INDEXING GAMBLE DESIRABILITY BY EXTENDING PROPORTIONAL STOCHASTIC DOMINANCE

ZIV HELLMAN AND AMNON SCHREIBER

ABSTRACT. We axiomatically characterise two new orders of desirability of gambles (risky assets) that are natural extensions of the proportional stochastic dominance order to complete orders. These orders are represented by indices with parallels to the recently introduced Aumann-Serrano index of riskiness and the Foster-Hart measure of riskiness. The new indices are shown to be related to the concept of coherent measures of risk and to the Sharpe ratio.

1. INTRODUCTION

Given a choice between two gambles (or risky assets, or securities), which one is more desirable? On which should one put one's money, and how much? The answer to such questions is often: it depends. That is, the answer is subjective and depends on the utility function for money of the agent charged with choosing.

There are many cases, however, in which a subjective answer is unsatisfactory. A pension management firm that pools funds from a broad range of clients, for example, needs in a sense to consider itself acting as a representative agent with an 'objective' capacity to sort potential gambles by desirability. But how is an objective ranking obtained?

In recent years, two numerical objective measures of riskiness introduced into the literature have garnered much attention: the Aumann–Serrano index (Aumann and Serrano (2008)) and the Foster–Hart measure (Foster and Hart (2009)). Both of these are based on the paradigm of acceptance or rejection of gambles; in other words agents are asked with respect to each

---

Date: August 16, 2016.

The authors thank Sergiu Hart for useful discussions and suggestions, as well as seminar participants at the Hebrew University of Jerusalem, Bar Ilan University, the University of Oxford and the Paris Game Theory Seminar.

Department of Economics, Bar Ilan University, Ramat Gan, 52900, Israel.
Email: amnonschr@gmail.com.

gamble g whether they are willing to accept the terms of the gamble or prefer to avoid it. Gambles g and h are then ranked, in a general sense, by asking whether gamble g is accepted more, and thus rejected less, than h.

In many realistic situations, however, a 'take it or leave it' approach to risky assets is not the norm; investors may instead select a *proportion* of an offered asset, with an attendant scaling of both positive and negative payoffs. This significantly shifts the perspective on the matter. For example, instead of asking about the certainty equivalent of a gamble g of an agent with utility u and wealth w, one inquires about the certainty equivalent of the *optimal proportion* of g taken by an agent with utility u at wealth w.

In effect, an agent is now not being asked to compare a gamble g directly with another gamble h. Instead, the focus changes to two *families* of gambles, namely positive scalar multiples of g versus positive scalar multiples of h, and the agent is in effect asked to select his or her optimal gamble in each of these two families and to compare *those* optimal gambles to each other.

Starting from this observation, we study here the topic of ranking gambles when agents may choose optimal proportions of gambles offered to them. Many of the same issues that are involved in comparing gambles in the non-proportional case arise in the proportional setting, namely finding an ordering that is both objective and *complete*, in the precise sense of enabling comparison of any pair of gambles.

Completeness is conspicuously lacking in several broadly used indices of gamble desirability. The Sharpe ratio, for example, fails to extend as an index for gambles that are not normally distributed. Similarly, the most widely accepted orderings on gambles, the n-th degree stochastic dominance orders (Hadar and Russell (1969), Hanoch and Levy (1969)), are objective but none of them are complete, in any degree. Stochastic dominance was long ago extended to the proportional gamble setting (see Levy (2006)), but this extended ordering is, again, not complete for comparing all gambles.

In Section 5, we propose two complete gamble desirability indices, denoted S and G, that are of homogeneity zero and extend stochastic dominance. The motivation is a list of axiomatic desiderata that one would reasonably want such desirability indices to satisfy. The S index, when suitably extended to continuum state spaces, extends the Sharpe ratio; the G index ranks gambles according to their optimal growth path.

The two indices are by their definitions naturally 'dual' to each other, with S a CARA-based index and G a CRRA-based index, but they are also dual in another, more interesting way. There are many possible complete indices on gambles that extend stochastic dominance, but S and G do so 'uniformly', as we now explain. Stochastic dominance, by design, ranks a gamble g higher than h if every agent with a utility function located within a specified class of utilities prefers g to h. One may regard this as almost defining a voting mechanism: g ranks higher than h if and only if every agent in a specified class 'votes' for g over h in preference.

Unfortunately, the broader the class of participating agents, the more difficult it is to attain unanimity. Denote by $CE^*(u, w, g)$ the certainty equivalence an agent with utility u and wealth w ascribes to his or her optimal proportion of g. Then if one were to try to rank gamble g over h by asking each agent to point to the optimal proportions of g and h and then checking whether $CE^*(u, w, g) > CE^*(u, w, h)$, uniformity of voting preferences will almost certainly not be attained; even worse, a particular agent may rank g above h and then h above g, depending on the wealth w.

To remedy this, in Section 6.1 we rank g and h under a much more restrictive condition. Suppose that an agent with utility u states that he considers a gamble g worthy of consideration if and only if it meets some minimum criterion, for example if and only if $CE^*(u, w, g) > c$, for some fixed c. The gamble g is then regarded as uniformly undesirable by the agent if it *never* meets that criterion, at any wealth, i.e., $CE^*(u, w, g) \leq c$ for all w.

Given a pair of gambles g and h we can then ask: is it the case that for any agent, with any utility u, if g is uniformly undesirable then h must also be uniformly undesirable? If so, then we say that g wealth uniformly dominates h. We thus define a new order on gambles. It is immediately clear that it is a partial order, but somewhat surprisingly it turns out that wealth uniform dominance is not only a complete order, it defines the same order as the S index and hence monotonically extends stochastic dominance to all orders.

This brings us back to the Aumann–Serrano index and the Foster–Hart measure. The Aumann–Serrano index is based on CARA utilities and the Foster–Hart measure on logarithmic utilities. Both indices, in the standard paradigm of acceptance or rejection of gambles, define complete orders on gambles that extend the stochastic dominance order. Moreover, Hart (2011) shows that the two orders are related to each other in a 'dual' type of relation, with Aumann–Serrano following from 'wealth uniformity' and Foster–Hart from 'utility uniformity'.

The S-index of this paper bears a similarity to the Aumann–Serrano index. Both indices are related to CARA utility functions, and both are related to concepts involving 'wealth uniformity'. This then leads to the following question: if the S index is the proportional gambles parallel to the Aumann–Serrano index, via a wealth uniformity concept, what is the index that parallels the Foster–Hart measure, via an appropriately defined utility uniformity concept? In Section 6.3, we define such a utility uniformity concept within our framework and show that it does indeed lead to a complete order extending stochastic dominance – which, it turns out, is representable by our G index, completing the parallelism.

We have been careful to describe what the S and G indices measure as 'gamble desirability' and not 'risk'. In this we take inspiration from the elementary CAPM model, in which an index such as the Sharpe ratio takes into account various factors, including but not exclusively limited to elements of risk, to identify an optimal portfolio, with our indices intended to play a similar role.

However, it is possible to use these indices to define coherent risk indices, in the sense of Artzner et al (1999). In Section 7 we introduce the concept of risk compensator, which intuitively measures the minimal amount of money that an agent would require to be uniformly added to the payoff of a particular gamble at each state of the world to compensate him or her for accepting an optimal proportion of the gamble in lieu of surely obtaining instead a risk free return. The risk compensators that are naturally defined using the S and G indices turn out to be coherent measures of risk that may be used to identify 'purely' risk elements of gambles, in complement to the desirability measurements of S and G themselves. We expand more on risk compensators in a companion paper.

2. Preliminaries

2.1. **Gambles.** As in Aumann and Serrano (2008), a *gamble* g is a real-valued random variable over a finite state space that satisfies:

(1) $\mathbf{P}[g < 0] > 0$ (losses are possible);
(2) $\mathbf{E}[g] > 0$ (has positive expectation).

This is motivated by considering that: i) an agent offered a gamble with no possible losses can be expected to prefer an unbounded share of such a gamble, hence we limit attention to gambles with some negative values; ii) similarly, an agent offered a gamble with negative expectation will optimally take zero shares of such a gamble. Let $\mathcal{G}$ denote the collection of all gambles, i.e., random variables satisfying conditions (1) and (2) listed above.

Unless otherwise stated, we assume that each gamble g takes finitely many values, $x_1, x_2, \ldots, x_m$ with respective probabilities $p_1, p_2, \ldots, p_m$, such that $p_i > 0$ for all i and $\sum_{i=1}^{m} p_i = 1$.

2.2. **Utilities and Risk Aversion.** A *utility function* is understood to refer a von Neumann-Morgenstern utility function for money that is strictly monotonicly decreasing, strictly concave, and twice continuously differentiable.

The *Arrow–Pratt coefficient of absolute risk aversion* (ARA), ρ of a utility function u at wealth w is defined

$$\rho_u(w) = -\frac{u''(w)}{u'(w)}.$$

The *Arrow–Pratt coefficient of relative risk aversion* (RRA),

$$\varrho_u(w) = -w\frac{u''(w)}{u'(w)} = w\rho_u(w).$$

Both $\rho_u(\cdot)$ and $\varrho_u(\cdot)$ are utility specific attributes, and both ρ and ϱ yield complete orders on utility-wealth pairs.

We will henceforth assume that all utility functions satisfy the condition that their coefficients of risk aversion are bounded away from zero, i.e., for each utility function u there exists an $\varepsilon > 0$ such that $\rho_u(w) > \varepsilon$ for all w.

A utility function that satisfies $\rho(w) = \rho(w')$ (respectively, $\varrho(w) = \varrho(w')$) for all pairs of wealth w and w' is said to satisfy CARA, or *constant absolute risk aversion* (respectively, CRRA, or *constant relative risk aversion*). A utility function that satisfies $\rho(w') \leq \rho(w')$ (respectively, $\varrho(w) \geq \varrho(w')$) for all pairs of wealth w and w' such that $w' > w$ is said to satisfy DARA, or *decreasing absolute risk aversion* (respectively, IRRA, or *increasing relative risk aversion*).

2.3. **Certainty Equivalence.** The *certainty equivalent* of a gamble g for an agent with utility u and wealth w, denoted $CE(u, w, g)$ is the real number implicitly defined by the solution to

$$\mathbf{E}[u(w+g)] = u(w + CE(u, w, g)). \tag{1}$$

Note that by the presumed concavity of u, the function $CE(u, w, g)$ is also concave.

The importance of the certainty equivalence of a gamble in the study of risk aversion is underscored by the well-known Arrow–Pratt Theorem, which asserts that for any pair u and v of utility functions, $\rho_v(w) > \rho_u(w)$ for all w if and only if $CE(u, w, g) > CE(v, w, g)$ for all gambles g and all wealth levels w.

3. Risk Adjusted Desirability Indices

3.1. **Indices of Gamble Desirability.** In greatest generality, an *index of desirability* of gambles is a function from the collection of gambles to the positive reals, $Q : \mathcal{G} \to \mathbb{R}_+$, with the intended intuitive interpretation that g is 'more desirable' than h if $Q(g) > Q(h)$. We will sometimes refer simply to an 'index' as opposed to an index of desirability for the sake of brevity.

The most immediate and naïve index of desirability ranks gambles by the first moments of their distributions, i.e., simply determines that $Q(g) > Q(h)$ if and only if $\mathbf{E}[g] > \mathbf{E}[h]$. Such an index clearly ignores any aspect of a gamble that might plausibly be termed the gamble's 'riskiness', and it is for this reason that '*risk adjusted*' indices were invented.

The concept of a risk adjusted index encompasses too broad a range of possibilities to be pinned down formally, but one can say that a risk adjusted index ideally strikes some balance between aspects that are considered to make a gamble desirable (often expected value, although maximum gain or any of several other attributes may be used) and a gamble's risk, which reduces desirability. One would roughly posit that, 'all else being equal', a risk averse agent will find a more risky gamble less desirable than a less risky gamble.

3.2. **The Mean-Variance Approach.** The mean-variance approach has been the most prominent approach in the literature to constructing an objective risk adjusted index for several decades. In this view, desirability of a gamble is given by the first moment of the distribution (i.e., the mean), and risk adjustment is accomplished by taking into account the second moment (i.e., the variance,or in practice, the square root of the variance). Objectivity is attained by postulating that risk is objectively measured by way of a gamble's variance.

The 'gold standard' index in this class is the Sharpe ratio. It often, however, yields satisfactory results only when distributions are normal, or utility functions are quadratic. Furthermore, the Sharpe ratio fails the test of monotonicity with respect to stochastic dominance, as detailed in the next section. There is a vast literature of attempts to expand on the Sharpe ratio indexing by taking into consideration higher order moments, such as skewness, kurtosis, hyperskewness, hyperflatness, etc.

3.3. **Stochastic Dominance.** An entirely different approach to risk ranking, the stochastic dominance ordering, takes the view that in order to take risk into account one naturally needs to ask how agents with risk averse utility functions would order gambles. This reflects the dictum that 'risk is that to which risk averse agents are averse'. Essentially, the idea is that if *all* risk averse agents find one gamble more preferable than another, then one has constructed a desirability ranking that implicitly takes risk preferences into account, without reference to any moments of distribution. Objectivity is attained by seeking the broadest common denominator amongst all utility functions within particular classes.

Stochastic dominance has been an enormously influential concept; as Aumann and Serrano (2008) and Hart (2011) note, stochastic dominance may be considered 'the most uncontroversial, widely accepted' order on gambles. Its main drawback is that it provides only a partial ordering on $\mathcal{G}$.

There are several equivalent ways to define stochastic dominance. The definition that arises most straight-forwardly from the idea of considering the preferences over gambles of risk averse agents proceeds as follows.

Denote by U_1 the collection of all strictly increasing utility functions, i.e., utilities u satisfying $u'(w) > 0$ for all w. Then g first order stochastically dominates h, denoted $g\, SD_1\, h$, if the expected utility that g yields is always at least as large as that of h, i.e.,

$$\mathbf{E}[u(w+g)] \geq \mathbf{E}[u(w+h)] \tag{2}$$

for all w and all $u \in U_1$. This does intuitively capture the arguably weakest possible requirement for a desirability index: if all agents who prefer more wealth to less wealth agree to rank g higher than h, then g is objectively more desirable than h.

By Equation (1), under the assumption of strictly increasing utility functions, Equation (2) holds if and only if $u(w + CE(u, w, g)) \geq u(w + CE(u, w, h))$, if and only if

$$CE(u, w, g) \geq CE(u, w, h). \tag{3}$$

We may therefore alternatively use Equation (3) as the definition of $g\, SD_1\, h$.

Continuing in this vein, denote by U_2 the collection of all strictly increasing risk averse utility functions, i.e., utilities u satisfying $u'(w) > 0$ and $u''(w) < 0$ for all w. Then g second order stochastically dominates h, denoted $g\, SD_2\, h$, if and only if Equation (2), i.e., $\mathbf{E}[u(w+g)] \geq \mathbf{E}[u(w+h)]$, holds for all w and all $u \in U_2$, alternatively, if and only if (3) holds for all w and all $u \in U_2$. This captures the arguably weakest requirement for a risk adjusted desirability index: if all *risk averse* agents agree to rank g higher than h, then g is objectively more desirable than h, even after adjusting for risk aversion.

Although first and second order stochastic dominance are by far the two orders of stochastic dominance that are used most often, there is no inherent reason to stop at the second order. In fullest generality, denote by U_n the collection of all utility functions u that satisfy

$$(-1)^m u^{(m)}(w) \geq 0 \tag{4}$$

for all w and all $0 < m \leq n$, where $u^{(m)}(w)$ is the m-th order derivative of u at w. Then g n-th order stochastically dominates h, denoted $g\, SD_n\, h$, if Equation (3), holds for all w and all $u \in U_n$.

The SD_n relation, for any n, is a partial order of $\mathcal{G}$, that is, it is always possible to find a pair $g, h \in \mathcal{G}$ such that neither $g\, SD_n\, h$ nor $h\, SD_n\, g$ holds. The collections of utility functions $\{U_n\}$ are monotonically ordered by reverse set inclusion, that is, $U_1 \supset U_2 \supset \ldots \supset U_n \supset \ldots$. Hence if $g\, SD_n\, h$ then $g\, SD_m\, h$ for all $m \geq n$. In this sense the SD_n relation becomes 'less partial' as n increases, but a complete order is never attained.

Letting n go to infinity, one defines infinite degree stochastic dominance. Denote by D_∞ the collection of all completely monotone utility functions, i.e. utility functions u that satisfy Equation (4) for all $m > 0$ and all w. Then g infinite order stochastically dominates h, denoted $g\, SD_\infty\, h$, if and only if Equation (2), i.e., $\mathbf{E}[u(w+g)] \geq \mathbf{E}[u(w+h)]$, holds for all w and all $u \in U_\infty$. As is the case for SD_n for any n, the SD_∞ relation is a partial order on $\mathcal{G}$. Thistle (1993) shows that $g\, SD_\infty\, h$ if and only if $g\, SD_n\, h$ for some n.

We note here that a well-known critique of the Sharpe ratio is that it may violate first order stochastic dominance. A simple example[1] is the following: let g yield -1 with probability 0.02 and 1 with probability 0.98, and h yield -1 with probability 0.02, 1 with probability 0.49 and 2 with probability 0.49. Then $h\, SD_1\, g$ but the Sharpe ratio of g, which is $0.28/0.96$ is higher than that of h, which is $(7\sqrt{3}/20)/1.45$. When a gamble g has a normal distribution this cannot happen: in that case, the ordering of the Sharpe ratio is identical to that of first order stochastic dominance.

4. The Proportional Approach

As noted above, in many cases an agent is not required either to accept or reject a gamble in a binary manner. Instead any proportion $\alpha > 0$ of the gamble may be chosen. Several indices of gambles in the literature (cf. Aumann and Serrano (2008) and Hart (2011)) are based on the 'accept or reject a gamble' paradigm. Our goal here is to construct a risk adjusted index of gambles in a context in which agents may choose proportions of gambles. In other words, it is not a gamble g that is compared to a gamble h directly by an agent for ordering gambles; rather, the optimal proportion of g is compared to the optimal proportion of h.

[1] This example is taken from Aumann and Serrano (2008).

4.1. **Optimal Proportions of Gambles.** An agent with utility function u and wealth w offered the opportunity to take any proportion α of a gamble g can be expected to choose the proportion α^* that optimises expected utility. In other words, the agent solves $\alpha^*_{g,w} := \arg\max_\alpha \mathbf{E}[u(w + \alpha g)] \in \mathbb{R}_+$ and then takes the gamble $\alpha^*_{g,w} g$. This optimal proportion is given by the solution to the first order condition $\mathbf{E}[u'(w+\alpha g)g] = 0$. In what follows we will write α^*_g when w is clear from context and furthermore allow ourselves to write α^* when both g and w are clearly identified by context.

One may regard the space of gambles $\mathcal{G}$ as being partitioned into equivalence classes for our purposes. That is, a gamble h is included in $[g]$, the equivalence class of g, if and only if $h = \beta g$ for some $\beta > 0$. An agent with utility u and wealth w makes a well-defined choice of an *optimal representative* g^* of $[g]$ for each equivalence class in the following sense: if $[g] = [h]$ then, denoting $g^* = \alpha^*_g g$ and $h^* = \alpha^*_h h$, one has $g^* = h^*$. Obviously, if $[g] \neq [h]$ then $g^* \neq h^*$.

It is tempting to express everything here in terms of the equivalence classes, which would mean that the focus would be on seeking a complete and objective ordering that would enable one to state that one equivalence class $[g]$ is more desirable than another class $[h]$. We will not fully adopt this approach because the optimal representative g^* of $[g]$ depends on u and w; different agents, and even the same agent at different wealth levels, may choose different representatives in each equivalence class, and their calculations will depend on the identity of their chosen optimal representative. Hence, although our ultimate goal *is* to find an ordering that can be considered an ordering of the equivalence classes, we will find it more convenient along the way to work with individual gambles g as the focus rather than $[g]$, while throughout keeping in mind the fact that agents are choosing representatives of equivalence classes.

4.2. **Proportional Certainty Equivalence.** Continuing in the vein of an agent regarding a gamble g as the equivalence class $[g]$ of all its possible positive scalar multiples, the certainty equivalence, as defined in Equation (1), needs to be reinterpreted accordingly.

We define the *proportional certainty equivalence* of g at u and w, denoted $CE^*(u, w, g)$, by $CE^*(u, w, g) := CE(u, w, \alpha^* g)$, where α^* is the optimal proportion of g according to u.[2] In parallel to Equation (1), CE^* can equivalently be defined implictly as the solution to

$$\max_\alpha \mathbf{E}[u(w + \alpha g)] = u(w + CE^*(u, w, g)). \tag{5}$$

[2] Given the assumption that risk aversion coefficients are bounded away from zero, it is easy to see that α^* is always well defined and finite.

The interpretation is straight-forward: $CE^*(u, w, g)$ is the certainty equivalence of the optimal representative of $[g]$ that is chosen by an agent with utility u and wealth w.

We say that agent i with utility u_i is *uniformly no less risk averse* than agent j with utility u_j, written $i \succeq j$, if $CE^*(u_j, w_j, g) \geq CE^*(u_i, w_i, g)$ for any value of w_i and w_j and any gamble g. We say that i is uniformly more risk averse than j, written $i \succ j$, if $i \succeq j$ but not $j \succeq i$.

In words, i is uniformly more risk averse than j if for any gamble g and any pair of wealth levels w_i and w_j, even after i and j choose their respective *optimal* representatives from $[g]$, i *always* perceives lower certainty equivalence than j. Lemma 1 relates this proportional gamble concept to the Arrow–Pratt measure of local risk aversion.

Lemma 1. *$i \succ j$ if and only if $\rho_i(w_i) > \rho_j(w_j)$ for all w_i and w_j.*

4.3. **Proportional Stochastic Dominance.** The concept of stochastic dominance similarly needs to be extended to take into account the freedom an agent has to take proportions of gambles. Working in analogy with Equation (3), we define a gamble *g n-th order proportionally stochastically dominating* a gamble h, denoted $g\, PSD_n\, h$, if

$$CE^*(u, w, g) \geq CE^*(u, w, h). \tag{6}$$

for all w and all $u \in U_n$.

Haim Levy introduced (see Levy (2006)) the concept of stochastic dominance with a riskless asset: a gamble *g n-th order stochastically dominates with a riskless asset* a gamble h, if for each $\beta > 0$ there is an $\alpha > 0$ such that αg n-th order stochastically dominates βh. It turns out that proportional stochastic dominance extends Levy's dominance concept:

Lemma 2. *Proportional stochastic dominance extends stochastic dominance with a riskless asset.*

As the next lemma, Lemma 3, shows (by setting $\alpha = 1$), stochastic dominance with a riskless asset extends stochastic dominance, to all orders. (Lemma 3 is proved in Levy (2006) for $n = 1$ and $n = 2$). This is then used to show that proportional stochastic dominance extends the standard stochastic dominance concept.

Lemma 3. *If, for two gambles f and g there exists $\alpha > 0$ such that αf n-th order stochastic dominates g, then for every $\beta > 0$, there is a γ such that γf n-th order stochastic dominates βg.*

Proposition 1. *Proportional stochastic dominance extends stochastic dominance to all orders.*

5. TWO INDICES

With the preliminaries in place, we can now specify a list of desiderata that we would like to see in a general proportional risk adjusted index Q.

(1) Obviously, the index should be proportional, that is, the desirability of a gamble g should be uniform for all $h \in [g]$. Another way of saying the same thing is to require that $Q(xg) = Q(g)$ for all $x > 0$.
(2) As a risk adjusted index, Q should consistently relate both to agents' subjective perceived desirability of gambles, as measured by CE^*, and to agents' subjective risk aversions, at least in cases where agents can unambiguously be compared in their risk aversions, as measured by the $i \succ j$ partial ordering of agents.
(3) The index Q ought to relate to both of the major approaches to risk adjusted indices adopted in the literature, stochastic dominance and the mean-variance approach. Thus we want Q to be monotonic with respect to proportional stochastic dominance and to extend the Sharpe ratio beyond gambles with normal distributions.
(4) Finally, we want the index Q to be a complete, not a partial order on $\mathcal{G}$.

5.1. **Axioms.**

Motivated by the above list of desiderata, we present the following formal axioms that we expect a complete index Q on $\mathcal{G}$ to satisfy.

Positive Homogeneity Zero. $Q(xg) = Q(g)$, for all real $x > 0$.

Continuity. If g_n are uniformly bounded and converge to g in probability[3] then $Q(g_n) \to Q(g)$.

Monotonicity. If g stochastically dominates h, to any order, then $Q(g) > Q(h)$.

Duality. If $i \succ j$ and $Q(g) > Q(h)$, then $CE^*(u_j, w_j, g) > CE^*(u_i, w_i, h)$ for all w_i and w_j.

While the interpretation of the initial three axioms is self-evident, the duality axiom requires some more explanation: duality starts from the observation (by definition) that if $i \succ j$ then $CE^*(u_j, w_j, g) \geq CE^*(u_i, w_i, g)$ and $CE^*(u_j, w_j, h) \geq CE^*(u_i, w_i, h)$, for all wealth levels $w_i, w_j \in \mathbb{R}$.

[3] As standard, we say that g_n converges to g in probability if for every $\varepsilon > 0$, there is an N such that $Prob(|g_n - g| > \varepsilon) < \varepsilon$ for all $n > N$.

It then goes on to posit that if the less risk averse agent values h more than the more risk averse agent does (as witnessed by $CE^*(u_j, w_j, h) \geq CE^*(u_i, w_i, h)$) then *a fortiori* the less risk averse agent should value the more desirable gamble g (as witnessed by $Q(g) > Q(h)$) all the more, hence $CE^*(u_j, w_j, g) > CE^*(u_i, w_i, h)$. [4]

5.2. **The S Index.** For every gamble $g \in \mathcal{G}$ we define the *S index* of g by

$$S(g) = -\log \mathbf{E}[e^{-\alpha^* g}] \tag{7}$$

where α^* is the optimal proportion of g taken by an agent with CARA utility of parameter 1. More formally, α^* is implicitly defined by

$$\mathbf{E}(e^{-\alpha g} g) = 0. \tag{8}$$

The most immediate interpretation of S is that $S(g)$ is the certainty equivalent that a CARA utility agent of parameter 1 ascribes to g when taking his or her optimal proportion, α^*, of g. That is, $S(g) = CE^*(u_1, w, g)$ where $u_1(g) = -e^{-g}$, i.e., the CARA utility with ARA parameter one.[5]

The reader might object that restricting to the ARA parameter 1 is completely arbitrary, which is true: in fact, the same ordering of gambles determined by S obtains for any CARA utility parameter, as Lemma 7 (in the appendix) shows.

It might furthermore be objected that restricting to representative agents with CARA utilities for determining an index of desirability is overly limiting as well, but it turns out that this is sufficient for our goals: the S index satisfies *all* the desiderata listed above for such an index:

Proposition 2. *The order imposed by S is well defined and is a complete order on $\mathcal{G}$ that extends proportional stochastic dominance to all orders.*

Theorem 1. *The S index satisfies the axioms of continuity, monotonicity and duality, and any index Q satisfying these axioms is ordinally equivalent to S.*

The S index, when applied to continuous distributions, also extends the Sharpe ratio. We have hitherto defined gambles only with respect to finitely many possible states. We can straightforwardly extend the definition of gamble to a random variable with a continuous probability measure for which there exists a positive value of α^* satisfying $\mathbf{E}[e^{\alpha^* g} g] = 0$. We also

---

[4] A similar concept of duality appears in Aumann and Serrano (2008).

[5] To see this, note that by Equation 5, the definition of u_1, and the fact that the CARA utility is wealth-independent, $\mathbf{E}[e^{-\alpha g}] = -u_1(CE^*) = e^{-CE^*}$. Hence $S(g) = -\log \mathbf{E}[e^{-\alpha^* g}] = -\log(e^{-CE^*}) = CE^*$.

define the function Sharpe(g) for such a gamble as the ratio of the expectation of g divided by the standard deviation of g. We can then state the following result for normally distributed gambles.

Proposition 3. *For any two normally distributed random variables g and h, $S(g) > S(h)$ if and only if Sharpe$(g) >$ Sharpe(h).*

An index very similar to the S index appears in Zakamouline and Koekebakker (2009), where it shown that it is an index within the family of Generalised Sharpe Indices, as defined in Hodges (1998). Zakamouline and Koekebakker (2009), however, does not include an axiomatic development of the index, nor comparisons to the stochastic dominance order.

5.3. **The G Index.** Recalling the interpretation $S(g) = CE^*(u_1, w, g)$ where u_1 is $u_1(w) = -e^{-w}$, we seek a naturally 'dual' index in the following sense: the mathematical dual to the exponential function is the logarithmic function, and similarly the dual to the CARA utilities are the $\log$-based utilities. This motivates us to seek an index that essentially measures the certainty equivalent of a gamble according to an agent with $\log$ utility.

In this context, we interpret the value of a gamble in terms of multiplicative as opposed to additive returns. For example, rather than saying that a gamble returns fixed prizes, we suppose that a gamble returns, say, $+20\%$ of the wealth placed on it with 0.5 probability and -10% of wealth on the gamble with 0.5 probability.[6]

In multiplicative terms, the maximisation problem encountered by an agent with utility u and wealth w is:

$$\max_{\alpha} u(w + \alpha w g) \tag{9}$$

In other words α here indicates an optimal fraction of wealth to place on the gamble (in multiplicative returns), as opposed to optimal proportion of gamble.

Definition 1. Let g be a gamble. Let $K(g)$ be defined implictly as the solution to

$$\mathbf{E}[(1 + K(g)g)^{-1}] = 1.$$

Then define

$$G(g) := e^{\mathbf{E}[\log(1+K(g)g)]} - 1 \tag{10}$$

♦

[6] Since we are concentrating on multiplicative returns, we need to restrict attention to gambles whose maximal possible loss for an agent is less than 100 percent of the wealth placed on the gamble.

The index G can be interpreted as $G(g) = CE^*(u_l, w, g)/w$ where $u_l(w) = \log(w)$. Operationally, the G index ranks gambles according to their optimal growth path, with $K(g)$ none other than the well known Kelly index of g (Kelly (1956)). Further details on the operational interpretation of the G index are in Appendix B.

Proposition 4. *G is a complete, homogeneity zero ordering on $\mathcal{G}$ that extends proportional stochastic dominance.*

6. Wealth Uniform and Utility Uniform Dominance

Both the S and G indices extend the proportional stochastic dominance order to a complete order, in line with one of our central desiderata. However, S and G are definitely not the only indices of gambles that have this property. In fact, the certainty equivalence of *any* utility function that has positive derivatives to all odd orders and negative derivatives to all even orders extends the proportional stochastic order.

In this section we show what makes S and G unique. A gamble g stochastically dominates h to order n if every agent with utility in U_n prefers g to h. Even if we let n increase to infinity, however, we do not get a complete order. If we try to correct this by extending dominance by declaring that g dominates h if *every* agent with a risk averse utility function prefers g to h then we again fail to attain a complete order: we have gone too far in the other direction, with an 'electorate' that is too broad to agree on preferences.

To get a complete order we need to make the comparison between the gambles only when the certainty equivalence preferences of the agents are expressed 'uniformly' in an appropriately defined manner, as detailed in this section. Since certainty equivalence preference depends on both a utility function and a wealth level, there are two main ways to impose uniformity: wealth uniformity (uniform preference at *all* wealth levels) or utility uniformity (uniform preference at a fixed wealth level by *all* utilities). We can then show that S is the only index[7] that satisfies wealth-uniform proportional dominance and G is the only index that satisfies utility uniform proportional dominance.

6.1. Wealth Uniform Dominance.

Stochastic dominance, by design, ranks a gamble g higher than h if *every* agent with a utility function located within a specified class of utilities prefers g to h, in other words, $CE(u, w, g) \geq CE(u, w, h)$ for all $u \in U_n$ and all $w \in U_n$ if $g\, SD_n\, h$. A theorem in Levy (2006) extends exactly the same property to the PSD_n ordering: $CE^*(u, w, g) \geq CE^*(u, w, h)$ for all $u \in U_n$ and all $w \in U_n$ if $g\, PSD_n\, h$.

[7] Up to ordinal equivalence.

Metaphorically, one may say that g dominates h to order n if one were to conduct a vote amongst all agents with utility in U_n and they unanimously vote to prefer g to h.

It is tempting to try to extend this conception of dominance to a complete order on $\mathcal{G}$. The relative ranking of g and h would then in a sense be determined by 'unanimous voting' with the electorate extended to all risk averse agents: each agent i, with utility function u_i, would be asked his or her opinion on the g and h by way of the relative values of $CE^*(u_i, w, g)$ versus $CE^*(u_i, w, h)$ for all w. Then g would be declared objectively more desirable than h if $CE^*(u, w, g) \geq CE^*(u, w, h)$ unanimously for all w by all the risk averse utility agents casting ballots in this election.

This is asking for too much. The gambles g and h may have different ranges with different probabilities and payoff values. Uniformity in ordering the proportional certainty equivalences of g and h among all utilities and wealth levels may be difficult to attain. And even the vote of a particular agent i may be ambiguous; as the wealth w changes, the utility function u_i might sometimes rank the certainty equivalent of g higher and sometimes that of h higher.

We therefore change the criteria of which voters will count and how we require that they order gambles. Suppose that an agent with utility u states that he considers a gamble g worthy of consideration if and only if it meets some minimum criterion, for example if and only if $CE^*(u, w, g) > c$, for some fixed c. We can then ascertain whether g fails this criterion uniformly, for all wealth levels w, i.e., $CE^*(u, w, g) \leq c$ for all w. If so, g can be regarded as uniformly undesirable by the agent.

We can then say that g wealth uniformly dominates h if whenever g is uniformly undesirable then h is also uniformly undesirable. Note that this does allow for some w such that $CE^*(u, w, h) > CE^*(u, w, g)$, but from the perspective of the agent with utility u this does not make a difference: since both g and h are uniformly undesirable when g wealth uniformly dominates h, that is, $CE^*(u, w, h) \leq c$ at all wealth levels, h is rejected along with g in any event.

Restricting our electorate to voters with wealth uniform preferences with respect to g and h is still insufficient if we work with the entire collection of *all* risk averse utility functions. Unanimity may fail to obtain in some cases, hence the goal of a complete order on $\mathcal{G}$ may not be available. Some regularity in the utility functions needs to be required.

Following Hart (2011), let U_{IR} denote the collection of utility functions that satisfy IRRA and let U_{DA} denote the collection of utility functions that satisfy DARA. Let U_{SR} be the collection of utilities that satisfy 'some rejection', meaning that for each $u \in U_{SR}$ no gamble will always be accepted by u, i.e. for each g there is some wealth level w at which $CE(u,g,w) < 0$. Perhaps more intuitively, this postulates that for each agent there is some level of 'poverty', or small enough wealth, below which he or she become so risk averse that he will reject g. Finally, denote $U^* = U_{DA} \cap U_{IR} \cap U_{SR}$.

The formal definition of wealth uniform dominance is then:

Definition 2. A gamble g *wealth uniformly dominates* h, denoted $g >_{WUD} h$, if for every constant real number c

$$\begin{aligned} &\text{if } CE^*(u,w,g) \le c \text{ for all wealth levels } w \\ &\text{then } CE^*(u,w,h) \le c \text{ for all wealth levels } w \end{aligned} \tag{11}$$

for every utility $u \in U^*$.

It is immediate that the relation defined in Definition 2 satisfies transitivity and relexivity, hence is a partial order. It is far from immediate whether this order is complete, and whether it extends stochastic dominance to all orders, but somewhat surprisingly it turns out that these properties are satisfied and that the WUD order is equivalent to the order defined by S index.

Theorem 2. *Wealth-uniform dominance is a complete order on risky gambles that extends proportional stochastic dominance. Indeed,* $g >_{WUD} h$ *if and only if* $S(g) > S(h)$.

6.2. **Wealth Bounded Dominance.** We can provide an alternative characterisation of the S index, based on a concept similar to wealth bounded domination as in Hart (2011). The advantage of this concept for our purposes is that in relation to wealth uniform domination it expands the collection of utility functions that can meet the conditions of the characterisation.

In this section, for a gamble g, denote by $M_g := \max g$ the maximal gain of g and by $L_g := \max(-g)$ the maximal loss of g. Definition 2 posits that for a number c, $CE^*(u,w,g) \le c$ for all wealth levels w. But for any specific w, the fact that $CE^*(u,w,g) \le c$ contains little information on u outside the interval $[w + \min \alpha^*_{g,w} g, w + \max \alpha^*_{g,w} g] = [w - \alpha^*_{g,w} L_g, w + \alpha^*_{g,w} M_g]$. A similar statement can be made about h. This motivates the following definition.

Definition 3. A gamble g *wealth boundedly dominates* h, denoted $g >_{WBD} h$, if for every constant real number c, every utility $u \in U^*$, and every wealth level w, there is a positive number b such that

$$\begin{aligned} &\text{if } CE^*(u,w',g) \le c \text{ for all wealth levels } w' \text{ with } |w'-w| \le b \\ &\text{then } CE^*(u,w',h) \le c. \end{aligned} \tag{12}$$

Note that the number b in Definition 3 depends on the utility function u and on the wealth level w under consideration, not only on g and h (compare with Hart (2011)). This is inevitable in the context of optimal proportions of gambles, as the optimal proportion of h is itself a function of u and w. (Interestingly however, as the proof of Theorem 3 shows, the dependence on u and w is only with respect to h, not g).

Theorem 3. *Wealth-uniform dominance and wealth-bounded dominance are equivalent, i.e.,* $g >_{WBD} h$ *if and only if* $S(g) > S(h)$.

6.3. **Utility Uniform Dominance.** Hart (2011) shows that the wealth uniformity concept of that paper is related to the Aumann–Serrano index, which is CARA based. That paper also contrasts wealth uniformity with a utility uniformity concept, which is shown to be related to the Foster–Hart index, a log based index.

Wealth uniform dominance here, as presented in Definition 2, was shown in the previous section to lead to an equivalent ordering as the S index, which is also CARA based. To complete the parallels, we work here with utility uniformity in the context of taking optimal proportions of gambles, and show that this is related to the G index, which is a log based proportional gamble index.

For wealth uniform dominance, a gamble g was regarded as uniformly undesirable, with respect to some fixed constant c and utility u, if $CE^*(u, w, g) \leq c$ for all w. For utility uniformity, we switch the roles of u and w, that is, c is again fixed, but now w is fixed while u ranges over all U^*, i.e., $CE^*(u, w, g) \leq c$ for all $u \in U^*$. Then g utility uniformly dominates h at w if the utility uniform undesirability of g, as witnessed by $CE^*(u, w, g) \leq c$ for all $u \in U^*$, implies the utility uniform undesirability of h, as witnessed by $CE^*(u, w, h) \leq c$ for all $u \in U^*$.

Formally:

Definition 4. A gamble g *utility uniformly dominates* h, denoted $g >_{UUD} h$, if for every constant real number $c > 0$

$$\begin{aligned} &\text{if } CE^*(u, w, g) \leq c \text{ for all } u \in U^* \text{ at } w \\ &\text{then } CE^*(u, w, h) \leq c \text{ for all } u \in U^* \text{ at } w \end{aligned} \tag{13}$$

for every wealth level $w > 0$.

Theorem 4. *Utility-uniform dominance is a complete order on risky gambles that extends proportional stochastic dominance. Indeed,* $g >_{UUD} h$ *if and only if* $G(g) > G(h)$.

7. Indices of Acceptability and Coherent Risk Measures

As emphasised above, our two indices, S and G, are indices of gamble desirability. They do not measure risk, although an element of risk can detract from the desirability of a gamble.

The relationship between indices of risk and indices of gamble desirability has been studied in the literature. In a seminal work, Artzner et al (1999) presented a set of four properties that an index of risk ought to satisfy. In contrast, Cherny and Madan (2008) put forward a set of properties that an index of gamble desirability should satisfy; indices that satisfy these properties are termed 'acceptability measures'. We show in this section that the S and G indices are indeed acceptability measures but not coherent risk measures. We can, however, use insights from Cherny and Madan (2008) to produce yet another interestingly motivated derivation of the S and G indices and along the way to identify two new coherent risk measures.

7.1. Acceptability Indices.

Let $\mathcal{L}$ denote the class of all bounded random variables over a finite probability space. Note that $\mathcal{L}$ extends $\mathcal{G}$, which is the subclass of elements of $\mathcal{L}$ that satisfy the two conditions listed in Section 2.1.

Cherny and Madan (2008) identify, using an axiomatic approach, a class of mappings from $\mathcal{L}$ to the real numbers that they term 'acceptability indices'. An index $\alpha : \mathcal{L} \to \mathbb{R}_+$ is an acceptability index if it satisfies:

(1) Quasi-concavity: For every $x \in \mathbb{R}_+$, if $\alpha(g_1) \geq x$ and $\alpha(g_2) \geq x$ then $\alpha(\lambda g_1 + (1-\lambda)g_2) \geq x$ for all $\lambda \in [0,1]$.
(2) Monotonicity: if $g_1 \leq g_2$ a.s. then $\alpha(g_1) \leq \alpha(g_2)$.
(3) Homogeneity zero: $\alpha(\lambda g_1) = \alpha(g_1)$ for all $\lambda > 0$.
(4) Fatou continuity: if (g_n) is a sequence of gambles satisfying $g_n \leq 1$ and $\alpha(g_n) \geq x$ for all n, and the sequence g_n converges in probability to $g \in \mathcal{G}$, then $\alpha(g) \geq x$.
(5) Law invariance: if g_1 and g_2 share the same probability distribution, then $\alpha(g_1) = \alpha(g_2)$.
(6) Consistency with second-order stochastic dominance: if g_1 second order stochastically dominates g_2, then $\alpha(g_1) \geq \alpha(g_2)$.
(7) Arbitrage consistency: $g \geq 0$ a.s. iff $\alpha(g) = \infty$.
(8) Expectation consistency: if $\mathbf{E}[g] < 0$ then $\alpha(g) = 0$; if $\mathbf{E}[g] > 0$ then $\alpha(g) > 0$.

If we expand the domain of S and G from $\mathcal{G}$ to $\mathcal{L}$ by setting $S(g) = 0$ (respectively, $G(g) = 0$) if $\mathbf{E}[g] < 0$ and $S(g) = \infty$ (respectively, $G(g) = \infty$) if $g \geq 0$, then the S and the G indices satisfy all the criteria for desirability indices: arbitrage consistency, expectation consistency, and homogeneity zero hold by definition, law invariance and monotonicity are obvious, continuity follows from the continuity of the exponential and logarithmic functions along with the continuity of the first-order conditions for the optimal proportions of a gamble, and consistency with stochastic dominance (to all orders) is proven in Propositions 2 and 4.

What remains is showing[8] compatibility with the axiom of quasi-concavity, which is the content of Proposition 5.

Proposition 5. *The indices S and G both satisfy quasi-concavity.*

One of the implications of an index $\alpha : \mathcal{L} \to \mathbb{R}_+$ satisfying quasi-concavity, together with the homogeneity zero property, is that the following set forms a convex cone for any $x \in \mathbb{R}_+$:

$$\mathcal{C}_x = \{g \mid \alpha(g) \geq x\}.$$

Hence if one chooses a minimal level x for S or G which one demands gambles to clear before one accepts them, then any linear combination of such gambles will also clear the minimal level. This has obvious significance for the composition of portfolios and the measurement of the index levels of portfolios.

There are still further implications. Concentrating specifically on the S index, denote the convex cone defined by S and any x by

$$\mathcal{C}_x^S = \{g \mid S(g) \geq x\}.$$

From this, for any gamble $g \in \mathcal{G}$ define

$$A_x^S(g) := \inf(y \in \mathbb{R} \mid (g+y) \in \mathcal{C}_x^S). \tag{14}$$

Equivalently,

$$A_x^S(g) = \inf(y \in \mathbb{R} \mid \{S(g+y) \geq x). \tag{15}$$

Finally, we may interpret $A_x^S(g)$ by: $A_x^S(g) = y$ such that y is the unique real number such that $S(g+y) = x$, or in other words, the minimal amount of money that one would need to add to (or subtract from) g in every state such that the newly created gamble $g+y$ satisfies $S(g+y) = x$.

The same reasoning can be applied to define

$$\mathcal{C}_x^G = \{g \mid G(g) \geq x\}.$$

and then for any gamble g, to define

$$A_x^G(g) := \inf(y \in \mathbb{R} \mid (g+y) \in \mathcal{C}_x^A), \tag{16}$$

where $y = A_x^G(g)$ would be the unique real number such that $G(g+y) = x$.

[8] The authors thank their colleague Ron Peretz for his assistance in composing the proof of Propostion 5.

This idea of measuring the minimal constant amount of money to be added to a gamble g in every state to attain a particular target can be generalised beyond the operators S and G. Section 7.2 explores this concept.

7.2. **Absolute Risk Compensator.** Let $A_x^{u,w}$ be a set of indices, parametrized by x, defined as follows. Given a utility function u, a wealth level w, and a positive number $x > 0$, the value of the function $A_x^{u,w}(g)$ is the real number y that solves

$$\sup_{\alpha} \mathbf{E}[u(w + \alpha(g + y))] = u(w + x). \tag{17}$$

Using the α^* notation introduced earlier to denote $\alpha^* := \arg\max_\alpha \mathbf{E}[u(w+\alpha(g+y))]$, Equation (17) can be rewritten as $\mathbf{E}[u(w+\alpha^*(g+y))] = u(w+x)$.

We may interpret y as the minimal amount of money that an agent with utility function u and wealth level w would require to be uniformly added to the payoff of g at each state of the world to compensate him or her for accepting an optimal proportion of the modified gamble $g + y$ in lieu of taking instead the sure sum of money x.[9] For this reason, we may call $A_x^{u,w}(g)$ the *absolute risk compensator* of g (with respect to u and x).

What $A_x^{u,w}(g)$ measures is, we argue, a natural candidate for a measure of 'risk'. If we interpret x as an absolute risk-free return, then $A_x^{u,w}(g)$ as defined in Equation (17) is in a sense what an agent would demand as payment to 'neutralise' the gamble g by making the expected utility equal to the utility of the risk free return. The greater $A_x^{u,w}(g)$, the more is needed to compensate for the risk of g, and hence the greater the risk inherent in the gamble.

Compare and contrast this to certainty equivalence, i.e., the solution to $\mathbf{E}[u(w + g)] = u(w + CE(u, w, g))$, which is inherently a measure of gamble desirability, measuring the expected utility of the gamble g (relative to current wealth). The greater $CE(u, w, g)$, the greater the expected utility of g, hence the greater the desirability of g.

Note that $A_x^{u,w}(g)$ is not necessarily positive. If $A_x^{u,w}(g)$ is negative, this may be interpreted as a situation in which the gamble g is sufficiently attractive (relative to x) to induce an agent to pay money for the privilege of taking on g.

This brings us close in spirit to the accept/reject criterion of Aumann and Serrano (2008) and Hart (2011). Under the accept/reject criterion, an agent

[9] Alternatively, $A_x^{u,w}(g)$ can be interpreted as the negative value of the price of g that would make the decision maker indifferent between investing optimally in g or receiving x. We choose to present it as 'compensation' rather than 'price' to emphasise the aspect of risk that it measures.

with utility u accepts g if $\mathbf{E}[u(w+g)] > u(w)$ and rejects g if $\mathbf{E}[u(w+g)] \leq u(w)$. Under the risk compensator approach, we attain a proportional version of accept/reject by setting $x = 0$ and determining that the agent should accept g if $A_0^{u,w}(g) \leq 0$ (i.e., if the agent would be willing to pay for taking g) and reject g if if $A_0^{u,w}(g) > 0$ (i.e., if the agent would need to be compensated for the risk in g).

7.3. **Relative Risk Compensator.** The right-hand side of Equation (17) is formulated in absolute terms, i.e., x can be interpreted as the absolute amount the agent can receive in a risk-free manner. It is sometimes more natural to regard the risk free return as the fraction of the wealth that the agent can gain from a risk-free investment. We can therefore reformulate Equation (17) in relative terms, as follows.

Let $R_x^{u,w}$ be a set of indices, parametrized by x. Given a utility function u, a wealth level w, and a positive number $x > 0$, the value of the function $R_x^{u,w}$ is the real number y that solves

$$\sup_{\alpha} \mathbf{E}[u(w + \alpha w(g+y))] = u((1+x)w). \tag{18}$$

Here, y may be interpreted as the minimal amount of money that an agent with utility function u and wealth level w would require to be uniformly added to the payoff of g at each state of the world to compensate him or her for accepting an optimal proportion of the modified gamble $g+y$ in lieu of surely obtaining instead $(1+x)w$. In this case, x represents the fraction of current wealth that is attained by a risk-free return. For this reason, we may call $R_x^{u,w}(g)$ the *relative risk compensator* of g (with respect to u and x).

7.4. **Risk Compensators as Coherent Risk Measures.** In the absolute case, if we choose u to be the CARA utility $u_1(x) = -e^{-x}$, for which $A_x^{u_1,w}(g)$ is independent of the wealth level, Equation (17) becomes[10]

$$S(g+y) = x. \tag{19}$$

In the relative case, if we choose u to be the CRRA utility $u_2(x) = log(1+x)w$ for $R_x^{u,w}(g)$, Equation (18) becomes[11]

$$G(g+y) = x. \tag{20}$$

[10] The derivation of Equation (19) is as follows: by Equation (17), using the utility $u_1(x) = -e^{-x}$, one has $\mathbf{E}[e^{-\alpha^*(g+y)}] = u_1(x) = e^{-x}$, hence $\log(\mathbf{E}[e^{-\alpha^*(g+y)}]) = -x$, therefore $S(g+y) = -\log(\mathbf{E}[e^{-\alpha^*(g+y)}]) = x$.

[11] The derivation of Equation (20) is as follows: by Equation (18), using the utility $u_2(x) = \log(1+x)w$, and recalling the definition of $K(g)$, one has $\mathbf{E}[\log(w + wK(g+y)(g+y))] = u_2(x) = \log(1+x)w$, hence $\mathbf{E}[\log(1+K(g+y)(g+y))] = \log(1+x)$, therefore $e^{\mathbf{E}[\log(1+K(g+y)(g+y))]} = e^{\log(1+x)}$, from which one concludes $G(g+y) = e^{\mathbf{E}[\log(1+K(g+y)(g+y))]} - 1 = x$.

The measures defined in Equations (19) and (20) are exactly the same as those of Equations (14) and (16), respectively, and we therefore denote them by A_x^S and R_x^G, respectively.

Artzner et al (1999) proposed an axiomatic definition of a 'coherent risk measure'. A measure $\rho : \mathcal{L} \to \mathbb{R}$ is a coherent risk measure according to Artzner et al (1999) if it satisfies the axioms:

(1) Translation Invariance: For all $g \in \mathcal{G}$ and real numbers λ, $\rho(g+\lambda) = \rho(g) - \lambda$.
(2) Positive Homogeneity: For all $g \in \mathcal{G}$ and real numbers $\lambda > 0$, $\rho(\lambda g) = \lambda \rho(g)$.
(3) Monotonicity: For all $g_1, g_2 \in \mathcal{G}$ with $g_1 \leq g_2$, $\rho(g_1) \geq \rho(g_2)$. (Intuitively,)
(4) Sub-additivity: For all $g_1, g_2 \in \mathcal{G}$, $\rho(g_1 + g_2) \leq \rho(g_1) + \rho(g_2)$.

The Artzner et al (1999) axioms have intuitive justification:

(1) Translation invariance states that adding a guaranteed amount λ to a gamble reduces its risk by that guaranteed amount.
(2) Positive homogeneity states that scaling both the gains and losses by the same scalar λ multiplies risk by that amount.
(3) Monotonicity states that if gamble g_1 pays less in every state of the world than g_2 then g_1 is riskier than g_2.
(4) Sub-additivity states that a merger can never create extra risk.

Proposition 6. *For all $x > 0$, the indices A_x^S and R_x^G are coherent measures of risk, i.e., they satisfy translation invariance, sub-additivity, positive homogeneity and monotonicity.*

More about the collections of indices A_x^S and R_x^G appears in a companion paper to this one.

8. Conclusion

We have used recent ideas that have appeared in the recent literature on indices of risk (initiated by Aumann and Serrano 2008)[12], to define two indices of gambles, representing orders of desirability (as opposed to orders of riskiness). Specifically, we use a duality axiom (inspired by a similar axiom in Aumann and Serrano (2008)) to characterise axiomatically an index of desirability, denoted by S. We then use a variation on the concepts of wealth and utility uniform dominance from Hart (2011) to characterise two

[12] This literature includes the works of Foster and Hart (2009, 2013) and Hart (2011), along with many others.

indices. The first one turns to be none other than S, the index that was previously defined axiomatically. The other index, which we denote by G, is in a sense parallel to the Foster and Hart (2009) operational index of riskiness.

The main properties of our indices S and G are: first, they are compatible with the concept of stochastic dominance and its natural extension defined in the present paper, which we call *proportional stochastic dominance*. Second, the S index can be viewed as a generalised Sharpe ratio. The G index is relevant for maximising the growth path of a portfolio.

Finally, both indices satisfy the axioms of acceptability indices of Cherny and Madan (2008). As such, they define in a natural way two new indices of riskiness which are 'coherent' in the sense of Artzner et al (1999). Those new indices are the subject of a companion paper to this paper.

APPENDICES

APPENDIX A. PROPERTIES OF THE S AND G INDICES

A.1. **Units.** The index S is the certainty equivalent of the optimal investment in a gamble for a parameter-one CARA agent. Hence, its units are dollars. In contrast, the index G is the certainty equivalent of the optimal investment in a gamble for a log-utility agent, in terms of fraction of his wealth. G is therefore a unit-less pure number.

A.2. **Homogeneity zero.** As one of the main motivations for our indices is studying the relative desirability of gambles in which agents may choose their optimal proportions of investment, homogeneity zero is a natural property, i.e., for all $\beta > 0$:

(1) $S(\beta g) = S(g)$.
(2) $G(\beta g) = G(g)$.

Homogeneity zero follows directly from the definitions of the indices.

A.3. **Diversification.** One of the fundamental insights of the study of investments, both theoretically and practically, is that diversification often grants investors significant advantages over investing in only one single security. This ought to be reflected in any index of desirability of gambles.

Let g and h be identically distributed gambles (not necessarily independent), implying that $S(g) = S(h)$. Let $z = w_g g + w_h h$, where $w_g > 0$ and $w_h > 0$. Then,

(1) $S(z) \geq S(g)$.
(2) $G(z) \geq G(g)$.

Proof. Let $w = w_g/(w_g + w_h)$. From homogeneity zero, $S(wg + (1 - w)h) = S(w_g g + w_h h)$ and $S(w_g g) = S(wg)$, therefore it suffices to show that claim 1 holds for weights w and $1 - w$. Now, let $\alpha^* = \arg\max_\alpha E - e^{-\alpha g}$. Since $-e^{-x}$ concave, it follows from Jensen's inequality that

$$-Ee^{\alpha^*(-wg-(1-w)h)} \geq -wEe^{-\alpha^* g} - (1-w)Ee^{-\alpha^* h}. \tag{21}$$

where the RHS equals to $-Ee^{-\alpha^* g}$ and by definition $-Ee^{-\alpha^* g} = -e^{-S(g)}$. With respect to the LHS of (21), by definition $E - e^{\alpha^*(-wg-(1-w)h)} \leq -e^{-S(wg+(1-w)h)}$ and therefore

$$-e^{-S(wg+(1-w)h)} \geq -e^{-S(g)} \tag{22}$$

implying

$$S(wg + (1-w)h) \geq S(g). \tag{23}$$

The log function is concave as well and therefore the same reasoning works for G. ∎

A.4. **Compound Gambles.** Let g and h be two gambles and let z be the compounded gamble formed by these two gambles, by which is meant the gamble that with probability p goes on to implement gamble g with probability $1 - p$ goes on to implement h. Then,

(1) $S(z) \leq pS(g) + (1-p)S(h)$.
(2) $G(z) \leq pG(g) + (1-p)G(h)$.

Proof. The claim is a corollary of the following lemma:

Lemma 4. *For any concave function u, gambles g and h, wealth level w, and a combined gamble z taking g with probability p and h with probability $1 - p$,*

$$CE^*(u, w, z) \leq pCE^*(u, w, g) + (1-p)CE^*(u, w, h).$$

Proof. For notational compactness denote $C_k := CE(u, w, k)$ and $C^*_k := CE^*(u, w, k)$. Now, for all $\alpha > 0$

$$\begin{aligned} u(w + C_{\alpha z}) &= Eu(w + \alpha z) \\ &= pEu(w + \alpha g) + (1-p)Eu(w + \alpha h) \\ &= pu(w + C_{\alpha g}) + (1-p)u(w + C_{\alpha h}) \\ &\leq pu(w + C^*_g) + (1-p)u(w + C^*_h) \qquad (24) \end{aligned}$$

Since Equation (24) is true for every $\alpha > 0$ it is true also for the optimal α. Therefore,

$$u(w + C^*_z) \leq pu(w + C^*_g) + (1-p)u(w + C^*_h), \tag{25}$$

and by the concavity of u

$$pu(w + C^*_g) + (1-p)u(w + C^*_h) \le u(w + pC^*_g + (1-p)C^*_h). \tag{26}$$

Since u is increasing

$$C^*_z \le pC^*g + (1-p)C^*h. \tag{27}$$

∎

A.5. **Subadditivity of independent gambles.** A diversified portfolio of independent gambles is desirable. However its desirability is limited by the following:

$$S(\alpha g + (1-\alpha)h) \le S(g) + S(h)$$

Proof. To see that S is subadditive, let g be a gamble that takes the value g_i with probability p_i and let h be a gamble that takes the value h_j with probability q_j, and assume that g and h are independent. Let z be the combined gamble $z = \alpha g + (1-\alpha)h$. For all $\gamma > 0$

$$\begin{aligned} \mathbf{E}(e^{-\gamma z}) &= \sum_i p_i \sum_j q_j e^{-\gamma(\alpha g_i + (1-\alpha)h_j)} \\ &= \sum_i p_i e^{-\gamma \alpha g_i} \sum_j q_j e^{-\gamma(1-\alpha)h_j} \\ &\ge e^{-(S(g)+S(h))}. \end{aligned} \tag{28}$$

Since Equation (28) is true for all α it is also holds true for the optimal α, therefore

$$e^{-S(z)} \ge e^{-(S(g)+S(h))}, \tag{29}$$

implying

$$S(z) \le S(g) + S(h). \tag{30}$$

∎

Appendix B. The G Index as an Optimal Growth Path

The G index of Definition 1 can be given an operational interpretation: it ranks gambles according to their optimal growth path.[13]

Following a similar definition in Foster and Hart (2009), a *process* $\mathscr{G}$ is a sequence $(g_t)_{t\ge 1}$ of gambles such that $g_t \in \mathcal{G}$ for all t. The intuitive interpretation is of an agent offered a gamble g_t at each time t. We will assume that the gambles in a process are independent as random variables

[13] $K(g)$ is none other than the well known Kelly index of g (Kelly (1956)).

but not necessarily that they have identical distributions. As a special case, we may consider a constant process, which consists of a countably infinite repetition of a constant gamble g, i.e., $g_t = g$ for all t. A *bounded process* is a process $(g_t)_{t\geq 1}$ satisfying the condition that there exist $b, a \in \mathbb{R}$ such that the payoffs of g_t for all t are bounded from below by b and bounded from above by a.

A *rebalancing policy* for a process $\mathscr{G} = (g_t)_{t\geq 1}$ is a corresponding sequence $A = (\alpha_t)_{t\geq 1}$ satisfying the condition that $\alpha_t > 0$ for all t. Intuitively, a rebalancing policy is a choice of the proportion of gamble g_t that an agent chooses at time t.

For a process $\mathscr{G} = (g_t)_{t\geq 1}$ and rebalancing policy $A = (\alpha_t)_{t\geq 1}$ a *growth path* of length n with initial wealth w, denoted $P_n(\mathscr{G})$, is a sum $w + \alpha_1\overline{g}_1 + \alpha_2\overline{g}_2 + \ldots\ldots + \alpha_n\overline{g}_n$, where $\overline{g}_t$ denotes a realization of g_t. When $\mathscr{G}$ is a constant process $g_t = g$ for all t, we will write $P_n(g)$ in place of $P_n(\mathscr{G})$.

A *simple rebalancing policy* is a rebalancing policy $A = (\alpha_t)_{t\geq 1}$ satisfying the following two conditions: (a) ‘no bankruptcy’, that is, we require $P_t(\mathscr{G}) \geq 0$ for all t, hence it must be that $P_{t-1}(\mathscr{G}) + \alpha_t \min(g_t) > 0$; (b) the rebalancing multiplier α_t is calculated as $\alpha_t = \beta_t P_{t-1}(\mathscr{G})$ where $\beta_t > 0$ for all t and β_t is a function solely of the gamble at time t. As a special case, we call the simple rebalancing policy that sets $\alpha_t = K(g_t)P_{t-1}(\mathscr{G})$ the *Kelly policy* for $\mathscr{G}$

We will particularly be interested in the asymptotic growth rates of growth paths as functions of the choices of rebalancing policies. Let $\mathscr{G} = (g_t)_{t\geq 1}$ be a bounded process and let $A = (\alpha_t)_{t\geq 1}$ be a simple rebalancing policy such that $\alpha_t = \beta_t P_{t-1}(\mathscr{G})$ for all t. Then for any n, $P_n(\mathscr{G}) = w\prod_{t=1}^{n}(1 + \beta_t\overline{g}_t)$. Applying the logarithm function,

(31) $$\log(P_n(\mathscr{G})) = \log(w) + \sum_{t=1}^{n}\log(1 + \beta_t\overline{g}_t),$$

hence

(32) $$\frac{\log(P_n(\mathscr{G})) - \log(w)}{n} = \frac{\sum_{t=1}^{n}\log(1 + \beta_t\overline{g}_t)}{n}$$

and therefore

(33) $$\lim_{n\to\infty}\frac{\log(P_n(\mathscr{G}))}{n} = \lim_{n\to\infty}\frac{\sum_{t=1}^{n}\log(1 + \beta_t\overline{g}_t)}{n}.$$

Every pairing of a process $\mathscr{G}$ and a simple rebalancing policy A induces an *expected asymptotic growth rate*, which we denote as $X^{\mathscr{G},A} := \mathbf{E}\left(\lim_{n\to\infty}\frac{\log(P_n(\mathscr{G}))}{n}\right)$.

Lemma 5. *Let $\mathscr{G} = (g_t)_{t\geq 1}$ be a bounded process. Then the Kelly policy almost surely yields the highest expected asymptotic growth amongst all possible simple rebalancing policies.*

Proof. Denote

$$\overline{X}_n = \frac{\sum_{t=1}^n \log(1+\beta_t \overline{g}_t)}{n}$$

Then by the Hoeffding inequality, $\mathbb{P}(\overline{X}_n - \mathbf{E}(\overline{X}_n) \geq \varepsilon) \leq e^{-2n\varepsilon^2}$, for every $\varepsilon > 0$.

The expected value $\mathbf{E}(\overline{X}_n)$ is given by $\mathbf{E}(\overline{X}_n) = \frac{1}{n}\mathbf{E}(\sum_{t=1}^n \log(1 + \beta_t g_t)) = \frac{1}{n}\sum_{t=1}^n \mathbf{E}(\log(1+\beta_t g_t))$. The optimal value of $\mathbf{E}(\log(1+\beta_t g_t))$ at every t is attained at $\beta_t^* := K(g_t)$. From this one can conclude that almost surely the highest expected asymptotic growth is attained with the Kelly policy. ∎

Proposition 7. *Let $\mathscr{G} = (g_t)_{t\geq 1}$ and $\mathscr{H} = (h_t)_{t\geq 1}$ be two bounded processes satisfying the property that $G(g_t) > G(h_t)$ for all t. Then applying the Kelly policy for $\mathscr{G}$ guarantees almost surely that the expected asymptotic growth of $\mathscr{G}$ is greater than the expected asymptotic growth of $\mathscr{H}$.*

Proof. Choose the Kelly policies A and D for $\mathscr{G}$ and $\mathscr{H}$, i.e., the rebalancing policies $\alpha_t = \beta_t^* P_{t-1}(\mathscr{G})$ and $\delta_t = \varphi_t^* P_{t-1}(\mathscr{H})$ with $\beta_t = K(g_t)$ and $\varphi_t^* = K(h_t)$, respectively. Then by Lemma 5, with probability exponentially close to 1 the expected asymptotic growths $X^{\mathscr{G},A}$ and $X^{\mathscr{H},D}$ attain the maximal possible values for $\mathscr{G}$ and $\mathscr{H}$, respectively. Since $G(g_t) > G(h_t)$, one has $\mathbf{E}(\log(1 + \beta_t^* g_t)) > \mathbf{E}(\log(1 + \varphi_t^* g_t))$ for all t and it follows that $X^{\mathscr{G},A} > X^{\mathscr{H},D}$. ∎

We may interpret Proposition 7 in the following way: suppose that an agent is given the option of choosing, at each time t, between gamble g_t and gamble h_t, which he or she may take in any proportions. Then if the agent always chooses the gamble with higher G-index value and follows the Kelly policy, the agent is guaranteed to have higher expected overall wealth in the long run than is the gamble with lower G-index value is selected.

When processes are constant, we obtain stronger results.

Theorem 5. *If $G(g) > G(h)$ then an agent can choose a simple rebalancing policy such that*

$$\textit{Prob}(\lim_{n\to\infty}(P_n(g) > P_n(h))) = 1,$$

where g and h respectively define two constant processes.

Proof. Suppose the agent chooses a simple rebalancing policy for g that is always a constant multiple of current wealth, i.e., there is a constant α_g such that $\alpha_t = \alpha_g P_t(g)$ for all t. Working with

$$\overline{X}_n = \frac{\sum_{t=1}^{n} \log(1+\alpha_g \overline{g})}{n}$$

we can this time apply the strong law of large numbers, which implies that $\lim_{n\to\infty} \overline{X}_n = \mathbf{E}(\log(1+\alpha_g g))$ almost surely.

The expectation $\mathbf{E}(\log(1+\alpha_g g))$ is optimised by using the Kelly policy for g, i.e., $\alpha_g^* = K(g)$. Similarly, the expectation $\mathbf{E}(\log(1+\alpha_h h))$ is optimised by using the Kelly policy for h, i.e., $\alpha_h^* = K(h)$.

Since $G(g) > G(h)$, one has $\mathbf{E}(\log(1+\alpha_g^* g)) > \mathbf{E}(\log(1+\alpha_h^* h))$ Hence if the agent uses the Kelly policy for g, the expected asymptotic growth of g is almost surely greater than the expected asymptotic growth of h for any possible policy, and the result follows. ∎

APPENDIX C. AUXILIARY LEMMAS

In this section we prove some auxiliary lemmas that will be useful for proving the other claims of the paper. Denote, throughout, the CARA utility with parameter γ by u_γ and the CARA utility with parameter 1 by u_1.

Lemma 6. $CE^*(u_\gamma, w, g) = CE^*(u_1, w, g)/\gamma$.

Proof. $CE^*(u_1, w, g) = CE(u_1, w, \alpha^* g)$, where α^* is the optimal proportion of g taken by an agent with CARA 1 utility at wealth w. From the first-order condition, α^* is determined as the solution to $\mathbf{E}(e^{-\alpha g} g) = 0$.

Next consider u_γ. We wish to find the optimal proportion of g taken by an agent with CARA γ utility at wealth w, which we will denote here by β^*. This is determined as the solution to $\mathbf{E}(e^{-\gamma\beta g} g) = 0$.

Set $\beta := \alpha^*/\gamma$. Then

$$\mathbf{E}(e^{-\gamma\beta g} g) = \mathbf{E}(e^{-\gamma(\alpha^*/\gamma) g} g) = \mathbf{E}(e^{-\alpha^* g} g) = 0,$$

hence $\beta^* = \alpha^*/\gamma$. It follows that $CE^*(u_\gamma, w, g) = CE(u_\gamma, w, \beta^* g)$.

Next (recalling that we may assume that $w = 0$), note that

$$\begin{aligned} \mathbf{E}(u_\gamma(\beta^* g)) &= \mathbf{E}(-e^{-\gamma\beta^* g}) \\ &= \mathbf{E}(-e^{-\gamma\alpha^*/\gamma g}) \\ &= \mathbf{E}(-e^{-\alpha^* g}) \\ &= \mathbf{E}(u_1(\alpha^* g)). \end{aligned} \tag{34}$$

Denote $CE_1 = CE(u_1, w, \alpha^* g)$ and $CE_\gamma = CE(u_\gamma, w, \beta^* g)$. By the definition of certainty equivalence,

$$\mathbf{E}(u_1(\alpha^* g)) = u_1(CE_1) = -e^{-CE_1}, \tag{35}$$

and

$$\mathbf{E}(u_\gamma(\beta^* g)) = u_\gamma(CE_\gamma) = -e^{-\gamma CE_\gamma}. \tag{36}$$

Putting together Equations (34), (35), and (36) yields $-e^{-CE_1} = -e^{-\gamma CE_\gamma}$, from which one concludes that $CE^\gamma = CE^1/\gamma$. ∎

Lemma 7. *For all $\gamma > 0$*

$$S(g) > S(h) \iff CE^*(u_\gamma, w_g, g) > CE^*(u_\gamma, w_h, h) \tag{37}$$

for all w_g and w_h.

Proof. By definition, $S(g) > S(h) \iff CE^*(u_1, w_g, g) > CE^*(u_1, w_h, h)$. By Lemma 6, $CE^*(u_\gamma, w_g, g) = CE^*(u_1, w_g, g)/\gamma$ and $CE^*(u_\gamma, w_h, h) = CE^*(u_1, w_h, h)/\gamma$. The result follows immediately. ∎

Lemma 8 is the statement of the well-known Arrow–Pratt Theorem, which we need for Lemma 9.

Definition 5. Fix an interval I of the real line

(1) A gamble g is *admissible* for I at a wealth level w if

$$w + g \subset I.$$

(2) For a utility function u, a gamble g is **-admissible* for I and u at a wealth level w if $w + \alpha^*_{u,w} g$ is admissible for I at w, i.e.,

$$w + \alpha^*_{u,w} g \subset I.$$

Lemma 8. *Let I be an interval on the real line. Let u_i and u_j be a pair of utility functions with domain that includes I. Then the following statements are equivalent.*

(1) $u_j = \hat{f} \circ u_i$ *for some concave function* $\hat{f}$
(2) $CE(u_i, w, g) > CE(u_j, w, g)$ *for all gambles g and all wealth levels $w \in I$ such that g is admissible for I at w.*
(3) $\rho_j(w) > \rho_i(w)$ *for all w in the interior of I*

Lemma 9. *Let I be an interval on the real line. Let u_i and u_j be a pair of utility functions with domain that includes I. Then*

$$\rho_j(w) > \rho_i(w)$$

for all w in the interior of I implies that

$$CE^*(u_i, w, g) > CE^*(u_j, w, g)$$

*for all gambles g and wealth levels w such that g is *-admissible for I and u_j at w.*

Note that the Lemma does not asserts whether $\alpha_i^* \geq \alpha_j^*$ or the opposite.

Proof. Let $w \in I$ satisfy that g is *-admissible for I and both u_i and u_j at w. Let α_j^* denote the optimal proportion of g for u_j at w, and similarly α_i^* denote the optimal proportion of g for u_i at w.

Then on one hand, by definition of CE^*

$$CE^*(u_j, w, g) = CE(u_j, w, \alpha_j^* g) \tag{38}$$

and on the other hand, again by definition of CE^*

$$CE(u_i, w, \alpha_j^* g) \leq CE(u_i, w, \alpha_i^* g) = CE^*(u_i, w, g) \tag{39}$$

Since by assumption $\alpha_j^* g$ is admissible for I at w, Lemma 8 implies that

$$CE(u_i, w, \alpha_j^* g) > CE^*(u_j, w, \alpha_j^* g) \tag{40}$$

Putting together Equations (38), (39), and (40) yields $CE^*(u_i, w, g) > CE^*(u_j, w, g)$. ∎

Lemma 10. *Let u_i and u_j be a pair of utility functions. Then*

$$\rho_j(w_j) > \rho_i(w_i)$$

for all w_j, w_i implies that

$$CE^*(u_i, w_i, g) > CE^*(u_j, w_j, g)$$

for all gambles g and wealth levels w_i, w_j.

Proof. Define $\hat{u}_i(w) = u_i(w + w_i - w_j)$. Hence, $\hat{u}_i(w_j) = u_i(w_i)$. Since for any value of g, say x, $\hat{u}_i(w_j + x) = u_i(w_i + x)$, $CE^*(\hat{u}_i, w_j, g) = CE^*(u_i, w_i, g)$. In addition, it is easy to verify that $\hat{\rho}_i(w) = \rho_i(w + w_i - w_j)$. Now let I be an interval such that g is *-admissible for I and u_j at w. From Lemma 9 it follows that $CE^*(\hat{u}_i, w_j, g) > CE^*(u_j, w_j, g)$ implies that $CE^*(u_i, w_i, g) > CE^*(u_j, w_j, g)$. ∎

Aumann and Serrano (2008) prove that the Aumman–Serrano index extends stochastic dominance of the first and second order; similarly, Foster and Hart (2009) prove that the Foster–Hart index extends stochastic dominance of the first and second order. Both of these indices actually extend stochastic dominance to any order n; for completeness, we present the proof of this in the next proposition.

Proposition 8. *The Aumman-Serrano index, $AS(g)$, and the Foster–Hart measure, $FH(g)$, both extend stochastic dominance to every order n. In other words, $g_1 D_\infty g_2$ implies both $AS(g_2) > AS(g_1)$ and $FH(g_2) > FH(g_1)$.*

Proof. For any gamble g, $AS(g)$ satisfies the equation $E[e^{-g/AS(g)}] = 1$. For $\alpha > 0$, let $\phi_1(\alpha) := E[e^{-\alpha g_1}]$ and let $\phi_2(\alpha) := E[e^{-\alpha g_2}]$. By Proposition 4 of Thistle (1993), if $g_1 D_\infty g_2$ then $\phi_1(\alpha) < \phi_2(\alpha)$ for all $\alpha > 0$.

Since both $E[e^{-g_1/AS(g_1)}] = 1$ and $E[e^{-g_2/AS(g_2)}] = 1$ while $\phi_1(\alpha) < \phi_2(\alpha)$ for all $\alpha > 0$, it must be that $1/AS(g_2) < 1/AS(g_1)$, hence $AS(g_2) > AS(g_1)$.

For any gamble g, $FH(g)$ satisfies the equation $E[\log(1 + \frac{g}{FH(g)})] = 0$. Let $q := FH(g_1)$ and define the function $u_q(x) := \log(1 + \frac{x}{q})$.

Note that u_q is completely monotone. Hence $g_1 D_\infty g_2$ implies $E[u_q(g_1)] > E[u_q(g_2)]$. But since $q = FH(g_1)$, by the Foster-Hart equation one has the identity $E[u_q(g_1)] = 0$, which immediately means that $E[u_q(g_2)] = E[\log(1 + \frac{g_2}{q})] < 0$. From this one concludes that $FH(g_1) = q < FH(g_2)$. ∎

APPENDIX D. PROOFS OF PROPOSITIONS AND THEOREMS

Proof of Lemma 1. Lemma 10 claims exactly the first direction, namely, that $\rho_i(w_i) > \rho_j(w_j)$ for all w_i, w_j implies $CE^*(u_j, w_j, g) > CE^*(u_i, w_i, g)$.

In the opposite direction, assume that $i \succ j$ and assume by contradiction that there are w_i and w_j such that $\rho_j(w_j) > \rho_i(w_i)$. Without loss of generality, assume that $w^* = w_i = w_j$. By continuity of ρ_i and ρ_j there is an interval I of values of w around w^* in which $\rho_j(w) > \rho_i(w)$ for all $w \in I$.

Suppose there is a gamble g satisfying $w^* + g \in I$ such that the optimal proportion taken by u_j at w^* is 1. As $\rho_j(w) > \rho_i(w)$ for all $w \in I$, it follows from Lemma 8 that for all α such that $w^* + \alpha g \subset I$:

$$CE^*(u_i, w^*, g) \geq CE(u_i, w^*, \alpha g) > CE(u_j, w^*, \alpha g) \tag{41}$$

In particular, Equation (41) holds for $\alpha = 1$. Hence

$$CE^*(u_i, w^*, g) \geq CE(u_i, w^*, g) > CE(u_j, w^*, g) = CE^*(u_j, w^*, g),$$

contradicting the assumption that $i \succ j$.

It remains to show that such a g exists. Let g be a gamble $(\varepsilon, -\varepsilon; p, 1-p)$, with $\varepsilon > 0$ and $0 < p < 1$ arbitrary, for the moment. The first order condition for agent j, when selecting his or her optimal proportion of g at w^*, is the solution for α of

$$pu_j'(w^* + \alpha\varepsilon)\varepsilon + (1-p)u_j'(w^* - \alpha\varepsilon)(-\varepsilon) = 0. \tag{42}$$

If we set

$$p = \frac{u_j'(w^* - \varepsilon)}{u_j'(w^* + \varepsilon) + u_j'(w^* - \varepsilon)}.$$

and take $\alpha = 1$, we satisfy Equation (42). Since this is correct for any value of ε, it is true in particular for ε such that $w^* + g \subset I$. This completes the proof. ∎

Proof of Lemma 2. Let g and h be gambles. Let $u \in U_n$ be chosen arbitrarily and let β_u^* be the optimal proportion of h according to u

Suppose that g n-th order stochastically dominates with a riskless asset h. By the assumption of stochastic dominance with a riskless asset, there is an $\alpha > 0$ such that $CE(u, w, \alpha g) \geq CE(u, w, \beta_u^* h) = CE^*(u, w, h)$. From this it follows immediately that $CE^*(u, w, g) \geq CE(u, w, \alpha g) \geq CE^*(u, w, h)$, hence $CE^*(u, w, g) \geq CE^*(u, w, h)$. ∎

Proof of Lemma 3. Let $f(\cdot)$ and $g(\cdot)$ be two gambles and let $F(\cdot)$ and $G(\cdot)$ be their respective cumulative distribution functions. Following Levy (2006), one has that f stochastically dominates g to n-th order if $I_m(x) \geq 0$ for all x and all $1 \leq m \leq n$, where I_n is the $n-1$-fold integral defined by

$$I_n(x) := \int_a^x \int_a^z \ldots \int_a^v G(t) - F(t) dt dv \ldots dz,$$

(with a any point at which both $G(a) = 0$ and $F(a) = 0$, and by the assumptions in this paper we may assume that $a < 0$).

Let $\alpha f(\cdot)$ denote the gamble f scaled by some $\alpha > 0$, and then denote by $F_\alpha(\cdot)$ the cumulative distribution function of αf. Since we are assuming discrete gambles, both $F(\cdot)$ and $F_\alpha(\cdot)$ are linear step functions.

For $n = 2$, this means that both $\int_a^x F(t)\,dt$ and $\int_a^x F_\alpha(t)\,dt$ are measures of areas under step functions, appropriately scalars of each other. From this it is easy to see that

$$\int_a^x F_\alpha(t)\,dt = \alpha \int_{\alpha^{-1}a}^{\alpha^{-1}x} F(t)\,dt.$$

Similarly, for $n = 3$, areas under triangles are being measured, and

$$\int_a^x \int_a^z F_\alpha(t)\,dt\,dv = \alpha^2 \int_{\alpha^{-1}a}^{\alpha^{-1}x} \int_{\alpha^{-1}a}^{z} F(t)\,dt\,dv.$$

and in general for n,

$$\int_a^x \int_a^z \ldots \int_a^v F_\alpha(t) dt\,dv \ldots dz =$$
$$\alpha^{n-1} \int_{\alpha^{-1}a}^{\alpha^{-1}x} \int_{\alpha^{-1}a}^{z} \ldots \int_{\alpha^{-1}a}^{v} F(t) dt\,dv \ldots dz$$

Now, suppose that $\alpha f \, SD_n \, g$. Then

$$\int_a^x \int_a^z \ldots \int_a^v G(t) dt \, dv \ldots dz \geq \int_a^x \int_a^z \ldots \int_a^v F_\alpha(t) dt \, dv \ldots dz, \tag{43}$$

for all values of x.

Let $\beta > 0$ be arbitrary, and let $\gamma := \alpha\beta$. Then

$$\begin{aligned} &\int_a^x \int_a^z \ldots \int_a^v G_\beta(t) dt \, dv \ldots dz \\ &= \beta^{n-1} \int_{\beta^{-1}a}^{\beta^{-1}x} \int_{\beta^{-1}a}^{z} \ldots \int_{\beta^{-1}a}^{v} G(t) dt \, dv \ldots dz \end{aligned} \tag{44}$$

for all x while

$$\begin{aligned} &\int_a^x \int_a^z \ldots \int_a^v F_\gamma(t) dt \, dv \ldots dz \\ &= \beta^{n-1} \int_{\beta^{-1}a}^{\beta^{-1}x} \int_{\beta^{-1}a}^{z} \ldots \int_{\beta^{-1}a}^{v} F_\alpha(t) dt \, dv \ldots dz \end{aligned} \tag{45}$$

for all x.

Putting together Equations (43), (44), and (45) yields $\gamma f \, SD_n \, \beta g$. ∎

Proof of Proposition 1. This follows directly from Lemma 2, which shows that proportional stochastic dominance extends stochastic dominance with a riskless asset, and Lemma 3 shows that stochastic dominance with a riskless asset extends stochastic dominance. ∎

Proof of Proposition 2. First we show that α^* exists and that it is finite. Let g be a gamble. The first order condition for a parameter-one CARA agent is

$$\sum_i p_i e^{-\alpha g_i} g_i = 0, \tag{46}$$

where the sum is over the set of payoffs of g. It is easy to see that the derivative of the left hand side is negative, that its value at $\alpha = 0$ is positive, and that it is negative as α increases to infinity. There is therefore only one value of α that satisfies the first order condition. It follows that the S ordering is well-defined and a complete ordering.

Next, suppose that g proportional stochastically dominates h at order n. By definition, this means that $CE^*(u, w, g) \geq CE^*(u, w, h)$ for all w and all $u \in U_n$. Since the utility function $u_1(g) = -e^{-g}$ is in U_n for all values of n, if follows immediately that $g \, PSD_n \, h$ implies that $S(g) > S(h)$. ∎

Proof of Theorem 1. In one direction we have to show that S satisfies the three axioms. First, S is continuous by definition and therefore satisfies continuity. Second, if g first order stochastic dominates h, its certainty equivalent is higher, and therefore S satisfies monotonicity. Finally, to see that S satisfies duality, suppose that $i \succ j$ and $S(g) > S(h)$; we need to show that $CE^*(u_j, w, g) > CE^*(u_i, w, h)$. Since $i \succ j$, by definition $\inf_w \rho_i(w) > \sup_w \rho_j(w)$. Let γ^* be a number satisfying

$$\inf_w \rho_i(w) > \gamma^* > \sup_w \rho_j(w). \tag{47}$$

We denote by u_{γ^*} the utility function of a CARA agent with parameter γ^*. By the assumption that $S(g) > S(h)$ and Lemma 7 one has

$$CE^*(u_{\gamma^*}, w, g) > CE^*(u_{\gamma^*}, w, h). \tag{48}$$

In addition, Equation (47) and Lemma 9 imply that $CE^*(u_i, w, g) < CE^*(u_{\gamma^*}, w, g)$ and $CE^*(u_{\gamma^*}, w, h) < CE^*(u_j, w, h)$. Putting it all together yields the desired result, i.e.,

$$CE^*(u_j, w, g) > CE^*(u_i, w, h). \tag{49}$$

In the other direction, we need to show that every index Q that satisfies the three axioms is ordinally equivalent to S. Let g and h be two gambles, and assume by contradiction that although $S(g) > S(h)$, $Q(g) \leq Q(h)$. Since Q satisfies continuity and monotonicity, without loss of generality we can assume that $Q(g) < Q(h)$. Otherwise, $Q(g) = Q(h)$, and we can define another gamble $\hat{h}$ such that $S(g) > S(\hat{h})$ but $Q(g) < Q(\hat{h})$.[14]

Now, $S(g) > S(h)$ implies that $CE^*(u_1, w, g) > CE^*(u_1, w, h)$. Let u_γ be a CARA agent with parameter γ, where

$$1 < \gamma < \frac{CE^*(u_1, w, g)}{CE^*(u_1, w, h)}. \tag{50}$$

Obviously, $u_\gamma \succ u_1$. From $Q(h) > Q(g)$ and the duality axiom we have $CE^*(u_1, w, h) > CE^*(u_\gamma, w, g)$. From Lemma 6 we have $CE^*(u_\gamma, w, g) = CE^*(u_1, w, g)/\gamma > CE^*(u_1, w, h)$, where the last inequality comes from (50). Combining all together implies $CE^*(u_1, w, h) > CE^*(u_1, w, h)$ which is obviously incorrect. ∎

Proof of Proposition 3. We use the following well-known equation regarding to the exponent of a normally-distributed random variable y:

$$\mathbf{E}(e^y) = e^{\mathbf{E}y + 0.5\sigma_y^2}. \tag{51}$$

[14] Specifically, let $\hat{h}$ be a gamble that takes exactly the same values as h plus ε on each event. Since S is continuous there exists ε sufficiently small such that $S(g) > S(\hat{h})$. By monotonicity $Q(g) < Q(\hat{h})$.

It follows from this equation that the certainty equivalent of a proportion of γ of a normally distributed gamble g for a CARA agent with ARA parameter of one is

$$CE(u_1, \gamma g, 0) = \gamma\mu_g + 0.5\gamma^2\sigma_g^2. \tag{52}$$

The maximum of this expression is at $\gamma = \mu_g/\sigma_g^2$. Inserting this value in Equation (52) we get

$$CE^*(u_1, g, 0) = \frac{\mu_g^2}{2\sigma_g^2}. \tag{53}$$

Recall that the Sharpe ratio of g is defined as μ_g/γ_g. Since $S(g) = CE^*(u_1, g, 0)$ and its Sharpe ration is positive, we get that $S(g) > S(h)$ if and only if the Sharpe ratio of g is larger than that of h. ∎

Proof of Proposition 4. First we show that the value of K exists and that it is unique. Any value of K that satisfies $\sum_i p_i(1+Kg_i)^{-1} = 1$ also satisfies

$$\sum_i p_i \frac{g_i}{1+Kg_i} = 0. \tag{54}$$

The left hand side of Equation (54) is positive when $K = 0$ and negative when K gets close to $-1/\min(g)$ from left (recall that $-\frac{1}{\min(g)} > 0$).[15] Since the left hand side is (positively) monotonic, there is only one value of K that satisfies the equation. It follows that the G ordering is well-defined and a complete ordering.

Next, suppose that g proportional stochastically dominates h at order n. By definition, this means that $CE^*(u, w, g) \geq CE^*(u, w, h)$ for all w and all $u \in U_n$. Since the log utility function is in U_n for all values of n, it follows immediately that $g\, PSD_n\, h$ implies that $G(g) > G(h)$. ∎

Proof of Theorem 2. For the proof of Theorem 2, we first introduce some convenient notation and a lemma.

As before, we denote by u_γ the CARA utility function with risk aversion parameter γ. Hence u_1 is the CARA utility function of parameter 1. Following this, ρ_{u_γ} denotes the Arrow–Pratt parameter of absolute risk aversion associated with u_γ. Finally, denote $S^\gamma(h) := CE^*(u_\gamma, w, h)$ for any wealth w (so that $S^1(h) = S(h)$ in the special case of the CARA utility function of parameter 1). Using this notation, one has $S^\gamma(g) > S^\gamma(h)$ for any γ if and only if $S(g) > S(h)$, by Lemma 7.

Lemma 11. *Let g be a gamble and let $c > 0$ be a real number. Then there is a CARA parameter γ such that*

$$S^\gamma(g) = CE^*(u_\gamma, w, g) = c$$

[15] For $K \geq -\frac{1}{\min(g)}$, the function $\log(1 + K(g))$ is not well defined.

for all w.

Proof. First calculate $S(g) = CE^*(u_1, w, g)$. Since by Lemma 6, $S^\gamma(g) = CE^*(u_\gamma, w, g) = S(g)/\gamma$ for any γ, and the goal is to find γ such that $S^\gamma(g) = c$, all that needs to be done is to solve $c = S(g)/\gamma$, which yields $\gamma = S(g)/c$. ∎

Now, assume g wealth uniformly dominates h but $S(h) > S(g)$. By definition,

$$CE^*(u_1, w, h) > CE^*(u_1, w, g)$$

where $u_1(w) = -e^{-w}$.

Let $c = CE^*(u_1, w, g)$. Let $u \in U^*$ be a utility function satisfying the condition that $\rho_u(x) = \max(1/x, 1)$ for $x > 0$.[16] Since $\rho_u(x) \geq \rho_{u_1}(x)$ for every $x > 0$, by Lemma 9 one has $CE^*(u, w, g) \leq c$ for all w. On the other hand, in the interval $(1, \infty)$, the risk aversion of u is the same as that of u_1, and therefore there exists some $d > 1$ such that for w in the range (d, ∞), one has

$$CE^*(u, w, h) = CE^*(u_1, w, h) > CE^*(u_1, w, g) = c.$$

This directly contradicts $g >_{WUD} h$, since $CE^*(u, w, g) \leq c$ for all w, yet there are wealth levels at which $CE^*(u, w, h) > c$. The contradiction establishes that $g >_{WUD} h$ implies $S(g) \geq S(h)$.

Conversely, assume that $S(g) \geq S(h)$. Let $u \in U^*$ be arbitrarily chosen and suppose that for some c, the inequality $c \geq CE^*(u, w, g)$ holds for all $w > 0$. Using Lemma 11, find the CARA paramater γ such that

$$S^\gamma(g) = CE^*(u_\gamma, w, g) = c$$

for all w.

Then for all w,

$$\rho_u(w) \geq \rho_{u_\gamma}(w). \tag{55}$$

To see why, suppose instead that there is a w' such that $\rho_u(w') < \rho_{u_\gamma}(w')$. Then, since $u \in U^*$, the risk aversion of u cannot increase and hence $\rho_{u_\gamma}(w) > \rho_u(w)$ for all $w \geq w$.

Let α^*_γ be the proportion of g that is optimal for u_γ. Since u_γ satisfies CARA, α^*_γ is constant for all wealth levels, as is $CE^*(u_\gamma, w, g) = CE(u_\gamma, w, \alpha^*_\gamma g)$.

Denoting $I = (w', \infty)$, there is a sufficiently large wealth level w'' such that $w + \alpha^*_\gamma g \subset I$ for all $w > w''$. Hence by Lemma 8, $CE(u, w, \alpha^*_\gamma g) >$

[16] For instance, let $u(x) := (\log(x) - 1)/e$, for $x \leq 1$ and $u(x) := -e^{-x}$ for $x > 1$. Then $\rho_u(x) = 1/x$ for $x \leq 1$ and $\rho_u(x) = 1$ for $x > 1$. (A similar example can be found in Hart (2011)).

$CE(u_\gamma, w, \alpha^*_\gamma g)$ for all $w > w''$. From this one gets $CE^*(u, w, \alpha^*_\gamma g) \geq CE(u, w, \alpha^*_\gamma g) > CE(u_\gamma, w, \alpha^*_\gamma g)$. But by definition, $CE(u_\gamma, w, \alpha^*_\gamma g) = CE^*(u_\gamma, w, g)$, while by homogeneity zero one has $CE^*(u, w, \alpha^*_\gamma g) = CE^*(u, w, g)$. We conclude that $CE^*(u, w, g) > CE^*(u_\gamma, w, g)$ for $w > w''$, contradicting $CE^*(u_\gamma, w, g) = c \geq CE^*(u, w, g)$.

But if Equation (55) holds, then by Lemma 9, one has

$$CE^*(u_\gamma, w, h) \geq CE^*(u, w, h) \tag{56}$$

for all w. Recalling the initial assumption that $S(g) \geq S(h)$ and noting that by Lemma 7 this is equivalent to $S^\gamma(g) \geq S^\gamma(h)$ yields

$$c = S^\gamma(g) \geq S^\gamma(h) = CE^*(u_\gamma, w, h) \geq CE^*(u, w, h).$$

This is the condition for $g >_{WUD} h$. The conclusion is that $S(g) \geq S(h)$ implies $g >_{WUD} h$. ∎

Proof of Theorem 3. As in the proof of Theorem 2, let γ be the CARA parameter such that $S^\gamma(g) = CE^*(u_\gamma, w, g) = c$.

In one direction, the proof that $g >_{WBD} h$ implies $S(g) \geq S(h)$ is much the same as in the proof of Theorem 2: supposing $S(h) > S(g)$ enables the construction of a utility function u_1 such that $CE(u, w, h) = CE^*(u_1, w, h) > CE^*(u_1, w, g) = c$, for all w sufficiently large, contradicting any possibility that $g >_{WBD} h$.

Conversely, assume that $S(g) \geq S(h)$, let $u \in U^*$ be arbitrarily chosen and let

$$b := \alpha^*_\gamma L_g + \alpha^*_{h,w} M_g.$$

Suppose that for some c, the inequality $c \geq CE^*(u, w', g)$ holds for all w' such that $|w' - w| \leq b$.

We show that $\rho_u(w') \geq \rho_{u_\gamma}(w')$ for all $w' \in [w - \alpha^*_{h,w} L_h, w + \alpha^*_{h,w} M_h]$. To see this, suppose by contradiction that for some $w' < w + \alpha^*_{h,w} M_h$, one has that $\rho_{u_\gamma}(w') > \rho_u(w')$. Then by the assumption of non-increasing of ARA for all $u \in U^*$ and the fact that ρ_{u_γ} is constant, $\rho_{u_\gamma}(w'') > \rho_u(w'')$ for all $w'' \in I := [w', \infty]$.

Denote $\overline{w} := w' + \alpha^*_\gamma L_g$. By construction, $\overline{w} + \alpha^*_\gamma g \subset I$ (because $\overline{w} - \alpha^*_\gamma L_g = w'$). At the same time, $\overline{w} \in [w - b, w + b]$. Since $\rho_{u_\gamma}(w'') > \rho_u(w'')$ for all $w'' \in I := [w', \infty]$, by Lemma 9, $CE^*(u, \overline{w}, g) > CE^*(u_\gamma, \overline{w}, g) = c$. This is a contradiction to the assumption of the inequality $c \geq CE^*(u, w', g)$ for all w'' such that $|w'' - w| \leq b$.

The conclusion is that $\rho_u(w') \geq \rho_{u_\gamma}(w')$ for all $w' \in [w - \alpha^*_{h,w} L_h, w + \alpha^*_{h,w} M_h]$. Appealing again to Lemma 9, we conclude that h is *-admissible for $[w - \alpha^*_{h,w} L_h, w + \alpha^*_{h,w} M_h]$ and u_γ at w, and hence $CE^*(u, w, h) \leq$

$CE^*(u_\gamma, w, h)$. But by the assumption that $S(g) > S(h)$, one has $CE^*(u_\gamma, w, h) < CE^*(u_\gamma, w, g) = c$. Therefore, $CE^*(u, w, h) < c$. ∎

Proof of Theorem 4. First note that by Corollary 9 in Hart (2011), $\tilde{\rho}_u(w) \geq 1$ for all $u \in U^*$ and all $w > 0$. Since $\tilde{\rho}_{\log}(w) = 1$ for all $w > 0$, it follows that $\tilde{\rho}_u(w) \geq \tilde{\rho}_{\log}(w)$ for all $w > 0$.

Since for all $w > 0$, $\tilde{\rho}_u(w) \geq \tilde{\rho}_{\log}(w)$ for all $u \in U^*$, equivalently $w\rho_u(w) \geq w\rho_{log}(w)$, it follows that

$$\rho_u(w) \geq \rho_{\log}(w)$$

for all $u \in U^*$ and all $w > 0$. Lemma 9 then implies that

$$CE^*(u_{\log}, w, g) \geq CE^*(u, w, g) \tag{57}$$

for all $w > 0$, all $u \in U^*$ and all gambles g.

Suppose that $G(h) > G(g)$. Then $CE^*(u_{\log}, w, h) > CE^*(u_{\log}, w, g)$ for all $w > 0$. Let $c = CE^*(u_{\log}, w, g)$. Clearly $CE^*(u, w, g) \leq c$ for all $u \in U^*$ and for all w, but the same does not hold for h, since $CE^*(u_{\log}, w, h) > c$. In other words, $g >_{UUD} h$ is contradicted. Hence $g >_{UUD} h$ implies $G(g) \geq G(h)$.

Conversely, suppose that $G(g) \geq G(h)$. Let some w and c satisfy the property that $CE^*(u, w, g) \leq c$ for all $u \in U^*$. Then in particular $CE^*(u_{\log}, w, g) \leq c$. As $G(g) \geq G(h)$, it follows that $CE^*(u_{\log}, w, h) \leq c$ and therefore $CE^*(u, w, h) \leq c$ for all $u \in U^*$.

The conclusion is that $G(g) \geq G(h)$ implies $g >_{UUD} h$. ∎

Proof of Proposition 5. In what follows, fix u to be a utility function such that for any gamble g, the derivative $du(w + \lambda g)/d\lambda$ exists for all wealth levels w, and similarly fix a wealth level w. What will be allowed to vary will be the identity of the gamble g.

To simplify notation, denote $\rho(g) := CE(u, w, g)$ and $\rho^*(g) := CE^*(u, w, g)$ and let α^*_g denote the optimal proportion of g taken by an agent with utility u at wealth w.

Suppose that $\rho^*(g_1) > x$ and $\rho^*(g_2) > x$, for two gambles g_1 and g_2. Then, for any $\lambda \in [0, 1]$,

$$\rho^*(\lambda g_1 + (1 - \lambda)g_2) = \rho^*\left(\frac{\lambda}{\alpha^*_{g_1}}\alpha^*_{g_1}g_1 + \frac{1-\lambda}{\alpha^*_{g_2}}\alpha^*_{g_2}g_2\right) \tag{58}$$

Let $D := \frac{\lambda}{\alpha^*_{g_1}} + \frac{1-\lambda}{\alpha^*_{g_2}}$, and write $\beta := \lambda/D\alpha^*_{g_1}$, $1-\beta = (1-\lambda)/D\alpha^*_{g_2}$. Then by homogeneity zero,

$$\rho^*\left(\frac{\lambda}{\alpha^*_{g_1}}\alpha^*_{g_1}g_1 + \frac{1-\lambda}{\alpha^*_{g_2}}\alpha^*_{g_2}g_2\right) = \rho^*\left(\frac{1}{D}\frac{\lambda}{\alpha^*_{g_1}}\alpha^*_{g_1}g_1 + \frac{1}{D}\frac{(1-\lambda)}{\alpha^*_{g_2}}\alpha^*_{g_2}g_2\right) \tag{59}$$

$$\geq \rho\left(\frac{\lambda}{D\alpha^*_{g_1}}\alpha^*_{g_1}g_1 + \frac{(1-\lambda)}{D\alpha^*_{g_2}}\alpha^*_{g_2}g_2\right) \tag{60}$$

$$\geq \frac{\lambda}{D\alpha^*_{g_1}}\rho\left(\alpha^*_{g_1}g_1\right) + \frac{(1-\lambda)}{D\alpha^*_{g_2}}\rho\left(\alpha^*_{g_2}g_2\right) \tag{61}$$

$$= \frac{\lambda}{D\alpha^*_{g_1}}\rho^*\left(\alpha^*_{g_1}g_1\right) + \frac{(1-\lambda)}{D\alpha^*_{g_2}}\rho^*\left(\alpha^*_{g_2}g_2\right) \tag{62}$$

$$= \beta\rho^*(g_1) + (1-\beta)\rho^*(g_2) \tag{63}$$

$$\geq \min\{\rho^*(g_1), \rho^*(g_2)\}. \tag{64}$$

where Equation (60) follows from the fact that $\rho * (g) \geq \rho(g)$ by definition, Equation (61) from the concavity of the function ρ, and Equation (62) from the definition of ρ^* and α^*. ∎

Proof of Proposition 6.

(1) *Translation Invariance*. Let $h = g+\lambda$, and let y_0 satisfy $S(g+y_0) = x$. Let $y_1 = y_0 - \lambda$. Then

$$S(h + y_1) = S(g + \lambda + y_0 - \lambda) = S(g + y_0) = x.$$

This proves that A^S_x is translation invariant. The same reasoning holds for R^G_x.

(2) *Positive Homogeneity*. Let $h = \lambda g$, and let y_0 satisfy $S(g+y_0) = x$. Let $y_1 = \lambda y_0$. Then

$$S(h + y_1) = S(\lambda g + y_1) = S(\lambda(g + y_0)) = x,$$

with the last equality following from the zero homogeneity of S. This proves that A^S_x satisfies positive homogeneity. The same reasoning holds for R^G_x.

(3) *Monotonicity*. This follows directly from the definitions.

(4) *Sub-additivity*. By first order stochastic dominance, both A^S_x and R^S_x monotonically increase with increasing x. Recall that by Proposition 5, both S and G are quasi-concave, i.e., letting α denote either one, if $\alpha(g_1) \geq x$ and $\alpha(g_2) \geq x$ then $\alpha(\lambda g_1 + (1-\lambda)g_2) \geq x$ for all $0 \leq \lambda \leq 1$.

Now, fix x and suppose that $S(g_1+y_1)=x$, $S(g_2+y_2)=x$ and $S(g_1+g_2+y_3)=x$. Then by homogeneity zero of S,

$$S\bigg(\frac{1}{2}(g_1+y_1)+\frac{1}{2}(g_2+y_2)\bigg)=S\bigg(\frac{1}{2}(g_1+g_2+y_1+y_2)\bigg)$$
$$=S(g_1+g_2+y_1+y_2).$$

At the same time, by quasi-concavity, $S\big(\frac{1}{2}(g_1+y_1)+\frac{1}{2}(g_2+y_2)\big)\geq x$. Hence $S(g_1+g_2+y_1+y_2)\geq x$ while $S(g_1+g_2+y_3)=x$, leading to the conclusion that $y_1+y_2\geq y_3$.

The same reasoning holds for G. ∎

REFERENCES

Arrow, J. K.: Aspects of the Theory of Risk-Bearing. Yrjo Jahnssonin Saatio, Helsinki (1965)

Artzner, P., F. Delbaen, J .M. Eber, and D. Heath, Coherent Measures of Risk. Mathematical Finance 9, 203-228 (1999)

Aumann, R. J. and Serrano, R.: An Economic Index of Riskiness. J. Polit. Econ. 116, 810–836 (2008)

Cherny, A. and Madan, D.: New Measures for Performance Evaluation. Journal of Financial Studies, 22 (7), 2571–2606 (2008)

Foster, D. P. and Hart, S.: An operational measure of riskiness. J. Polit. Econ., 117, 785–814 (2009)

Foster, D. P. and Hart, S.: A wealth-requirement axiomatization of riskiness. Theor. Econ. 8, 591–620 (2013)

Hadar, J. and Russell, R. W.: Rules for Ordering Uncertain Prospects. Am. Econ. Rev. 59, 25–34 (1969)

Hanoch, G. and Levy, H.: The Efficiency Analysis of Choices Involving Risk. Rev. Econ. Stud. 36, 335–346 (1969)

Hart, S.: Comparing Risks by Acceptance and Rejection. J. Polit. Econ. 119, 617–638 (2011)

Hodges, S.: A generalization of the Sharpe ratio and its applications to valuation bounds and risk measures. Working paper, Financial Options Research Centre, University of Warwick (1998)

Kelly, J. L.: A New Interpretation of Information Rate. AT&T Technical Journal (1956)

Levy, H.: *Stochastic Dominance*, Springer Science Publications, New York, (2006)

Li, M.: On Aumann and Serrano's Economic Index of Risk. Econo. Theory, forthcoming (2013)

Machina, M. J. and Rothschild, M.: Risk. The New Palgrave Dictionary of Economics, Stockton Press, New York (1987)

Markowitz, H.: Portfolio selection. J. of Finance, 7, 77–91 (1952)
Meilijson, I. On the adjustment coefficient, drawsowns and lundberg-type bounds for random walk. Ann. Appl. Probab. 19, 1015–1025 (2009)
Michaeli, M.: Riskiness for sets of gambles. Discussion Paper 603, Center for the Study of Rationality, Hebr. Univ. Jerus. (2012)
Palacios-Huerta, I., Serrano, R. and Volij, O.: On a controversy regarding expected utility and risk aversion: Individuals accept small gambles. Manuscript, Dept. Econ., Brown Univ. (2004)
Pratt, J. W.: Risk aversion in the small and in the large. Econometrica 32, 122–136 (1964)
Rothschild, M. and Stiglitz, E. J.: Increasing risk i: a definition. J. Econ. Theory, 2, 225–243. (1970)
Rothschild, M. and Stiglitz, E. J.: Increasing risk ii: its economic consequences. J. Econ. Theory, 3, 66–84 (1971)
Schreiber, A.: Comparing local risks by acceptance and rejection. Math. Finance, forthcoming (2013)
Thistle, P. D.: Negative Moments, Risk Aversion, and Stochastic Dominance. J. Financial Quant Anal, 28, 301–311 (1993)
Zakamouline, V. and S. Koekebakker: Portfolio performance evaluation with generalized Sharpe ratios: Beyond the mean and variance. Journal of Banking and Finance, 33, 1242-1254 (2009)