon \in (0, 1/3)$.

   Then there does not exist a finite $(N, \{f\}, \varepsilon)$-approximation of this type space. Take an arbitrary finite type space $((\Omega', \mathcal{M}'), \{(\Omega', \mathcal{M}'_i)\}_{i \in N}, \{t'_i\}_{i \in N})$ and $\psi \colon \Omega' \to \Omega$ that is $(\mathcal{M}', \mathcal{M})$, $(\mathcal{M}'_{\text{Anne}}, \mathcal{M}_{\text{Anne}})$, and $(\mathcal{M}'_{\text{Bob}}, \mathcal{M}_{\text{Bob}})$ measurable. Since Ω' is finite, there exists $n^* \in \mathbb{N}$ such that $\psi^{-1}(\{n^*\}) = \emptyset$ and either $\psi^{-1}(\{n^*-1\}) \neq \emptyset$ or $\psi^{-1}(\{n^*+1\}) \neq \emptyset$. Without loss of generality, suppose that $\psi^{-1}(\{n^*+1\}) \neq \emptyset$. Let $\omega' \in \Omega'$ be such that $\psi(\omega') = n^*+1$.

If $\{n^*, n^*+1\} \in \mathcal{M}_{\text{Anne}}$, then

$$\left| \int f \ \mathrm{d}\, t_{\text{Anne}}(\psi(\omega'), \cdot) - \int f \circ \psi \ \mathrm{d}\, t'_{\text{Anne}}(\omega', \cdot) \right| \geq \frac{1}{3}.$$

If $\{n^*, n^*+1\} \in \mathcal{M}_{\text{Bob}}$, then

$$\left| \int f \ \mathrm{d}\, t_{\text{Bob}}(\psi(\omega'), \cdot) - \int f \circ \psi \ \mathrm{d}\, t'_{\text{Bob}}(\omega', \cdot) \right| = \frac{1}{2}.$$

In both cases, the condition for a sufficiently close approximation fails. ♦

The next example (inspired by Hellman and Levy (2019)) shows that even if there are sufficiently many finite approximations to a type space with a common prior, those finite approximations themselves may fail to admit an ε-common prior for sufficiently small ε.

Example 28. Let $\Omega = \{\omega_1, \omega_2\}$, $N = \{1, 2\}$, $t_1 = \delta_{\{\omega_1\}}$, $t_2 = \delta_{\{\omega_2\}}$, $\mathcal{M} = \mathcal{P}(\Omega)$, and $\mathcal{M}_i = \{\emptyset, \Omega\}$ for each player $i \in N$. Then only the pair of functions

$$f_1(\omega) = \begin{cases} \alpha & \text{if } x = \omega_1, \\ -\beta & \text{if } x = \omega_2, \end{cases} \tag{4}$$

and $f_2 = -f_1$ form an agreeable bet for the type space $((\Omega, \mathcal{M}), \{\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$, for all $\alpha, \beta > 0$.

Let $\Omega_i = \Omega \times \{i\}$ for each $i \in \mathbb{R}$, yielding continuum-many copies of Ω. Let $\Omega^* = \cup_{i \in I} \Omega_i$, let $\mathcal{M}^*$ be the field generated by the sets $\{\omega\}$, for $\omega \in \Omega^*$ (i.e., the coarsest field containing the singletons of Ω^*), and let

$$t^*_j(\omega, \{x\}) = \begin{cases} t_j((\omega|_\Omega), \{x|_\Omega\}) & \text{if } \exists i \in I \text{ such that } \omega, x \in \Omega_i, \\ 0 & \text{otherwise}, \end{cases}$$

$j = 1, 2$.

Define $\mathcal{M}^*_j$ to be the field generated by Ω_i, $i \in I$, $j \in N = \{1, 2\}$. Then $T^* := ((\Omega^*, \mathcal{M}^*), \{(\Omega^*, \mathcal{M}^*_j)\}_{j \in N}, \{t^*_j\}_{j \in N})$ forms a type space.

Suppose by contradiction that there exists an agreeable bet (f_1, f_2) on T^*. Then for each $i \in I$, the functions f_1 and f_2 restricted to Ω_i must be defined on Ω_i as in Equation (4) (the specific values of α and β might depend on i).

If f_1 is not bounded, then (f_1, f_2) cannot be an agreeable bet. Hence there must exist $\varepsilon > 0$ such that $f_1^{-1}([\varepsilon, \infty))$ and $f_1^{-1}((-\infty, -\varepsilon))$ are both unions of infinitely many singletons. But then either $f_1^{-1}([\varepsilon, \infty)) \notin \mathcal{M}^*$ or $f_1^{-1}((-\infty, -\varepsilon)) \notin \mathcal{M}^*$, meaning that f_1 is not $\mathcal{M}^*$-integrable, hence (f_1, f_2) cannot be an agreeable bet.

As there is no agreeable bet on T^*, by Theorem 15 T^* admits a common prior. Indeed, it is easy to check that the probability charge on $\mathcal{M}^*$ which assigns 0 to each finite set is a common prior.

Moreover, for any $f \in B(\Omega^*, \mathcal{M}^*)$, for any finite copy of $(\Omega, \mathcal{M})$, for each $\omega' \in \Omega'$, $n \in N$

$$\left| \int f \, \mathrm{d}t_n(\psi(\omega'), \cdot) - \int f \circ \psi \; \mathrm{d}\, t'_n(\omega') \right| = 0,$$

where $\Omega' = \cup_{i \in J} \Omega_i$, $J \subseteq I$ is an arbitrary nonempty, finite set, $\mathcal{M}' = \mathcal{M}|_{\Omega'}$ and $t'_n = t_n \circ \psi$, $i \in N$, $\psi \colon \Omega' \to \Omega$ is such that $\psi(\omega) = \omega$, $\omega \in \Omega'$.

In words, T^* has many finite, even 'perfect', approximations. Despite this, none of these finite approximations admits a common prior. ♦

7 Comparisons

Most of the literature on common priors deals with σ-type spaces. As noted above, in the σ-additive case the result of our main theorem, Theorem 15, may not hold when the state space is not compact: in that case, a common prior implies no betting but not necessarily the converse. In this section we briefly compare our results with known results in the σ-additive case.

7.1 σ-type spaces: the non-compact case

In σ-type spaces (Definition 3) the general epistemic field $\mathcal{M}$ is a σ-field, therefore, the bets, which are $\mathcal{M}$-integrable functions, are $\mathcal{M}$-measurable functions. On this fact the results of this subsection are based on.

Definition 29. *Let $T = ((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$ be a σ-type space, and let $\varepsilon > 0$. A prior $p \in \Delta(\Omega, \mathcal{M})$ is a strong common ε-prior of T if for each $i \in N$ there exists $p_i \in \Pi_i \cap \Delta(\Omega, \mathcal{M})$ such that*

$$||p_i - p||_{TV} < \varepsilon.$$

What we term here a strong common ε-prior is called common ε-prior in Lehrer and Samet (2014).

Lemma 30. *Let $(X, \mathcal{A})$ be a measurable space and convex sets $P_1, P_2 \subseteq \mathrm{ca}(X, \mathcal{A})$. Then $\overline{P_1}^{TV}$ and $\overline{P_2}^{TV}$ are strongly separable by a linear functional if and only if $\overline{P_1}^{*}$ and $\overline{P_2}^{*}$ are strongly separable by a linear functional, where the closures are meant in $\mathrm{ba}(X, \mathcal{A})$.*

By the following theorem we can relate Theorem 2 on p. 170 of Lehrer and Samet (2014) to our results.

Theorem 31. *A σ-type space $((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$ admits a common prior if and only if it admits a strong common ε-prior for all $\varepsilon > 0$.*

Proof. If: If the σ-type space admits a strong common ε-prior p^ε for all $\varepsilon > 0$, then by that $\mathrm{pba}(\Omega, \mathcal{M})$ is weak* compact $(p^{1/n})_{n \in \mathbb{N}}$ has a cluster point. Any cluster point of $(p^{1/n})_{n \in \mathbb{N}}$ is a common prior for the σ-type space.

Only if: Suppose that there exists $\varepsilon > 0$ such that the σ-type space does not admit a strong common ε-prior. Then by Lemma 30 it holds that the σ-type space does not admit a common prior either. □

In the light of Theorem 31 the following corollary of Theorem 15 is a generalisation of Theorem 2 on p. 170 of Lehrer and Samet (2014).

Proposition 32. *Let $T = ((\Omega, \mathcal{M}), \{(\Omega, \mathcal{M}_i)\}_{i \in N}, \{t_i\}_{i \in N})$ be a σ-type space. Then only one of the following two cases is possible:*

- *T admits a strong common ε-prior for all $\varepsilon > 0$.*
- *There exists an agreeable bet.*

Theorem 22 is a generalisation of the above-mentioned result by Lehrer and Samet (2014) in the way that Lehrer and Samet (2014) restrict attention to countable state spaces with finitely many players, while here both the state space and the player set may be arbitrarily large. By defining the notion of strong common ε-prior for negative ε (similarly to that is done in Definition 6) a further generalisation of he above-mentioned result by Lehrer and Samet (2014) is possible: *Take a σ-type space, then for each $\varepsilon \in [-1, 1]$ only one of the following two cases is possible: either the σ-type space admits a strong common δ-prior for all $\delta > \varepsilon$, or there exists an ε-agreeable bet.*

7.2 σ-type spaces: the compact case

As noted above in Section 4, when the sets of priors are charges, consistent type spaces admit no agreeable bets; in the notation of that section, $\mathcal{C} = \mathcal{D}$. When priors are restricted to σ-priors, this may not hold in general: $\mathfrak{C} \subset \mathfrak{D}$ is the most that can be said. In contrast, as Feinberg (2000) and Heifetz (2006) show, when Ω is a compact state space $\mathfrak{C} = \mathfrak{D}$ holds when the only priors under consideration are σ-priors. It is of some interest to consider how this result of Feinberg (2000) and Heifetz (2006), which involves probability measures, relates to charges.

With respect to a topological space, an additive set function μ is defined to be *tight* if $\mu(A) = \sup\{\mu(K) \colon K \subseteq A,\ K \text{ is compact}\}$ for each event A. Every tight probability charge is a probability measure (see e.g. Theorem 10.13 on p. 378 in Aliprantis and Border (2006)). The probability measures used in both Feinberg (2000) (Theorem 5 on page 15) and Heifetz (2006) (Proposition 3 on page 112) are regular Borel probability measures, which are tight by definition.

Hence we have the following interpretation. Let T be a type space over a compact state space Ω that admits an agreeable bet. By Theorem 15 here, one deduces that $\cap_{i \in N} \Pi_i \neq \emptyset$, in other words there exists at least one probability charge that is a prior for all players i. The results of Feinberg and Heifetz specify further that in the compact case one of the charges in $\cap_{i \in N} \Pi_i$ is a tight probability charge, i.e., a probability measure.

Many foundational papers in the game theory literature (see e.g. Mertens and Zamir (1985); Brandenburger and Dekel (1993); Heifetz (1993); Mertens et al (1994) among others) restrict attention to beliefs that are tight probability measures. There may be scope to study the results of these and other similar studies from the perspective of charges.

8 Priors First or Posteriors First?

Definition 4 is reminiscent of a standard property of Bayesian updating, variously termed Bayesian consistency or the martingale property, often stated as 'the average posterior belief is equal to the prior'. More formally, for any event E, $P_i(E) = \int_\Omega t_i(\cdot, E) \, \mathrm{d} P_i$, that is, from knowing the posterior belief that player i has about event E at each state it is possible to reconstruct the prior measure of E.

In this paper we use the property specified in Definition 4 to define prior charges and measures; that is, the primitives in our model are the type functions and epistemic fields of each player, following which we can define which probability charges are priors. As the type functions are here playing the role of posteriors, we first identify the posteriors, and from there we define the priors ('posteriors first, then define the priors').

This approach to defining priors has ample precedent. However, there is a parallel (and older) 'prior first' approach that is more frequently seen in the literature and runs in the other direction: the primitives in this approach are prior probability measures, defined over all of Ω, and a partition Π^i of Ω associated with each player i. The epistemic field of each player is then defined to be the field generated by the partition elements, and $t_i(\omega, E)$ is defined to be the probability of event E at state ω given by using Bayesian updating to calculate the posterior probability of E conditional on $\Pi^i(\omega)$ and the prior.

When state spaces are finite, these two approaches are equivalent: the martingale property can easily be shown to obtain under the 'prior first' approach by the law of iterated expectations. In infinite state spaces this may no longer be true, which is why in this paper the types are the primitives and we define prior charges to be charges satisfying the martingale property with respect to the posteriors. Indeed, Equation 2 essentially states that a prior P_i is defined to be a charge such that the given type function t_i forms a conditional probability function of P_i with respect to the field $\mathcal{M}_i$.

We illustrate this first with an example[2] that indicates why charges in countable state spaces may fail to comply with the martingale property of Bayesian updating.

Example 33. Consider the following situation:

- The player set is $N = \{\text{Anne}, \text{Bob}\}$.
- The state space is $\Omega = (\{A\} \times \mathbb{N}) \cup (\{B\} \times \mathbb{N})$. In other words, every element $\omega \in \Omega$ is either of the form (A, n) or (B, n) for some $n \in \mathbb{N}$.
- The field of events is $\mathcal{M} = \mathcal{P}(\{A\} \times \mathbb{N}) \cup \{E \subseteq (\{B\} \times \mathbb{N})\colon |E| < \infty \text{ or } |E^{\complement}| < \infty\}$, where $\mathcal{P}(\{A\} \times \mathbb{N})$ is the power set of $\{A\} \times \mathbb{N}$, and $E^{\complement}$ denotes the complement of E.
- With respect to $(\Omega, \mathcal{M})$ define a probability charge P as follows: for each $E \in \mathcal{M}$,

  $$(5) \qquad P(E) = \begin{cases} \sum_{n \in E \cap (\{A\} \times \mathbb{N})} \frac{1}{2^{n+2}} & \text{if } |E \cap (\{B\} \times \mathbb{N})| < \infty, \\ \sum_{n \in E \cap (\{A\} \times \mathbb{N})} \frac{1}{2^{n+2}} + \frac{1}{2} & \text{if } |E \cap (\{B\} \times \mathbb{N})| = \infty. \end{cases}$$

  To gain some intuition, note that $P(\{A\} \times \mathbb{N}) = \frac{1}{2}$. This is because any event $E \in \mathcal{M}$ such that $E \subseteq (\{A\} \times \mathbb{N})$ satisfies $|E \cap (\{B\} \times \mathbb{N})| = 0$, hence $P(E)$ is the weight E receives under the upper line in Equation (5) alone. In fact, P restricted to $\{A\} \times \mathbb{N}$ is isomorphic to an honest-to-goodness σ-additive measure over $\mathbb{N}$, and $P(\{A\} \times \mathbb{N}) = \sum_{n \in \mathbb{N}} \frac{1}{2^{n+2}} = \frac{1}{2}$.

  In contrast, P restricted to $\{B\} \times \mathbb{N}$ is an additive but not σ-additive measure. For any $n \in \mathbb{N}$, the singleton event $E = \{(B, n)\}$ has $P(E) = 0$, since $|E \cap (\{B\} \times \mathbb{N})| = 1 < \infty$, but $|E \cap (\{A\} \times \mathbb{N})| = 0$. With similar reasoning, for any event E satisfying $E \subseteq (\{B\} \times \mathbb{N})$ and $|E| < \infty$, one has $P(E) = 0$.

  The over-all measure of $\{B\} \times \mathbb{N}$, however, satisfies $P(\{B\} \times \mathbb{N}) = \frac{1}{2}$. This is because, writing $E = \{B\} \times \mathbb{N}$, one has $|E \cap (\{B\} \times \mathbb{N})| = \infty$ and $|E \cap (\{A\} \times \mathbb{N})| = \emptyset$, yielding $P(E) = 0 + \frac{1}{2} = \frac{1}{2}$.

[2] Example 33 is inspired by elucidations of the notion of non-conglomerability of probability charges in Kadane et al (1996) and Dubey (1975).

- Anne's knowledge structure is the countably infinite partition: $\{(\{A\}\times\{n\}) \cup (\{B\} \times \{n\}) \colon n \in \mathbb{N}\}$. In words, every element of the partition of Anne is of the form $\{(A, n), (B, n)\}$ for some $n \in \mathbb{N}$. Bob's knowledge structure is the trivial partition consisting solely of one element: all of Ω.

Adopting a 'prior first' approach, let the charge P serve as a common prior, from which the posteriors are derived at each state.

Since for each n, the singleton (B, n) is associated with $P((B, n)) = 0$ and the singleton (A, n) with $P((A, n)) > 0$, it follows that in the posterior Anne's belief is $P(\{B\}\times\mathbb{N} \mid (A, n) \cup (B, n)) = 0$ and $P(\{A\}\times\mathbb{N} \mid \{(A, n)\} \cup \{(B, n)\}) = 1$. Bob's posterior is equal to the prior, since his partition is trivial.

Define a bet between the two as follows, where $f_A(\omega)$ indicates the payoff to Anne at state ω:

$$f_A(\omega) = \begin{cases} 1 & \text{if } \omega \in (\{A\} \times \mathbb{N}), \\ -2 & \text{if } \omega \in (\{B\} \times \mathbb{N}). \end{cases}$$

Bob's payoff is $f_B = -f_A$.

The above bet is an agreeable bet, since $E_{t_A(\omega)} f_A = 1$, and $E_{t_B(\omega)} f_B = 1$, for all $\omega \in \Omega$. This holds despite P being a 'common prior'. However, this does not contradict the theorems of the previous sections: given the construction, the martingale property does not hold here, since

$$P(\{B\} \times \mathbb{N}) = \frac{1}{2} \neq 0 = \int_\Omega 0 \, \mathrm{d}P = \int_\Omega t_A(\cdot)(\{B\} \times \mathbb{N}) \, \mathrm{d}P.$$

Hence P is not a prior by Definition 4, and is certainly not a common prior. ♦

Lest the reader get the impression that the failure of the martingale property in Example 33 is due solely to the use of additive but not σ-additive probability measures, the next example shows how a similar anomaly can occur even in the σ-additive setting.

Example 34. Consider the following situation:

- The player set is $N = \{\text{Anne}, \text{Bob}\}$.
- The state space is $\Omega = (\{A\} \times [0,1]) \cup (\{B\} \times [0,1])$. In other words, every element $\omega \in \Omega$ is either of the form (A, x) or (B, x) for some $x \in [0,1]$.
- Denote by $L([0,1])$ the standard Lebesgue σ-field over $[0,1]$ and by $C([0,1])$ the co-countable σ-field over $[0,1]$. Then let the σ-field of events in this example be $\mathcal{M} = \sigma(\{A\} \times L[0,1]) \cup (\{B\} \times C[0,1])$.
- For each set $S \in L[0,1]$, let the value $\ell(S)$ be the standard Lebesgue measure of S. By tolerable abuse of notation, we will also write $\ell(\{A\} \times S) = \ell(S)$, for $S \in L[0,1]$.
- With respect to $(\Omega, \mathcal{M})$ define a σ-additive probability measure P as follows:

  For each $E \in \mathcal{M}$,

  (6)
  $$P(E) = \begin{cases} \frac{1}{2}\ell(E \cap (\{A\} \times [0,1])) & \text{if } |E \cap (\{B\} \times [0,1])| \leq \aleph_0, \\ \frac{1}{2}\ell(E \cap (\{A\} \times [0,1])) + \frac{1}{2} & \text{if } |E^{\complement} \cap (\{B\} \times [0,1])| \leq \aleph_0. \end{cases}$$

  Note that for $E \subseteq (\{A\} \times [0,1])$, the value of $P(E)$ is essentially (one half of) the standard Lebesgue measure of an event of $L[0,1]$, while for $E \subseteq (\{B\} \times [0,1])$, the value of $P(E)$ is essentially (one half of) the standard co-countable measure of an of $C[0,1]$. In particular, $P(\{A\} \times [0,1]) = \frac{1}{2}$ and $P(\{B\} \times [0,1]) = \frac{1}{2}$.
- Anne's knowledge structure is $\{(\{A\} \times \{x\}) \cup (\{B\} \times \{x\}) \colon x \in \mathbb{R}\}$. In words, every element of the partition of Anne is of the form $\{(A, x), (B, x)\}$ for some $x \in [0,1]$. Bob's knowledge structure is the trivial partition consisting solely of one element: all of Ω.

Adopting a 'prior first' approach, let the charge P serve as a common prior, from which the posteriors are derived at each state.

To calculate Anne's posterior $P((\{B\} \times [0,1]) \mid (\{(A,x)\}, \{(B,x)\})$ for any particular $x \in [0,1]$, we first calculate $P((B \times [0,1]) \mid ((\{A\} \times (x -$

$\frac{1}{n}, x + \frac{1}{n})) \cup (B, x))$ for $n \in \mathbb{N}$, and then calculate this value as n goes to infinity. However, this value is zero for each n, hence the limit is also zero. It follows that in the posterior Anne's belief is $P((\{B\}) \times [0,1]) \mid (\{(A,x)\}, \{(B,x)\}) = 0$ and $P((\{A\} \times [0,1]) \mid (\{(A,x)\}, \{(B,x)\}) = 1$ Bob's posterior is equal to the prior, since his partition is trivial.

Define a bet between the two as follows, where $f_A(\omega)$ indicates the payoff to Anne at state ω:

$$f_A(\omega) = \begin{cases} 1 & \text{if } \omega \in (\{A\} \times [0,1]), \\ -2 & \text{if } \omega \in (\{B\} \times [0,1]), \end{cases} .$$

Bob's payoff is $f_B = -f_A$.

The above bet is an agreeable bet, since $E_{t_A(\omega)} f_A = 1$, and $E_{t_B(\omega)} f_B = 1$, for all $\omega \in \Omega$. This holds despite P being a 'common prior'. The martingale property does not hold here, since

$$P(\{B\} \times [0,1]) = \frac{1}{2} \neq 0 = \int_\Omega 0 \, \mathrm{d}P = \int_\Omega t_A(\cdot)(\{B\} \times [0,1]) \, \mathrm{d}P.$$

♦

Example 34 follows a similar pattern to Example 33, suitably reformulated to the continuum. We provide one more example, showing a failure of the martingale property even when the standard uniform Lebesgue measure is used throughout.

Example 35. Consider the following situation:

- The player set is $N = \{\text{Anne}, \text{Bob}\}$.
- The state space is $\Omega = [0,1)$.
- The σ-field of events $\mathcal{M}$ in this example is the standard Lebesgue σ-algebra over $[0,1)$.
- The σ-additive probability measure P is, similarly, the standard uniform Lebesgue measure.
- According to the Lebesgue measure $P(\{\omega\}) = 0$ for any $\omega \in \Omega$, and indeed $P(\{\omega_1, \ldots, \omega_n\}) = 0$ for any finite subset of Ω. This means that

we may define the conditional probability $P(\cdot \mid E)$ arbitrarily for any finite event E. Based on the principle of extending the uniformity of the over-all measure to the posteriors in the measure zero sets, define $P(\{\omega_i\} \mid \{\omega_1, \ldots, \omega_n\}) = 1/n$ if $\omega_i \in \{\omega_1, \ldots, \omega_n\}$, and zero otherwise.

- Let $H := \{x\colon 0 \leq x < 0.9\}$, hence $H^{\complement} = \{x\colon 0.9 \leq x < 1\}$. Let $g : H \to H^{\complement}$ be defined by $g(x) = 0.9 + \frac{x}{9}$.
- Anne's knowledge structure is the infinite partition: $\{\{\omega, g(\omega)\}\colon \omega \in H\}$. Bob's knowledge structure is the trivial partition consisting of all of Ω.

Define a bet between the two as follows, where $f_A(\omega)$ indicates the payoff to Anne at state ω:

$$f_A(\omega) = \begin{cases} -1 & \text{if } \omega \in H, \\ 2 & \text{if } \omega \in H^{\complement}. \end{cases}$$

Bob's payoff is $f_B = -f_A$.

At each state $\omega \in H$, Anne's posterior belief is as follows:

$$P(E \mid \omega) = \begin{cases} 1 & \text{if } \{\omega, g(\omega)\} \subseteq E, \\ \frac{1}{2} & \text{if } \omega \in E, \text{ and } g(\omega) \notin E, \\ \frac{1}{2} & \text{if } g(\omega) \in E, \text{ and } \omega \notin E, \\ 0 & \text{otherwise.} \end{cases}$$

At each state $\omega \in H^{\complement}$, Anne's posterior belief is as follows:

$$P(E \mid \omega) = \begin{cases} 1 & \text{if } \{\omega, g(\omega)\} \subseteq E, \\ \frac{1}{2} & \text{if } \omega \in E, \text{ and } g^{-1}(\omega) \notin E, \\ \frac{1}{2} & \text{if } g^{-1}(\omega) \in E, \text{ and } \omega \notin E, \\ 0 & \text{otherwise.} \end{cases}$$

At each state $\omega \in \Omega$, Bob's posterior belief is, of course, $t_B = P$.

The bet f is an agreeable bet, as at each state $\omega \in \Omega$, one calculates $E_{t_A(\omega)} f_A = 0.5$, and $E_{t_B(\omega)} f_B = 0.7$.

However, note that at each $\omega \in \Omega$, it is the case that $P(H \mid \omega) = 0.5$, hence

$$P(H) = 0.9 \neq 0.5 = \int_\Omega 0.5 \, \mathrm{d}P = \int_\Omega t_A(\cdot)(H) \, \mathrm{d}P.$$

♦

All the above examples share the essential element that the anomalies stem from partition elements of measure zero (or with significant intersection with zero measure sets). Probability charges over the integers are sometimes regarded as 'unintuitive' because, as in Example 33, they can assign probability zero to any point (and any finite set of points) yet measure one to the entire space, and indeed this property is key to the anomaly in that example.

However, as Examples 34 and 35 indicate, adopting σ-additive measures only pushes the anomalies to another level when considering atomless measures over the continuum. Indeed, in that setting it is also true that single points have zero measure, yet the measure of the entire space is non-zero. When models involve structures that exist within sets of measure zero, σ-additive measure theory can also fail to contend with critical aspects of the models. Several counter-examples in the recent literature, involving subjects such as common priors, betting, and Bayesian equilibria existence, depend in their constructions on considerations of measure zero subsets of spaces with continuum many elements.

In Bayesian updating models, measure zero sets are often problematic; naïve application of Bayes' rule can in certain situations break down to meaninglessness when measure zero sets are used. Indeed, as a close reading of Examples 34 and 35 shows, there is no canonical way to define posteriors at states within measure zero sets. Using Definition 4 for defining the relationship between priors and posteriors, instead of defining posteriors by Bayesian updating from priors, avoids many anomalies, as the examples in this section illustrate.

9 Conclusion

We have shown that expanding the concept of priors to include probability charges as priors is an endeavour yielding strong results: the No Betting Theorem is restored to its full power as a characterisation of common priors in any state space Ω of any cardinality, and a fine-grained decomposition of the space of type spaces is uncovered, revealing a continuum of classes in each of which an equivalence between epistemic and behavioural properties obtains. Further insight into the structure of the space of type spaces, via approximation theorems, appears in the sections above.

The role of priors in the Bayesian context, beyond the specific topic of the characterisation of common priors by way of agreeable bets, is of course extremely broad in many sub-fields of economic theory and far beyond. Few research studies on the potential of the use of probability charges as priors in a variety of settings have been conducted. The results here indicate there may be scope to consider further such studies.

A Proofs

Proof of Lemma 8. It comes directly from Definition 6 that Π_i^ε is convex for $\varepsilon \in [-1, 1]$.

Next we show that $\Pi_i = \overline{\mathrm{conv}(\{t_i(\omega, \cdot) : \omega \in \Omega\})}^*$. It is easy to see that $\overline{\mathrm{conv}(\{t_i(\omega, \cdot) : \omega \in \Omega\})}^* \subseteq \Pi_i$. Here we show that $\Pi_i \subseteq \overline{\mathrm{conv}(\{t_i(\omega, \cdot) : \omega \in \Omega\})}^*$.

Suppose by contradiction that there exists $\mu \in \Pi_i$ such that $\mu \notin \overline{\mathrm{conv}(\{t_i(\omega, \cdot) : \omega \in \Omega\})}^*$. Then μ is separated from $\overline{\mathrm{conv}(\{t_i(\omega, \cdot) : \omega \in \Omega\})}^*$ in the weak* topology, meaning that there exists $f \in B(\Omega, \mathcal{M})$, $f \neq 0$, and $\varepsilon > 0$ such that

$$\langle f, \mu \rangle := \int f \,\mathrm{d}\,\mu \geq \alpha + \varepsilon \text{ and } \langle f, \nu \rangle := \int f \,\mathrm{d}\,\nu \leq \alpha - \varepsilon \tag{7}$$

for all $\nu \in \overline{\mathrm{conv}(\{t_i(\omega, \cdot) : \omega \in \Omega\})}^*$. Let K be a bound of f, that is, $|f| \leq K$.

There exist a partition $B_1, \ldots, B_n \in \mathcal{M}$ of Ω and a simple function s_f on it such that for all $\omega \in \Omega$

$$|f(\omega) - s_f(\omega)| < \frac{\varepsilon}{2}\,. \tag{8}$$

Since $\mu \in \Pi_i$, for each B_m, $m = 1, \ldots, n$

$$\mu(B_m) = \mu(B_m \cap \Omega) = \int_\Omega t_i(\cdot, B_m)\,\mathrm{d}\mu\,. \tag{9}$$

Since $t_i(\cdot, B_m)$ is bounded and $\mathcal{M}_i$-integrable for each B_m, there exist a partition $C_1^m, \ldots, C_{n_m}^m \in \mathcal{M}_i$ of Ω and a simple function $s_{t_i(\cdot,B_m)}$such that for each $\omega \in \Omega$

$$\left|t_i(\omega, B_m) - s_{t_i(\cdot,B_m)}(\omega)\right| < \frac{\varepsilon}{2Kn}\,. \tag{10}$$

Let $C_1, \ldots, C_l$ be the common refinement of the partitions $B_1, \ldots, B_n$, $C_1^m, \ldots, C_{n_m}^m$, $m = 1, \ldots, n$. It is easy to see that $C_1, \ldots, C_l \in \mathcal{M}$ and that $\{C_1, \ldots, C_l\}$ forms a partition of Ω. Let $\omega_k \in C_k$, $k = 1, \ldots, l$ be arbitrarily fixed. Then by Equations (9) and (10) for all $m = 1, \ldots, n$

$$\left|\mu(B_m) - \sum_{k=1}^{l} t_i(\omega_k, B_m)\mu(C_k)\right| < \frac{\varepsilon}{2Kn}\,,$$

that is,

$$\left|\sum_{m=1}^{n} s_f(B_m)\mu(B_m) - \sum_{m=1}^{n} s_f(B_m)\sum_{k=1}^{l} t_i(\omega_k, B_m)\mu(C_k)\right| < \frac{\varepsilon}{2}\,.$$

Note that since $\mu(C_k) \geq 0$, $k = 1, \ldots, l$ and $\sum_{k=1}^{l}\mu(C_k) = 1$ the term $\sum\limits_{k=1}^{l} t_i(\omega_k, B_m)\mu(C_k)$ is a convex combination of $t_i(\omega_k, B_m)$, $k = 1, \ldots, l$, $m = 1, \ldots, n$. Recall that for each ω, $t_i(\omega, \cdot) \in \mathrm{pba}(\Omega, \mathcal{M})$ satisfies the conditions for being a prior of player i. Hence, writing $\nu := \sum\limits_{k=1}^{l} \mu(C_k)t_i(\omega_k, \cdot)$, we have $\nu \in \Pi_i$. Finally, by Equation (8)

$$\left|\int f \,\mathrm{d}\,\mu - \int f \,\mathrm{d}\,\nu\right| < \varepsilon\,,$$

which contradicts Equation (7). □

Proof of Theorem 12. If: Suppose by contradiction that $\cap_{m=1}^{n}\mathrm{cone}(K_m) \neq \{0\}$ and that at the same time there exist linear functionals $f_1, \ldots, f_n$ such that $f_m(x) \geq \alpha > 0$, for each $m = 1\ldots, n$ and for each $x \in K_m$, with $\sum_{m=1}^{n} f_m = 0$. Since $\cap_{m=1}^{n}\mathrm{cone}(K_m) \neq \{0\}$, there is an $x \neq 0$ such that $x \in \cap_{m=1}^{n}\mathrm{cone}(K_m)$; such an x then satisfies the property that there exist $\beta_1, \ldots, \beta_n > 0$ such that $\beta_m x \in K_m$ for $m = 1, \ldots, n$, and $\sum f_m(x) \geq \alpha\sum_{m=1}^{n}\frac{1}{\beta_m}$. This contradicts $\sum f_m(x) = 0$.

Only if: Let $K = K_2 \times \ldots \times K_n$ and $\hat{K} = \{x \in \mathrm{ba}(X, \mathcal{A})^{n-1} \colon x_1 = \ldots = x_{n-1},\ x_1 \in K_1)$ (so that $\hat{K}$ is an $n-1$ copy of K_1). Let $\tilde{K} = \mathrm{cone}(K_2) \times \ldots, \times \mathrm{cone}(K_n)$. It is clear that $\tilde{K}$ is weakly* closed and convex and that $\hat{K}$ is weakly* compact and convex.

Suppose that $\cap_{m=1}^n \mathrm{cone}(K_m) = \{0\}$. Then it follows from the definitions that $\tilde{K} \cap \hat{K} = \emptyset$, which implies that there exists a linear functional g, a real number β, and $\varepsilon > 0$ such that $g(x) \geq \beta + \varepsilon$ for all $x \in \tilde{K}$ and $g(x) \leq \beta - \varepsilon$ for all $x \in \hat{K}$. Since $\tilde{K}$ contains the origin, it must be the case that $\beta + \varepsilon \leq 0$, which implies that $\beta < 0$. Moreover, by definition of $\tilde{K}$, $g = g_2 + \ldots + g_n$, and $g_m(x) \geq 0$ for each $x \in K_m$ and each $m = 2, \ldots, n$.

Let $\delta = \frac{-\beta}{2(n-1)}$. Then $(g_m + \delta)(x) \geq \delta > 0$ for each $m = 2, \ldots, n$ and for each $x \in K_m$. Furthermore, $\sum_{m=2}^n (g_m + \delta)(x) \leq \frac{\beta}{2}$ for all $x \in \hat{K}$. Note that K_1 and $\hat{K}$ are isomorphic, hence $\sum_{m=2}^n (g_m + \delta)(x) \leq \frac{\beta}{2}$ for all $x \in K_1$.

Finally, let $\alpha := \min\{\delta, \frac{-\beta}{2}\} > 0$. For each $m = 2, \ldots, n$ let $f_m = g_m + \delta$. We now have all the ingredients for defining a zero-sum agreeable bet: let $f_1 = -\sum_{m=2}^n f_m$. Then $f_m(x) \geq \alpha > 0$ for each $m = 1, \ldots, n$ and each $x \in K_m$, and $\sum f_m = 0$. □

Proof of Lemma 13. In this proof, for $f \in B(\Omega, \mathcal{M})$ and $x \in \mathrm{ba}(\Omega, \mathcal{M})$ we will write $\langle f, x \rangle$ to denote $f(x)$; this underscores the duality between $B(\Omega, \mathcal{M})$ and $\mathrm{ba}(\Omega, \mathcal{M}) = B^*(\Omega, \mathcal{M})$.

Let $(x_n) \subseteq \Omega$ satisfy $|f(x_n)| \nearrow ||f||_{\mathrm{sup}}$.

For each $x \in A \subset \mathrm{ba}(\Omega, \mathcal{M})$, it is the case that $x + \varepsilon\delta_{\{x_n\}}, x - \varepsilon\delta_{\{x_n\}} \in \overline{A + \mathcal{O}_{TV}(0, \varepsilon)}^*$ for all n.

Suppose that $\lim f(x_n) \geq 0$. Note that $x - \varepsilon\delta_{\{x_n\}} \in \overline{A + \mathcal{O}_{TV}(0, \varepsilon)}^*$ for all n and for all $x \in A$, and that in addition $\langle f, x - \varepsilon\delta_{\{x_n\}} \rangle = \langle f, x \rangle - \varepsilon \langle f, \delta_{\{x_n\}} \rangle = \langle f, x \rangle - \varepsilon f(x_n)$ for all $x \in A$. It follows that $\langle f, x \rangle - \varepsilon \lim f(x_n) = \langle f, x \rangle - \varepsilon ||f||_{\mathrm{sup}}$ for all $x \in A$. From this we can deduce that $\langle f, x \rangle \geq \alpha$ for all $x \in \overline{A + \mathcal{O}_{TV}(0, \varepsilon)}^*$ if and only if $\langle f, x \rangle \geq \alpha + \varepsilon ||f||_{\mathrm{sup}}$ for all $x \in A$.

Suppose that $\lim f(x_n) \leq 0$. We have that $x + \varepsilon\delta_{\{x_n\}} \in \overline{A + \mathcal{O}_{TV}(0, \varepsilon)}^*$ for all n and for all $x \in A$ and that $\langle f, x + \varepsilon\delta_{\{x_n\}} \rangle = \langle f, x \rangle + \varepsilon f(x_n)$ for all n and for all $x \in A$, hence $\langle f, x \rangle + \varepsilon \lim f(x_n) = \langle f, x \rangle - \varepsilon ||f||_{\mathrm{sup}}$ for all

$x \in A$. We can deduce that $\langle f, x\rangle \geq \alpha$ for all $x \in \overline{A + \mathcal{O}_{TV}(0,\varepsilon)}^*$ if and only if $\langle f, x\rangle \geq \alpha + \varepsilon ||f||_{\sup}$ for all $x \in A$. □

Proof of Lemma 14. Since $\Pi_i^\varepsilon = \overline{\Pi_i + \mathcal{O}_{TV}(0,\varepsilon)}^* \cap \text{pba}(\Omega, \mathcal{M})$, $i \in N$, if $\cap_{i\in N}\overline{\Pi_i + \mathcal{O}_{TV}(0,\varepsilon)}^* = \emptyset$, we deduce $\cap_{i\in N}\Pi_i^\varepsilon = \emptyset$.

Since $\Pi_i^\varepsilon \subseteq \text{pba}(\Omega, \mathcal{M})$ if $x \in \cap_{i\in N}\overline{\Pi_i + \mathcal{O}_{TV}(0,\varepsilon)}^*$ we deduce $\cap_{i\in N}\Pi_i^\varepsilon \neq \emptyset$, i.e., if $\cap_{i\in N}\Pi_i^\varepsilon = \emptyset$ then $\cap_{i\in N}\overline{\Pi_i + \mathcal{O}_{TV}(0,\varepsilon)}^* = \emptyset$. □

Proof of Lemma 20. Only if: First we construct finite index sets. Let $I = \{N_0 \subseteq N \colon |N_0| < \infty\} \times \{F \subseteq B(\Omega, \mathcal{M}) \colon |F| < \infty\} \times (0,1)$. Intuitively, from the set $\{N_0 \subseteq N \colon |N_0| < \infty\}$ we can select an arbitrary finite collection of players, from $\{F \subseteq B(\Omega, \mathcal{M}) \colon |F| < \infty\}$ an arbitrary finite set of functions in $B(\Omega, \mathcal{M})$, and from $(0,1)$ an arbitrary $\varepsilon > 0$.

For a pair $(N_0, F, \varepsilon), (N_0', F', \varepsilon') \in I$, define $(N_0, F, \varepsilon) \leq (N_0', F', \varepsilon')$ if $N_0 \subseteq N_0'$, $F \subseteq F'$ and $\varepsilon' \leq \varepsilon$. Then $(I, \leq)$ so defined is a directed set, which can serve as the index set of a generalised sequence (i.e., a net).

Now, suppose that there exist approximate common σ-priors with finite support. By definition,this means that for any finite set $N_0 \subseteq N$, any finite set of functions $F \subseteq B(\Omega, \mathcal{M})$, $|F| < \infty$, and any $\varepsilon > 0$, there exists a measure $P^{(N_0,F,\varepsilon)} \in D(\Omega, \mathcal{M})$ such that for each $i \in N$

$$\mathcal{O}^*(P, F, \varepsilon) \cap \Pi_i \neq \emptyset.$$

Since $\text{pba}(\Omega, \mathcal{M})$, the set of probability charges (which includes the probability measures as a subset), is a weak* compact set, the set of measures $\{P^{(N_0,F,\varepsilon)}\}_{(N_0,F,\varepsilon)\in I}$ has a cluster point $P^* \in \text{pba}(\Omega, \mathcal{M})$. As Π_i is a weak* closed set for each $i \in N$ (by Lemma 8), P^*, as a cluster point, satisfies $P^* \in \cap_{i\in N}\Pi_i$. Hence we have identified a common prior, P^*, of T.

If: Suppose that T admits a common prior P. Since $\text{da}(X, \mathcal{A})$ is weak* dense in $\text{ba}(X, \mathcal{A})$, for any $F \subseteq B(\Omega, \mathcal{M})$, $|F| < \infty$, and any $\varepsilon > 0$, the corresponding ε weak* neighbourhood of P intersects the set $D(\Omega, \mathcal{M})$ of probability measures of finite support, i.e., there exists $P' \in \mathcal{O}^*(P, F, \varepsilon) \cap D(\Omega, \mathcal{M})$. Then for all $i \in N$

$$\mathcal{O}^*(P', F, \varepsilon) \cap \Pi_i \neq \emptyset,$$

which is what is required by Definition 19. □

Proof of Theorem 26. Suppose by contradiction that the type space T does not have a common prior. Then by Theorem 15 there exists an agreeable bet, that is, there exists a set of functions $f_{i_1}, \dots, f_{i_n} \in B(\Omega, \mathcal{M})$, for a finite index set $i_1, \dots, i_n \in N$, and a number $\alpha \in \mathbb{R}$ such that $\sum_{m=1}^n f_{i_m} = 0$ and $\int f_{i_m} \, \mathrm{d}\, t_{i_m}(\omega, \cdot) \geq \alpha > 0$, for every every state $\omega \in \Omega$ and every player i_m with $m = 1, \dots, n$. Without loss of generality we may assume that $||f_{i_m}||_{\text{sup}} \leq 1$, for all $m = 1, \dots, n$.

Let $N' = \{i_1, \dots, i_n\}$, $F = \{f_{i_1}, \dots, f_{i_n}\}$, $\varepsilon = \delta = \alpha/3$. Let $T' = ((\Omega', \mathcal{M}'), \{(\Omega', \mathcal{M}'_i)\}_{i \in N'}, \{t'_i\}_{i \in N'})$ be an $(N', F, \varepsilon, \delta)$ finite approximation of T. Then

- $\sum_{m=1}^n f_{i_m} \circ \psi = 0$,
- $\int f_{i_m} \circ \psi \, \mathrm{d}\, t'_{i_m} \geq \frac{2}{3}\alpha > \delta$.

Since T' achieves a δ-common prior, by Theorem 15 there does not exist a δ-agreeable bet. This is a contradiction. □

The proof of Lemma 30. If: Since $\overline{P_i}^{TV} \subseteq \overline{P_1}^*$, $i = 1, 2$, if $\overline{P_1}^*$ and $\overline{P_2}^*$ are strongly separable by a linear functional, then the very same linear functional strongly separates $\overline{P_1}^{TV}$ and $\overline{P_2}^{TV}$.

Only if: Suppose that $\overline{P_1}^{TV}$ and $\overline{P_2}^{TV}$ are strongly separable, and let f be the non-trivial strongly separating linear functional, i.e, there exist $c \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$f(x) \geq c + \varepsilon \quad \text{and} \quad c - \varepsilon \geq f(y),$$

for all $x \in \overline{P_1}^{TV}$ and for all $y \in \overline{P_2}^{TV}$.

Since $\mathcal{A}$ is a σ-field $f \in B(X, \mathcal{A})$ (f is $\mathcal{A}$-integrable) and it holds that

$$f(x) \geq c + \varepsilon \quad \text{and} \quad c - \varepsilon \geq f(y),$$

for all $x \in \overline{P_1}^*$ and for all $y \in \overline{P_2}^*$. □

B Why Bets Must Be Bounded Away From Zero

Our main theorem (Theorem 15) characterises the existence of common priors using agreeable bets (Definition 11) which are not only give rise to positive expectations at each state but have expectations bounded away from zero. We show here by an example why this presumption is necessary.

There are two players, Anne and Ben. The state space and partition is the basic partition space, that is, $\Omega = \{1, 2, \ldots\}$, Anne's knowledge partition, Π^A, is given by

$$\{\{1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \ldots\}$$

and Ben's knowledge partition, Π^B, is given by

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots\}.$$

The epistemic events: $\mathcal{M}$ is the field generated by the singleton sets, $\mathcal{M}_i$ is the field generated by the partition Π^i, $i = A, B$.

Anne's type function, t_A, is given by

$$t_A(n, \{n\}) = \begin{cases} 1 & \text{if } n = 1 \\ \frac{1}{2} & \text{if } n \text{ is even} \\ \frac{1}{2} & \text{if } n \text{ is odd, } n > 1. \end{cases}$$

Ben's type function, t_B, is given by

$$t_B(n, \{n\}) = \begin{cases} \frac{1}{2} & \text{if } n \text{ is odd} \\ \frac{1}{2} & \text{if } n \text{ is even.} \end{cases}$$

Anne	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	...
Ben	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	...

Figure 4: The type space of the example in this section.

Notice that the probability charge which assigns zero to each finite set is a common prior for this type space.

Suppose that f_A, with $f_B = -f_A$, is an agreeable bet, with $|f_A|$ a bounded and strictly increasing function and $f_A(n) = (-1)^{n+1}|f_A(n)|$, $n \in \Omega$. For $i = A, B$, for each $n \in \Omega$,

$$\int f_i \, \mathrm{dt_i(n, \cdot)} > 0,$$

but there does not exist $\alpha \in \mathbb{R}$ such that for each $n \in \Omega$

$$\int f_i \, \mathrm{dt_i(n, \cdot)} \geq \alpha > 0.$$

In words, this type space has an agreeable bet in sense of Lehrer and Samet (2014) but it does not have an agreeable bet in sense of Definition 11.

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