Topological Conditions for Uniqueness of Equilibrium in Networks

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Abstract. Equilibrium flow in a physical network with a large number of users (e.g., transportation, communication, and computer networks) may not be unique if the costs of the network elements are not the same for all uses. Such differences among users may arise if they are not equally affected by congestion or have different intrinsic preferences. Whether or not, for all assignments of cost functions, each user's equilibrium cost is the same in all Nash equilibria can be determined from the network topology. Specifically, this paper shows that in a two-terminal network, the equilibrium costs are always unique if and only if the network is one of several simple networks or consists of several such networks connected in series. The complementary class of all two-terminal networks with multiple equilibrium costs for some assignment of (user-specific) cost functions is similarly characterized by an embedded network of a particular simple type.

Keywords: Congestion, externalities, equilibrium flow, network topology, uniqueness of equilibrium.

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1 INTRODUCTION

Different kinds of networks, such as transportation, communication, and computer networks, exhibit congestion effects whereby increased demand for certain network elements (e.g., roads, telecommunication lines, servers) tends to downgrade their performance or increase the cost of using them. In such networks, the users' decisions (e.g., their choice of routes) are interdependent in that their optimal choices (e.g., the fastest routes) depend on what the others are doing. If everyone chooses optimally, given the others' choices, then the users' choices constitute a Nash equilibrium. Even if the users are identical in all respects, due to the congestion externalities, their choices at equilibrium may differ. However, if the number of users is very large and each of them has a negligibly small effect on the others, they have equal equilibrium payoffs or costs. Moreover, the payoffs or costs in different equilibria are the same. With a heterogeneous population of users, this need not be so. As the following example shows, when users are not identical, and are differently affected by congestion, equilibrium costs may vary not only across users but also from one Nash equilibrium to another.

Example 1. A continuum of three classes of users travels on the two-terminal network shown in Figure 1(a). Each user has to choose one of the four routes connecting the users' common point of origin *o* and the common destination *d*. The cost of each route is the sum of the costs of its edges. For each user class, the cost of edge e_j is given by an increasing affine function of the fraction x of the total population with a route that includes e_j. The fraction of the population in each class and the corresponding cost functions are given in the following table, where blank cells indicate prohibitively high costs:

	Fraction of Population	Cost functions				
		e_1	e_2	e_3	e_4	e_5
Class I	4/13	$3.1 + x$	$8x$			x
Class II	5/13		x	$0.5 + x$		x
Class III	4/13			$8x$	$2.1 + x$	x

Clearly, users in each class only effectively have a choice of two routes from *o* to *d*: e_1 and e_2 e_5 for class I users, e_2 e_5 and e_3 for class II, and e_3 and e_4 e_5 for class III. (The costs of the other two routes are prohibitively high.) If all the users choose the <u>first-mentioned</u> route for their class, their choices constitute a <u>strict</u> Nash equilibrium in that each user's cost is strictly less than it would be on the alternative route. The same is true if everyone chooses the <u>second-mentioned</u> route. However, the costs in the first equilibrium (≅ 3.41, 0.77, and 2.46 for class I, II, and III users, respectively) are different from those in the second equilibrium (≅ 3.08, 0.88, and 3.02, respectively), and similarly for the average costs (≅ 2.10 in the first equilibrium and 2.22 in the second).

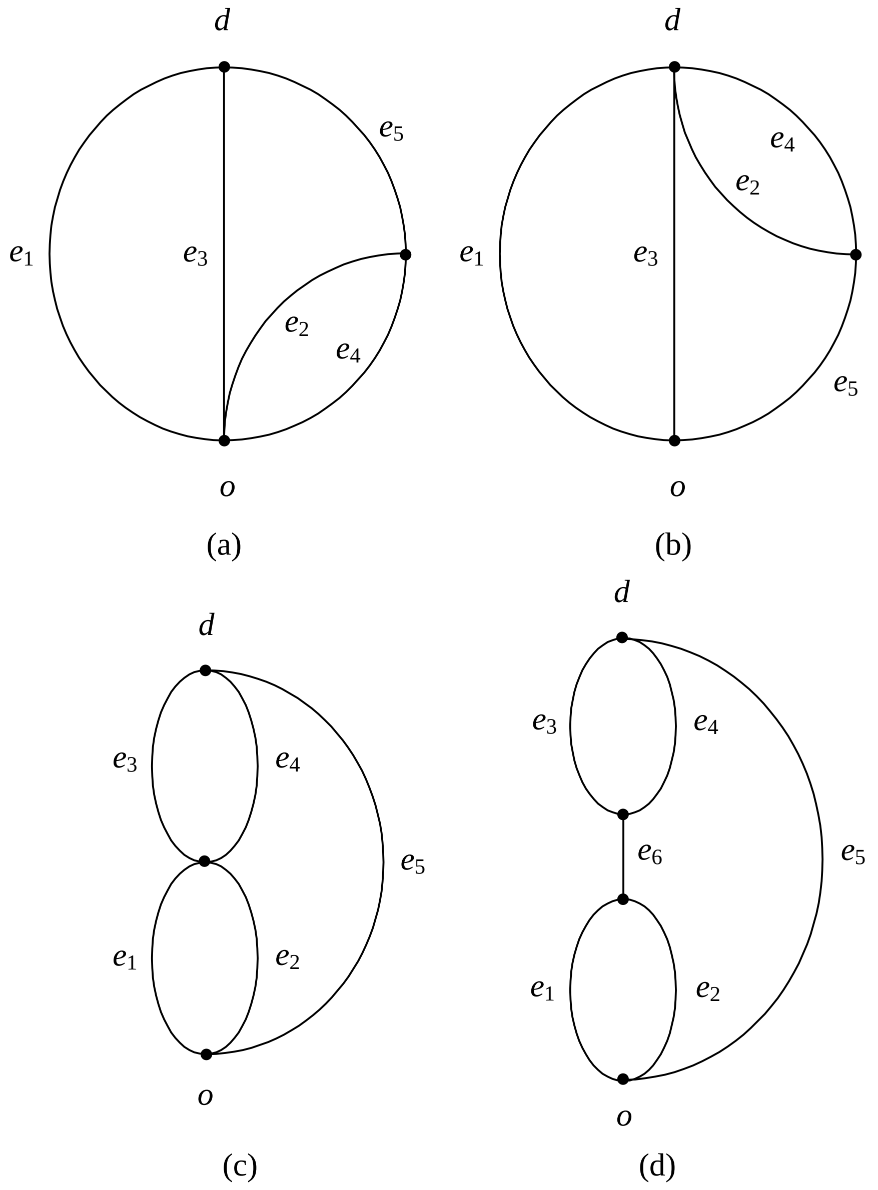

Figure 1. Two-terminal networks allowing for multiple equilibrium costs.

In Example 1, neither of the two Nash equilibria Pareto dominates the other: for class I users, the first equilibrium cost is higher, and for class II and III, the second is higher. In fact, for the network in Figure 1(a), this would be so for any assignment of cost functions. The reason is that the routes in this network, as well as in the almost identical one 1(b), are independent in the sense that each of them has an edge that is not in any other route. As shown in [8, Theorem 3], this topological property implies that, for any assignment of cost functions, all the Nash equilibria are Pareto efficient. In the other two networks in Figure 1, routes are not independent. In these networks, some Nash equilibria may be strictly Pareto dominated by others.

Example 2. A continuum of three classes of users travels from o to d on the network in Figure 1(c). The fraction of the population in each user class and the corresponding cost functions are given in the following table, where blank cells indicate prohibitively high costs:

	Fraction of Population	Cost functions				
		e_1	e_2	e_3	e_4	e_5
Class I	3/7		$6/7 + 3x$	$3x$		$6x$
Class II	2/7		$8x$		x	$6x$
Class III	2/7	x		$8x$		$6x$

Each user can only effectively choose between e_5 and a single alternative route, which is $e_2\,e_3$ for class I users, $e_2\,e_4$ for class II, and $e_1\,e_3$ for class III. There is one Nash equilibrium in which class I users take e_5 and class II and III take the respective alternative routes. In this equilibrium (which can be shown to be Pareto efficient), everyone's cost is the same, 18/7. There is another Nash equilibrium in which class I users take their alternative route $e_2\,e_3$ and class II and III take e_5. Again, the equilibrium cost is the same for all users, but this time it is higher, and equals 24/7. Any convex combination of these two equilibria (in terms of the proportion of users in each class taking each route) is also a Nash equilibrium, with costs given by the corresponding convex combination of the above costs. Thus, there is a continuum of Nash equilibria, which can be Pareto ranked since, in each equilibrium, the costs for all users are the same.

The main result of this paper is that, if the costs to users are allowed to differ, whether or not there exit some cost functions with multiple equilibrium costs for some users depends on the network topology. For example, it is shown in [7] that in a network with parallel routes (like the one in Figure 2(a)), the equilibrium costs are unique for any assignment of cost functions. It is shown below that, indeed, the same is true for all five networks in Figure 2, as well as the networks created by connecting two or more of them in series. Moreover, these are essentially the only two-terminal networks in which uniqueness of each user's equilibrium cost is guaranteed. For any other two-terminal network, it is possible to find an example with multiple equilibrium costs, very similar to Examples 1 or 2 above. Indeed, any such network has one of those in Figure 1 embedded in it, in a sense made precise below. Therefore, in a sense, the four networks in Figure 1 represent the only kinds of networks for which multiple equilibrium costs are possible. Thus, this paper gives two equivalent topological characterizations of two-terminal networks that may or may not have a multiplicity of equilibrium costs. The first directly identifies all networks with unique equilibrium costs for any assignment of cost functions, and the second all the networks in which, for some assignment of cost functions, the equilibrium costs are not unique. Moreover, the results below show that, in the first kind of networks, not only are the equilibrium costs unique but also the fraction of each class of users traversing each edge at equilibrium is generically unique. This means that, unless certain special relations exist among the cost functions, this fraction is the same in all Nash equilibria.

In this paper, networks are always assumed to be undirected, in contrast to the more common practice in the literature of assuming that edges are directed, and can be traversed in one direction only. Here, such traveling restrictions, if they exist, are considered part of the cost functions, which may assign a very high cost to one of the two directions. The merits of this approach are demonstrated by the results in this paper (and [8]). These results show that the uniqueness of the equilibria (and their Pareto efficiency) are, indeed, linked to the topology of the underlying undirected network.

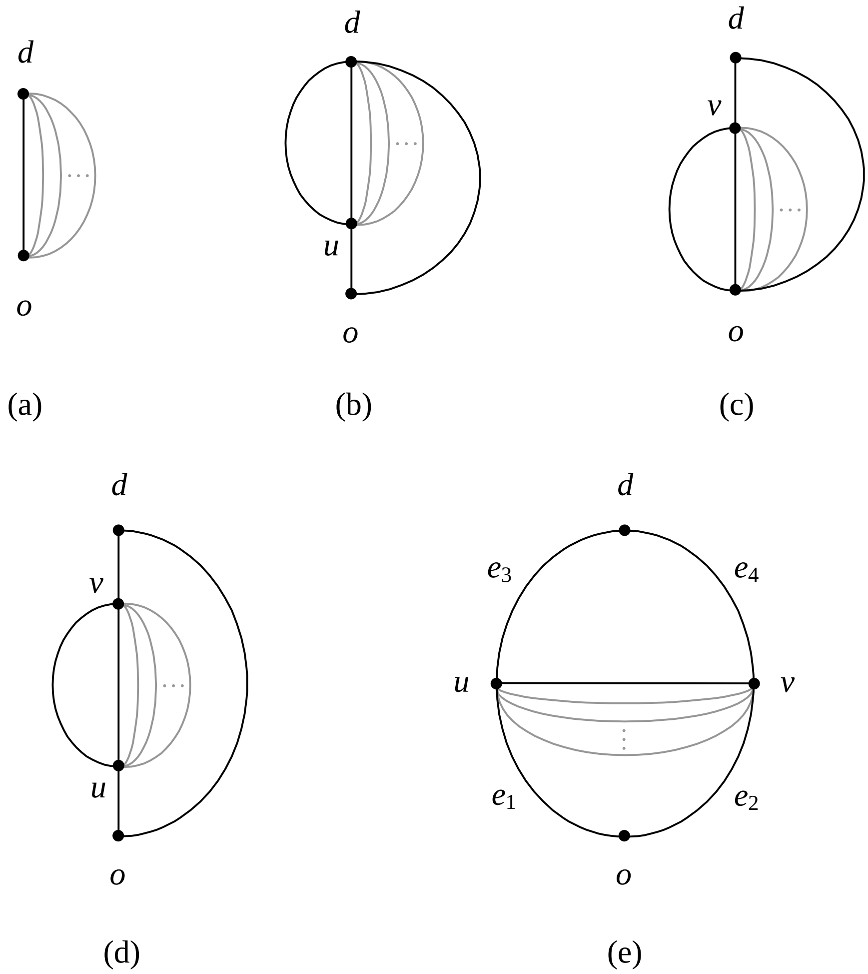

Figure 2. Two-terminal networks in which the equilibrium costs are always unique. In (a), one or more edges are connected in parallel. This network is embedded in each of the other four.

2 GRAPH-THEORETIC PRELIMINARIES

An undirected multigraph consists of a finite set of vertices together with a finite set of edges. Each edge *e* joins two distinct vertices, *u* and *v*, which are referred to as the end vertices of *e*. Thus, loops are not allowed, but more than one edge can join two vertices. An edge *e* and a vertex *v* are said to be incident with each other if *v* is an end vertex of *e*. The degree of a vertex is the number of edges incident with it. A path of length *n* is an alternating sequence $v_0\, e_1\, v_1 \;\cdots\; v_{n-1}\, e_n\, v_n$ of vertices and edges, beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it, and all the vertices (and necessarily all the edges) are distinct. The first and last vertices v_0 and v_n are called the initial and terminal vertices of the path, respectively. If these are clear from the context, the path may be written more simply as $e_1 e_2 \cdots e_n$. An arc is a path of length one, consisting of a single edge and its two end vertices. It may be viewed as a specification of a particular direction to the edge. Obviously, each edge can be directed in two ways, which differ from each other in the identity of the end vertex chosen as the initial vertex and that chosen as the terminal vertex. One, and only one, of these two arcs is part of any path that includes the edge. In this sense, any such path specifies a particular direction for the edge. The set of all arcs in a network is denoted by $\mathcal{A}$.

A two-terminal network (network, for short) is an undirected multigraph together with a distinguished ordered pair of distinct vertices, *o* (for "origin") and *d* (for "destination"), such that each vertex and each edge belong to at least one path with the initial vertex *o* and terminal vertex *d*. Any such path will be called a route. The set of all routes in a network is denoted by $\mathcal{R}$.

Two networks G' and G'' may be identified if they are isomorphic in the sense that there is a one-to-one correspondence between the vertices of G' and G'' and between their edges such that (i) the incidence relation is preserved and (ii) the origin and destination in G' are paired with the origin and destination in G'', respectively. A network G' will be said to be embedded in the wide sense in a network G'' if the latter can be obtained from the former by a sequence of operations of the following three types (see Figure 3):

1. The subdivision of an edge; i.e., its replacement by two edges with a single common end vertex.
2. The addition of a new edge joining two existing vertices.
3. The subdivision of a terminal vertex, *o* or *d*; i.e., the addition of a new edge joining that vertex to a new vertex *v*, such that at least one (and possibly all[1]) of the edges originally incident with the terminal vertex are incident with *v* instead.

[1] The special case in which *v* replaces the terminal vertex (*o* or *d*) as the end vertex of all the edges originally incident with the latter is called "terminal extension" in [8]. The qualifier "in the wide sense" used in this paper is meant to distinguish the present notion of embedding from that in [8], which does not allow general terminal subdivisions but only terminal extensions.

Two networks that can be obtained from the same network by a sequence of subdivisions of edges are said to be homeomorphic. For present purposes, such networks, which can be obtained from each other by the insertion and removal of non-terminal degree-two vertices, are close to being identical.

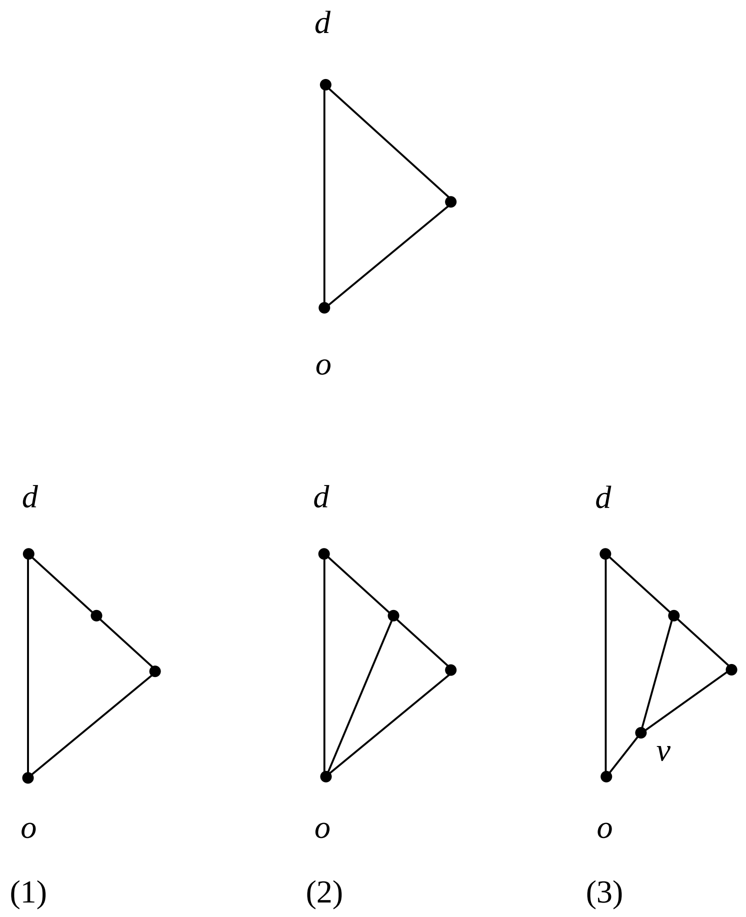

Figure 3. The upper network is embedded in the wide sense in each of the lower three, which are obtained from it by carrying out the following operations: (1) subdividing an existing edge, (2) adding a new edge, and, finally, (3) subdividing the origin.

Two networks *G′* and *G″* with the same origin–destination pair, but no other common vertices or edges, may be connected in parallel. The set of vertices in the resulting network *G* is the union of the sets of vertices in *G′* and *G″*, and similarly for the set of edges. The origin and destination in *G* are the same as in *G′* and *G″*. Two networks *G′* and *G″* with a single common vertex (and, hence, without common edges), which is the destination in *G′* and the origin in *G″*, may be connected in series. The set of vertices in the resulting network *G* is the union of the sets of vertices in *G′*

and G'', and similarly for the set of edges. The origin in G coincides with the origin in G' and the destination is the destination in G''. The connection of an arbitrary number of networks in series or in parallel is defined recursively. Each of the connected networks is embedded in the wide sense in the network resulting from their connection.

The following graph-theoretic result plays an important role in this paper. The proofs of this and of the other results in the paper are given in the Appendix.

Proposition 1. *For every network G, one, and only one, of the following conditions holds*:

(i) *G is homeomorphic to one of the networks in Figure 2 or it consists of two or more such networks connected in series.*

(ii) *One (or more) of the networks in Figure 1 is embedded in the wide sense in G.*

3 THE MODEL

The population of users is an infinite set I, endowed with a nonatomic probability measure m. The measure m assigns values between zero and one to a s-algebra of subsets of I, the measurable sets. These values are interpreted as the set sizes relative to the total population. A strategy profile is a mapping $s : I \to \mathcal{R}$ (from users to routes) such that, for each route r, the set of all users i with $s(i) = r$ is measurable. For each arc a, the measure of the set of all users i such that a is part of $s(i)$ is called the flow through a in s and is denoted by f_a. Note that each edge e is associated with a pair of arc flows, one giving the flow though e in one direction, and the other in the opposite direction. However, if all the routes in the network pass though e in the same direction, then one of these flows would always be zero, and in this case, there is no ambiguity in associating e with a single arc flow, which may be denoted by f_e.

The cost of each arc a for each user i is given by a nonnegative and strictly increasing cost function $c_a^i : [0, 1] \to [0, \infty)$. When the flow through arc a equals f_a, the cost for user i of traversing a is $c_a^i(f_a)$. Note that, for each user, each edge e is associated with a pair of cost functions, one for each direction. In each direction, the cost only depends on the flow through the edge in this direction. However, if all the routes in the network pass though e in the same direction (and, hence, the flow in the opposite direction is always zero), then only one cost function has to be associated with e for each user i, which may be denoted by c_e^i. The same would be true for all edges if passing through each edge were allowed in one direction only, as is the assumption in much of the literature (e.g., [4], [9], [10], [13], but not [3]). Clearly, this is equivalent to assigning a very high cost to the opposite direction. In this paper, for the sake of generality and simplicity of notation, the cost functions are not required to have the property that traversing each edge in a particular direction is very costly for all users.[2] However, requiring this would not affect any of the results below.

The cost of each route r for each user i is defined as the sum of the costs for user i of the arcs forming part of r. The cost thus depends on the flow through each of r's edges in the direction specified by this route. A strategy profile s is a Nash equilibrium if each route is only used by those for whom it is a least cost route. Formally, the equilibrium condition is:

$$\text{For each user } i, \quad \sum_{\substack{a \in \mathcal{A} \\ a \text{ is part of } s(i)}} c_a^i(f_a) = \min_{r \in \mathcal{R}} \sum_{\substack{a \in \mathcal{A} \\ a \text{ is part of } r}} c_a^i(f_a), \tag{1}$$

where, for each arc a, f_a is the flow through a in s. In an equilibrium s, the minimum in (1) is user i's equilibrium cost.

4 EXISTENCE AND UNIQUENESS OF EQUILIBRIUM

Under weak assumptions on the cost functions, at least one Nash equilibrium exists. Specifically, a sufficient condition for the existence of equilibrium is that, for every arc a, $c_a^i(x)$ is a continuous function of x for each user i and a measurable function of i for each $0 \le x \le 1$.[3] However, the main concern of this paper is with uniqueness. If all the users are identical, then the equilibrium itself is typically not unique. This is

[2] Other than that, the definition of cost function in this paper is standard. Note that this definition is considerably less general than in [8], where the costs are not required to be nonnegative and strictly increasing, are allowed to depend on the opposite flows as well as the flows through junctions, and are not assumed to be additively separable in the sense that the cost of each route is the sum of the costs of its arcs.

[3] The proof of this assertion, which is very similar to that of [7, Theorem 3.1], is omitted. Its validity can also be deduced from more general results, [11, Theorems 1 and 2] or [12, Theorem 1].

because, at equilibrium, any two groups of users of equal size taking different routes may interchange their choice of routes without affecting the equilibrium. However, the equilibrium flow through each arc in the network is the same in all Nash equilibria, and, therefore, each user's equilibrium cost is unique ([1]). In fact, as the following proposition shows, this result extends to the case in which the users' cost functions are only identical up to additive constants in the sense that, for each arc a and each pair of users i and i', the difference $c_a^i(x) - c_a^{i'}(x)$ is a constant that does not depend on x. For a further extension of the uniqueness result, see [2].

Proposition 2. *If the users' cost functions are identical up to additive constants, then, for every two Nash equilibria, the flow through each arc in the network in the first equilibrium is equal to that in the second, and the same is true also for each user's equilibrium cost.*

The main question this paper is concerned with is whether, for a given network, uniqueness of the equilibrium arc flows and the equilibrium costs holds for arbitrary cost functions. A network will be said to have the uniqueness property if, for any assignment of cost functions, the flow through each arc is the same in all Nash equilibria. As the following proposition shows, this property can also be defined in terms of the equilibrium costs.[4]

Proposition 3. *For every network G, the following conditions are equivalent*:

(i) *G has the uniqueness property.*

(ii) *For any assignment of cost functions, each user's equilibrium cost is the same in all Nash equilibria.*

(iii) *For any assignment of cost functions, some user's equilibrium costs are the same in all Nash equilibria.*

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[4] In the case of a homogeneous population of users (and, more generally, cost functions that are identical up to additive constants), the equilibrium costs are also unique if the cost functions are not strictly increasing but only nondecreasing. However, with a heterogeneous population of users, the assumption of strict monotonicity (which is part of the definition of cost function in Section 3) cannot be dispensed with. This is because, if only some users are not affected by congestion, changing these users' choice of routes in an equilibrium may result in another equilibrium in which the costs to the other users are different from those in the first equilibrium.

As mentioned in the Introduction, whether the uniqueness property holds for a network depends on the network topology. In [5] and [7], this property is shown to hold for any network consisting of one or more edges connected in parallel, as in Figure 2(a). Such a network will be called a parallel network. Clearly, connecting two or more networks with the uniqueness property in series results an a network that also has this property, since the users' choice of routes in each constituent network does not restrict the choices or affect the costs in the other networks. This paper's main result shows that the uniqueness property holds, in fact, for all the networks in Figure 2, and, moreover, these networks and those that are constructed by connecting several of them in series are essentially the only networks with this property. Any network homeomorphic to one of those in Figure 2 will be said to be nearly parallel. Note that a networks is nearly parallel if and only if it has a single route, two parallel routes, or can be constructed from a network with two parallel routes by adding one or more parallel paths, with the same initial and terminal vertices.

Theorem 1. *A network has the uniqueness property if and only if it is nearly parallel or it consists of two or more nearly parallel networks connected in series.*

It follows immediately from Theorem 1 and Proposition 1 that the networks without the uniqueness property are precisely those in which one of the networks in Figure 1 is embedded in the wide sense.

Corollary 1. *For every network G, there exists some assignment of cost functions with multiple equilibrium costs if and only if one of the networks in Figure 1 is embedded in the wide sense in G.*

For the network in Figure 1(a), Example 1 in the Introduction specifies cost functions with two strict Nash equilibria such that each user's equilibrium cost is not the same in both equilibria. These cost functions give rise to the same equilibrium costs in the second network in Figure 1, which differs from the first only in that the origin and destination are interchanged. For the network in Figure 1(c), Example 2 specifies cost functions with two Nash equilibria, the first of which strictly Pareto dominates the second. In that example, both equilibria are not strict: any user taking e_5 would incur the same cost on his alternative route, and vice versa. However, it is easy to modify Example 2 in such a way that the two equilibria become strict. For example, if the two cost functions of the form 8*x* are changed to 8.7 *x*, those of the form 6*x* to 6.5 *x*, and the constant 6/7 to 1.1, each user's route in the first equilibrium still has a lower cost than in the second, but, in both equilibria, the cost of the user's equilibrium route is strictly less than those of the other routes. It follows that the modified cost functions can also be used for the network in Figure 1(d). That is, two strict Nash equilibria with costs arbitrarily close to those in the previous network can be obtained simply by assigning a sufficiently low cost (e.g., a cost function of *x*/50) to edge e_6.

Another assignment of cost functions with a multiplicity (indeed, a continuum) of equilibrium costs for the network in Figure 1(d) is given in the next example. Note the very simple form of the cost functions in this example: linear, without constant terms. In Examples 1 and 2, differences among the users involve both the slopes of the cost curves, which reflect the degree by which different users are affected by congestion, and the intercepts, which represent their innate preferences. Proposition 2 shows that differences of the first kind are necessary for the existence of multiple equilibrium costs. The following example shows that differences of the second kind are not necessary.

Example 3. A continuum of three classes of users travels from *o* to *d* on the network in Figure 1(d). The fraction of the population in each user class and the corresponding cost functions are given in the following table:

	Fraction of Population	Cost functions					
		e_1	e_2	e_3	e_4	e_5	e_6
Class I	9/20	x	$8x$	$8x$	$3x$	x	x
Class II	9/20	$8x$	$3x$	x	$8x$	x	x
Class III	1/10	$8x$	x	$8x$	x	x	$2x$

In this example, there is one Nash equilibrium in which 5/18 of class I users (1/8 of the total population) take the route $e_1 e_6 e_4$, 5/18 of class II users take the route $e_2 e_6 e_3$, and all the other users take e_5. In this equilibrium, everyone's cost is the same, 0.75. There is another Nash equilibrium in which 5/36 of class I users (1/16 of the total population) take the route $e_1 e_6 e_4$, 5/36 of class II users take the route $e_2 e_6 e_3$, the rest of class I and II users take e_5, and all class III users take the route $e_2\, e_6\, e_4$. In this equilibrium, the cost for all users is 0.775.

From these examples, it is easy to construct an assignment of cost functions with multiple equilibrium costs for any given network that does not have the uniqueness property. Of course, other examples also exist. For example, in [5], such an example is given for a network in which the network in Figure 1(b) is embedded in the wide sense.

5 EQUIVALENCE OF EQUILIBRIA

The uniqueness result for a heterogeneous population of users (Theorem 1) can be taken one step further. If the network has the uniqueness property, then not only is the flow through each arc the same in all Nash equilibria but, generically, it is also made up of the same mixture of user types. While it is very easy, even for networks with the uniqueness property, to construct examples in which different types of users traverse a given arc in different equilibria, the generic uniqueness result entails that such examples depend on the existence of certain special relations among cost functions. If there is no a priori reason to assume that such relations exist, then a unique composition of user types would be expected for each arc.

The theorem below is an extension of a similar result for parallel networks ([7, Theorem 4.3]). It is based on a model very similar to that used in this special case. A partition of the population is a finite disjoint family of sets $I_1, I_2, \ldots, I_n$, the user classes, such that $m(I_m) > 0$ for all m and $\bigcup_m I_m = I$.[5] A user class is interpreted as a collection of users who are known to have the same type. For a given partition of the population and a given network G, denote by $\mathcal{G}$ the set of all assignments of continuous and strictly increasing cost functions $c_a^i : [0, 1] \to [0, \infty)$ with the property that, for every pair of users i and i' in the same class I_m, $c_a^i = c_a^{i'}$ for all arcs a. Since this property clearly implies that the mapping $i \mapsto c_a^i(x)$ is measurable for any fixed $0 \le x \le 1$, it follows from the remarks at the beginning of Section 4 that every element of $\mathcal{G}$ has a nonempty set of Nash equilibria. Two Nash equilibria s and t will be said to be equivalent if the contribution of each user class I_m to the flow through each arc a

[5] Extension to the case of an infinite family (and even a continuum) of user classes is possible. The formulation would closely follow that in [7].

is the same in s and t, i.e., the measure of the set of all users $i \in I_m$ such that a is part of $s(i)$ is equal to the measure of the set of users $i \in I_m$ such that a is part of $t(i)$. This condition clearly implies that the flow through each arc is the same in both equilibria. The distance between two elements of $\mathcal{G}$, one with cost functions c_a^i and the other with $\hat{c}_a^i$, is defined as max $|c_a^i(x) - \hat{c}_a^i(x)|$, where the maximum is taken over all users i, arcs a, and $0 \leq x \leq 1$. This defines a metric for $\mathcal{G}$.[6] In a metric space, a property is considered to be generic if it holds in an open dense set ([6, Section 8.2]). The following theorem asserts that the property that all Nash equilibria are equivalent is generic if and only if the network satisfies condition (i) in Proposition 1.

Theorem 2. *For every network G, the following conditions are equivalent*:

(i) *G is nearly parallel*, *or it consists of two or more nearly parallel networks connected in series*.

(ii) *For every partition of the population, there is an open dense set in* $\mathcal{G}$ *such that, for any assignment of cost functions that belongs to this set, every two Nash equilibria are equivalent*.

An equivalent way of stating condition (ii) is that, for every partition of the population, the set of all assignments of cost functions in $\mathcal{G}$ with two (or more) non-equivalent Nash equilibria is nowhere dense in $\mathcal{G}$.

The following corollary of Theorems 1 and 2 adds to Proposition 3.

Corollary 2. *Condition* (ii) *in Theorem 2 is equivalent to the uniqueness property*.

6 REMARKS

The results in this paper, which link network topology with the uniqueness of the equilibrium costs or the arc flows, are similar in spirit to the results in [8], which link network topology with Pareto efficiency of the equilibria. However, uniqueness and Pareto efficiency are each equivalent to a different topological property. Specifically,

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[6] It is shown in [7] that the metric space $\mathcal{G}$ is topologically complete. In other words, the metric defined above is equivalent to some complete metric for $\mathcal{G}$ (i.e., the metric topologies are the same).

a two-terminal network has the property that, for any assignment of cost functions of the form considered here, all the equilibria are Pareto efficient if and only if the network has independent routes in the sense that any route has at least one edge that is not in any other route ([8, Theorem 3]). On the other hand, independence of the routes is neither a necessary nor sufficient condition for the network to have the uniqueness property, since it holds for the first two networks in Figure 1 but not the last two, and for the first four networks in Figure 2 but not the last one.

A weaker topological property than independent routes, which holds for all the networks in Figure 1, is a series-parallel network, i.e., one that can be built from single edges by sequentially connecting networks in series or in parallel. However, since the network in Figure 2(e) is not series-parallel, even this is not a necessary (or sufficient) condition for the network to have the uniqueness property. In [8, Theorem 1], a series-parallel network is shown to be a necessary and sufficient condition for Braess's paradox never to occur, with a population of identical users. Braess's paradox is said to occur when lowering the cost of one or more arcs increases everyone's equilibrium cost. As remarked in [8], with non-identical users, Braess's paradox can occur even in a series-parallel network (but never in a network with independent routes). For instance, replacing the constant 6/7 in Example 2 with any greater number would leave only the first equilibrium mentioned, in which each user's equilibrium cost is 18/7. Replacing it with any positive constant less than 6/7 would leave only the second equilibrium, in which all the users have equilibrium costs higher than 18/7.

Another variant of Example 2 may serve to show that, even with a population of identical users, equilibrium costs may not be unique if there are certain routes that users are not allowed to take. (This is a special case of non-additively-separable costs; see [8].) It differs from the original Example 2 in that the costs of the routes $e_2\,e_3$, $e_2\,e_4$, $e_1\,e_3$, and e_5 for all users are the same as their costs for class I, class II, class III, and all users, respectively, in the original example, and continuing from e_1 to e_4 is not allowed (or is very costly). This variant of Example 2 has the same continuum of equilibria as the original example, and thus a continuum of equilibrium costs.

This paper and [8] both consider undirected networks, and view directionality, if it exists, as part of the cost functions. In a series-parallel network, edges have intrinsic directions, since all routes pass through each edge in the same direction (see

[8]). Of the networks in Figures 1 and 2, only that in 2(e) is not series-parallel, and in this network, uniqueness of the equilibrium cost does not depend on how the edges joining *u* and *v* are directed. Nevertheless, the results in this paper would not hold if edges were viewed as having predetermined directions. This is demonstrated by the two directed networks in Figure 4. The undirected version of the one in Figure 4(a) is homeomorphic to the network in Figure 2(e). Therefore, for any assignment of cost functions, the equilibrium flow through each arc is unique, and, in addition, for every partition of the population, there is an open dense set in $\mathcal{G}$ such that, for any assignment of cost functions in this set, all Nash equilibria are equivalent. The same clearly holds for the second directed network in Figure 4, in which the directed routes are essentially the same as in the first in terms of their arcs. However, this directed network cannot be constructed by connecting in series directed versions of nearly parallel networks. This shows that Theorems 1 and 2 do not hold for directed networks. The directed network in Figure 4(b) also cannot be obtained by the subdivision or addition of edges or by the subdivision of terminal vertices from any of the networks in Figure 1, if these are directed as series-parallel networks. This shows that Proposition 1 also does not hold for directed networks, which demonstrates the usefulness of the present approach of linking the uniqueness of the equilibrium costs with the topology of undirected networks.

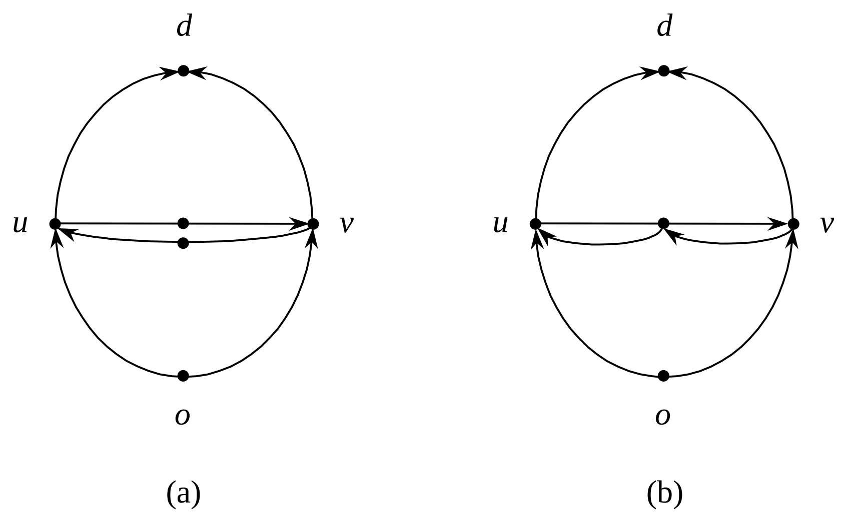

Figure 4. Directed two-terminal networks in which the equilibrium costs are always unique.

APPENDIX

The Appendix presents the proofs of the various results in the paper, as well as five lemmas required in the proofs. The term "uniqueness property" is defined in the paragraph preceding Proposition 3. The definition of nearly parallel networks is in the paragraph preceding Theorem 1.

Lemma 1. *For every network G, at least one of the two conditions in Proposition 1 holds*.

Proof. The proof proceeds by induction on the number of edges in *G*. If there is only one edge, condition (i) in Proposition 1 clearly holds. Suppose that *G* has more than one edge. The induction hypothesis will be that the assertion of the lemma holds for every network with a smaller number of edges than *G*. If *G* can be constructed by connecting two other networks in series, then, by the induction hypothesis, (i) both networks are nearly parallel or consist of several nearly parallel networks connected in series, or (ii) at least one of the two networks has one of those in Figure 1 embedded in it in the wise sense. In the first case, condition (i) in Proposition 1 holds for *G*, and in the second case, condition (ii) holds. Suppose, next, that *G* has more than one edge, but it cannot be constructed by connecting two other networks in series. Then, by [8, Lemma 1], there are two routes in *G* that do not have any common edges or vertices other than *o* and *d* (i.e., parallel routes). The edges and vertices in these two routes constitute a <u>sub-network</u> of *G* (i.e., a network obtained by deleting some of *G*'s edges and non-terminal vertices), which is homeomorphic to a two-edge parallel network. Consider the collection of <u>all</u> nearly parallel sub-networks of *G* with two or more routes. In this collection, consider a <u>maximal</u> sub-network *G′*, i.e., one which is not itself a sub-network of one of the other members of the collection. If *G′* = *G*, then *G* is nearly parallel, and the proof is complete. Suppose, then, that at least one edge *e* in *G* is not in *G′*. Let *r* be a route in *G* that includes *e*, *u* the last vertex before *e* in *r* that is also in *G′*, and *v* the first vertex after *e* in *r* that is also in *G′*. By construction, none of the edges and vertices in *r* that follow *u* and precede *v* are in *G′*. Adding these edges and vertices to *G′* results in a sub-network of *G* that is homeomorphic to a network *G″* obtained by adding a <u>single</u> edge to *G′*. The assumed maximality of *G′* implies that *G″* is not nearly parallel. Therefore, to complete the proof of the lemma, it suffices to establish the following.

CLAIM. *Let G'' be a network resulting from adding a single edge to a nearly parallel network G' with two or more routes. Then, one of the networks in Figure 1 is embedded in G'' in the wide sense or G'' is nearly parallel.*

The proof of the claim involves checking five cases, (a) through (e). In each case, *G'* is assumed to be homeomorphic to the corresponding network in Figure 2.

CASE (a). If the edge added to *G'* has the end vertices *o* and *d*, then *G''* is nearly parallel (specifically, homeomorphic to a parallel network). If at least one end vertex is not *o* or *d*, then, depending on whether the network *G'* has (i) only two routes or (ii) three or more routes, the network *G''* is (i) homeomorphic to one of the networks in Figures 2(b)–2(e), or (ii) homeomorphic to a network obtained by adding one or more edges joining *o* and *d* to one of these four networks. In the first case, *G''* is nearly parallel, and in the second, one of the networks in Figures 1(a) and 1(b) is embedded in it in the wide sense.

CASE (b). In this case, there is a unique non-terminal vertex *u* in *G'* of degree three or more, and a unique route *r* <u>not</u> containing *u*. If the end vertices of the edge added to *G'* are both in *r* (possibly coinciding with its initial or terminal vertices *o* or *d*), then the network in Figure 1(b) is embedded in the wide sense in *G''*. If only one of the two end vertices is in *r*, and this vertex is <u>not</u> *d*, then the other end vertex is in some route containing *u*. Depending on whether that vertex follows, coincides with, or precedes *u*, the network in Figure 1(b), 1(c), or 1(d), respectively, is embedded in the wide sense in *G''*. Exactly the same three possibilities exist if both end vertices of the added edge are in routes containing *u*, and at least one of them precedes *u*. If one end vertex coincides with *u* and the other one follows it, then, depending on where the latter vertex lies, *G''* is nearly parallel or has the network in Figure 1(a) embedded in the wide sense in it. Finally, if both end vertices follow *u*, then the network in Figure 1(b) is embedded in the wide sense in *G''*.

CASE (c). This case is very similar to the previous one, since the network in Figure 2(c) is obtained from that in 2(b) by interchanging *o* and *d*.

CASE (d). In this case, there are two non-terminal vertices in *G'* of degree three or more, *u* and *v*, and a unique route *r* not containing *u* or *v*. The analysis of the present case is identical verbatim to that of Case (b) except for the final sentence, which has to be modified as follows. If both end vertices of the edge added to *G'* follow *u*, then, depending on whether at least one of them precedes *v*, one of them coincides with *v* and the other one follows it, or both of them follow *v*, the network in Figure 1(b), 1(c), or 1(d), respectively, is embedded in the wide sense in *G''*.

CASE (e). The network in Figure 2(e) can be obtained from that in 2(b) by subdividing *d*, and from that in 2(c) by subdividing *o*. Therefore, it is not difficult to see that a network homeomorphic to *G''* can be obtained from one homeomorphic to either 2(b) or 2(c) by the addition of an edge followed by the subdivision of a terminal vertex. Clearly, if the network obtained in the interim stage, after the edge addition and before the terminal subdivision, is also homeomorphic to one of the networks in Figures 2(b) and 2(c), then *G''* is homeomorphic to the one in Figure 2(e). If the network obtained in the interim stage is not homeomorphic to one of these two, then it follows from the analysis of Cases (b) and (c) that one of the networks in Figure 1 is embedded in the wide sense in it, and in this case, the same is true for *G''*. ■

Lemma 2. *The uniqueness property holds for a network G if and only if it holds for every network homeomorphic to G, and in this case, it holds for any network obtained from G by removal of a single edge. If G can be constructed by connecting two other networks G' and G'' in series, then the uniqueness property holds for G if and only if it holds for both G' and G''.*

Proof. To prove the first part of the lemma, it clearly suffices to consider a network *G'* obtained from *G* by either removal or subdivision of a single edge *e*. In the former case, *G'* clearly has the uniqueness property if *G* has it, since removing an edge is equivalent to forcing its costs to be prohibitively high. In the latter case, only the <u>sum</u> of the costs of the two parts of *e* matters, since any route in *G'* containing one of them also contains the other. Therefore, given a cost function for *e* in a particular direction, subdividing this edge and assigning half the original cost function to each of its two parts has no effect on the routes' costs. It is, therefore, clear that the uniqueness property holds for *G'* if and only if it holds for *G*.

If *G* results from connecting two networks *G'* and *G''* in series, then there is an obvious one-to-one correspondence between the set of all routes in *G* and the set of all pairs consisting of a route in *G'* and a route in *G''*. Hence, there is a one-to-one correspondence between the set of all strategy profiles s in *G* and the set of all pairs consisting of a strategy profile s' in *G'* and a strategy profile s'' in *G''*. There is also a one-to-one correspondence, defined by restriction, between the set of all assignments of cost functions for G and the set of all pairs consisting of an assignment of cost functions for G' and an assignment of cost functions for G''. It is not difficult to see that a strategy profile s in G is a Nash equilibrium with respect to the first assignment if and only if the corresponding strategy profiles s' and s'' are both Nash equilibria with respect to the corresponding assignments of cost functions for G' and G''. If G does not have the uniqueness property, then there is an assignment of cost functions for G with two Nash equilibria s and t such that the flow through some arc a in s is different from that in t. If a is in G', say, then these arc flows are the same as in s' and t', respectively, and therefore G' does not have the uniqueness property. Conversely, if G' does not have the uniqueness property, then any assignment of cost functions for G' with two Nash equilibria that have different arc flows, and any assignment of cost functions for G'' with at least one Nash equilibrium, together define an assignment of cost functions for G for which the equilibrium arc flows are not unique. ■

Lemma 3. *The uniqueness property holds for all the networks in Figure 2.*

Proof. To prove that the network in Figure 2(a) has the uniqueness property, it suffices to show that those in 2(b) and 2(c) have this property. This is because, if the first network has more than one edge, connecting it in series with a network with a single edge creates a network that can also be obtained from that in Figure 2(b) or that in 2(c) by the removal of the edge joining o and d. Therefore, it follows from Lemma 2 that if the last two networks have the uniqueness property, so does the network in Figure 2(a). To prove that these two networks do, indeed, have the uniqueness property, it suffices to show that the one in 2(e) has it. This is because a network with a single edge connected in series with the network in Figure 2(b) or in 2(c) can be obtained by the removal of one of the edges incident with the origin or the destination, respectively, in 2(e). Finally, the network in Figure 2(d) has the uniqueness property if and only if the one in 2(c) has it. This is because, in the former network, the edge preceding u and the edge following v together affect route costs in the same way as the single edge following v does in the latter. In conclusion, it suffices to prove that the uniqueness property holds for the network in Figure 2(e).

Let s and $\hat{s}$ be two Nash equilibria with respect to the same assignment of cost functions for the network in Figure 2(e). For each arc a, let f_a be the flow through a

in s, $\hat{f}_a$ the flow in $\hat{s}$, and $\Delta f_a = f_a - \hat{f}_a$. Clearly, $\Delta f_{e_1} + \Delta f_{e_2} = \Delta f_{e_3} + \Delta f_{e_4} = 0$. It has to be shown that $\Delta f_a = 0$ for all arcs a.

CLAIM 1. *If* $\Delta f_{e_1} \geq 0 \geq \Delta f_{e_3}$, *then* $\Delta f_a = 0$ *for all arcs* a.

Suppose that the assumption of the claim holds, or equivalently $\Delta f_{e_2} \leq 0 \leq \Delta f_{e_4}$. Let $\mathcal{A}_1$ be the set of all arcs with the initial vertex u and terminal vertex v, and $\mathcal{A}_2$ the set of all arcs with the initial vertex v and terminal vertex u. Let $\mathcal{R}_1^+$ be the set of all routes in G containing some arc $a \in \mathcal{A}_1$ with $\Delta f_a > 0$, and $\mathcal{R}_2^-$ the set of all routes containing some arc $a \in \mathcal{A}_2$ with $\Delta f_a < 0$. For each user i, the cost of the equilibrium route $s(i)$ is less than or equal to the cost of any other route. Therefore, if some arc $a \in \mathcal{A}_1$ is part of $s(i)$, then $c^i_{e_1}(f_{e_1}) + c^i_a(f_a) \leq c^i_{e_2}(f_{e_2})$, $c^i_a(f_a) + c^i_{e_4}(f_{e_4}) \leq c^i_{e_3}(f_{e_3})$, and $c^i_a(f_a) \leq c^i_{a'}(f_{a'})$ for all $a' \in \mathcal{A}_1$. If, moreover, $s(i) \in \mathcal{R}_1^+$ (i.e., $\Delta f_a > 0$), then it follows from the assumption $\Delta f_{e_1}, \Delta f_{e_4} \geq 0 \geq \Delta f_{e_2}, \Delta f_{e_3}$ that $c^i_{e_1}(\hat{f}_{e_1}) + c^i_a(\hat{f}_a) < c^i_{e_2}(\hat{f}_{e_2})$, $c^i_a(\hat{f}_a) + c^i_{e_4}(\hat{f}_{e_4}) < c^i_{e_3}(\hat{f}_{e_3})$, and $c^i_a(\hat{f}_a) < c^i_{a'}(\hat{f}_{a'})$ for all $a' \in \mathcal{A}_1$ with $\Delta f_{a'} \leq 0$. In this case, the route $\hat{s}(i)$ <u>cannot</u> include the edge e_2, the edge e_3, or any arc $a' \in \mathcal{A}_1$ with $\Delta f_{a'} \leq 0$ (since less costly alternatives exist, and $\hat{s}$ is an equilibrium). This proves that, for all users i with $s(i) \in \mathcal{R}_1^+$, also $\hat{s}(i) \in \mathcal{R}_1^+$. Therefore, if $\mathcal{R}_1^+$ is not empty, there must be some $a \in \mathcal{A}_1$ with $\Delta f_a > 0$ such that the measure of the set of all users i for whom a is part of $s(i)$ is less than or equal to the measure of the set of users i for whom a is a part of $\hat{s}(i)$. However, this implies $f_a \leq \hat{f}_a$, which contradicts the assumption $\Delta f_a > 0$. This contradiction proves that $\mathcal{R}_1^+$ is empty. A very similar argument shows that $\mathcal{R}_2^-$ is empty. It follows that the total flow in all the arcs in $\mathcal{A}_1$ minus the total flow in all the arcs in $\mathcal{A}_2$ (i.e., the net flow from u to v) in s is less than or equal to that in $\hat{s}$, which implies $\Delta f_{e_1} - \Delta f_{e_3} \leq 0$. In addition, $\Delta f_{e_1} = \Delta f_{e_3}$ if and only if $\Delta f_a = 0$ for all arcs a in $\mathcal{A}_1$ and $\mathcal{A}_2$. Since, by assumption, $\Delta f_{e_1} \geq \Delta f_{e_3}$, this proves that $\Delta f_a = 0$ for all arcs a in the network.

CLAIM 2. *If* $\Delta f_{e_1} > 0$, *then* $\Delta f_{e_3} \leq 0$.

This will be proved by assuming that $\Delta f_{e_1}, \Delta f_{e_3} > 0$ (and, hence, $\Delta f_{e_2}, \Delta f_{e_4} < 0$), and showing that this assumption leads to a contradiction. Let $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{R}_1^+$, and $\mathcal{R}_2^-$ be as in the proof of Claim 1, I_s the set of all users i such that e_4 is in $s(i)$ but not in $\hat{s}(i)$, and $I_{\hat{s}}$ the set of all users i such that e_4 is in $\hat{s}(i)$ but not in $s(i)$. The difference between the measures of the last two sets equals Δf_{e_4} (i.e., $\Delta f_{e_4} = m(I_s) - m(I_{\hat{s}})$).

Consider any user $i \in I_{\hat{s}}$ such that some arc $a \in \mathcal{A}_1$ is part of $\hat{s}(i)$. Since $\hat{s}$ is an equilibrium, $c_a^i(\hat{f}_a) + c_{e_4}^i(\hat{f}_{e_4}) \leq c_{e_3}^i(\hat{f}_{e_3})$. Since s is an equilibrium and e_4 is not in $s(i)$, $c_{e_3}^i(f_{e_3}) \leq c_a^i(f_a) + c_{e_4}^i(f_{e_4})$. By nonnegativity and strict monotonicity of the cost functions and the assumption $\Delta f_{e_3} > 0 > \Delta f_{e_4}$, this implies that $c_{e_4}^i(f_{e_4}) < c_{e_3}^i(f_{e_3})$ and $c_a^i(\hat{f}_a) < c_a^i(f_a)$. Since e_4 is <u>not</u> in $s(i)$, the former inequality implies $s(i) = e_1\, e_3$. The latter inequality implies $\Delta f_a > 0$, and, hence, $\hat{s}(i) \in \mathcal{R}_1^+$. Consider now any user $i \in I_{\hat{s}}$ such that $\hat{s}(i) = e_2\, e_4$. Since $\hat{s}$ is an equilibrium, (i) $c_{e_2}^i(\hat{f}_{e_2}) + c_{e_4}^i(\hat{f}_{e_4}) \leq c_{e_1}^i(\hat{f}_{e_1}) + c_{e_3}^i(\hat{f}_{e_3})$ and (ii) $c_{e_4}^i(\hat{f}_{e_4}) \leq c_a^i(\hat{f}_a) + c_{e_3}^i(\hat{f}_{e_3})$ for all $a \in \mathcal{A}_2$. Since $\Delta f_{e_2}, \Delta f_{e_4} < 0 < \Delta f_{e_1}, \Delta f_{e_3}$, (i) implies $c_{e_2}^i(f_{e_2}) + c_{e_4}^i(f_{e_4}) < c_{e_1}^i(f_{e_1}) + c_{e_3}^i(f_{e_3})$, from which it follows that $s(i) \neq e_1\, e_3$. Since, by assumption, $s(i)$ does not include e_4, there must be some $a \in \mathcal{A}_2$ with $c_a^i(f_a) + c_{e_3}^i(f_{e_3}) \leq c_{e_4}^i(f_{e_4})$ that is part of $s(i)$. Since $\Delta f_{e_3} > 0 > \Delta f_{e_4}$, $c_a^i(f_a) + c_{e_3}^i(\hat{f}_{e_3}) < c_{e_4}^i(\hat{f}_{e_4})$. It follows, by comparison with (ii) above, that $\Delta f_a < 0$, and, hence, $s(i) \in \mathcal{R}_2^-$. Together, this and the previous conclusion prove that $I_{\hat{s}}$ decomposes into two disjoint sets: The set $I_{\hat{s}1}$ of all users i with $\hat{s}(i) \in \mathcal{R}_1^+$ and $s(i) = e_1\, e_3$, and the set $I_{\hat{s}2}$ of all users i with $\hat{s}(i) = e_2\, e_4$ and $s(i) \in \mathcal{R}_2^-$. Hence, $m(I_{\hat{s}}) = m(I_{\hat{s}1}) + m(I_{\hat{s}2})$. Let I_{s1} and I_{s2} be the subsets of I_s defined in a similar manner to $I_{\hat{s}1}$ and $I_{\hat{s}2}$ but with s and $\hat{s}$ interchanged. Since these sets are clearly disjoint, $m(I_{s1}) + m(I_{s2}) \leq m(I_s)$. Therefore, $[m(I_{s1}) - m(I_{\hat{s}1})] + [m(I_{s2}) - m(I_{\hat{s}2})] \leq m(I_s) - m(I_{\hat{s}}) = \Delta f_{e_4} < 0$, which implies that $m(I_{s1}) < m(I_{\hat{s}1})$ or $m(I_{s2}) < m(I_{\hat{s}2})$. However, as shown below, each of these two inequalities leads to a contradiction.

Suppose that $m(I_{s1}) < m(I_{\hat{s}1})$. This means that there are more users i with $\hat{s}(i) \in \mathcal{R}_1^+$ and $s(i) = e_1\, e_3$ than users i with $s(i) \in \mathcal{R}_1^+$ and $\hat{s}(i) = e_1\, e_3$. By definition of $\mathcal{R}_1^+$, for every $r \in \mathcal{R}_1^+$, there are more users i with $s(i) = r$ than users i with $\hat{s}(i) = r$. Therefore, there are more users i with $s(i) \in \mathcal{R}_1^+$ and $\hat{s}(i) \notin \mathcal{R}_1^+$ than users i with $\hat{s}(i) \in \mathcal{R}_1^+$ and $s(i) \notin \mathcal{R}_1^+$. Since the latter kind of users includes all those for whom $\hat{s}(i) \in \mathcal{R}_1^+$ and $s(i) = e_1\, e_3$, it follows from the assumption at the beginning of this paragraph that some users i exist with $s(i) \in \mathcal{R}_1^+$ and $\hat{s}(i) \notin \mathcal{R}_1^+$ for whom $\hat{s}(i) \neq e_1 e_3$. For each such user i, there is an arc $a \in \mathcal{A}_1$ with $\Delta f_a > 0$ which is part of $s(i)$. Since s is an equilibrium, $c_a^i(\hat{f}_a) < c_a^i(f_a) \leq c_{a'}^i(f_{a'}) \leq c_{a'}^i(\hat{f}_{a'})$ for every arc $a' \in \mathcal{A}_1$ with $\Delta f_{a'} \leq 0$, which implies that such an arc a' cannot be part of $\hat{s}(i)$. Therefore, <u>none</u> of the arcs in $\mathcal{A}_1$ is part of $\hat{s}(i)$. Since, also, $\hat{s}(i) \neq e_1 e_3$, the route $\hat{s}(i)$ must begin with e_2. Since s and $\hat{s}$ are equilibria, the inequalities $c_{e_1}^i(f_{e_1}) + c_a^i(f_a) \leq c_{e_2}^i(f_{e_2})$ and $c_{e_2}^i(\hat{f}_{e_2}) \leq c_{e_1}^i(\hat{f}_{e_1}) + c_a^i(\hat{f}_a)$ hold. However, since the cost functions are strictly increasing and $\Delta f_{e_1}, \Delta f_a > 0 > \Delta f_{e_2}$, these inequalities contradict each other. A similar contradiction is reached if it is assumed that $m(I_{s2}) < m(I_{\hat{s}2})$. In this case, there are more users i with $s(i) \in \mathcal{R}_2^-$ and $\hat{s}(i) = e_2 e_4$ than users i with $\hat{s}(i) \in \mathcal{R}_2^-$ and $s(i) = e_2 e_4$. Since, for every $r \in \mathcal{R}_2^-$, there are more users i with $\hat{s}(i) = r$ than users i with $s(i) = r$, this implies that some users i exist with $\hat{s}(i) \in \mathcal{R}_2^-$ and $s(i) \notin \mathcal{R}_2^-$ for whom $s(i) \neq e_2 e_4$. For each such user i, there is an arc $a \in \mathcal{A}_2$ with $\Delta f_a < 0$ which is part of $\hat{s}(i)$. Since $\hat{s}$ is an equilibrium, $c_a^i(f_a) < c_a^i(\hat{f}_a) \leq c_{a'}^i(\hat{f}_{a'}) \leq c_{a'}^i(f_{a'})$ for every arc $a' \in \mathcal{A}_2$ with $\Delta f_{a'} \geq 0$, which implies that such an arc a' cannot be part of $s(i)$. Together with the assumption concerning i, this implies that $s(i)$ must begin with e_1. Since s and $\hat{s}$ are equilibria, the inequalities $c_{e_1}^i(f_{e_1}) \leq c_{e_2}^i(f_{e_2}) + c_a^i(f_a)$ and $c_{e_2}^i(\hat{f}_{e_2}) + c_a^i(\hat{f}_a) \leq c_{e_1}^i(\hat{f}_{e_1})$ hold. However, since $\Delta f_{e_1} > 0 > \Delta f_{e_2}, \Delta f_a$, these inequalities contradict each other. This completes the proof of Claim 2.

CLAIM 3. $\Delta f_a = 0$ *for all arcs* a.

This is established in Claim 1 under the assumption that $\Delta f_{e_1} \geq 0 \geq \Delta f_{e_3}$. By symmetry, this also holds if $\Delta f_{e_1} \leq 0 \leq \Delta f_{e_3}$. By Claim 2, Δf_{e_1} and Δf_{e_3} cannot both be (strictly) positive. By symmetry, they cannot both be negative, either. Therefore, the claim <u>always</u> holds. ∎

Lemma 4. *For each of the networks in Figure 1, there is an assignment of cost functions with two strict Nash equilibria* s *and* t *such that each user's equilibrium cost in* s *is different from that in* t.

Proof. The cost functions given in Example 1 specify such an assignment for the networks in Figures 1(a) and 1(b). The modified version of Example 2 given in the paragraph that follows Corollary 1 (Section 4) specifies such assignments for the networks in Figures 1(c) and 1(d). ■

Lemma 5. *Let G be a network for which there exits an assignment of cost functions as in Lemma 4. Then a similar assignment exists for any network G' in which G is embedded in the wide sense. That is, there is an assignment of cost functions for G' with two strict Nash equilibria* s' *and* t' *such that each user's equilibrium cost in* s' *is different from that in* t'.

Proof. Since the definition of embedding in the wide sense is recursive, it suffices to consider the case in which *G'* is obtained from *G* by (1) the subdivision of an edge, (2) the addition of an edge, or (3) the subdivision of a terminal vertex. In the first two cases, the conclusion is almost obvious. In case (1), the assignment of cost functions chosen for *G'* has to be the same as for *G* except that, for each user, the cost in each direction of the edge that was subdivided is equally divided between its two parts. In case (2), the cost for each user of the added edge has to be set higher than the user's equilibrium costs in s and in t. It remains to consider case (3). Terminal subdivision adds an edge, incident with the origin or the destination, which some users have to traverse in order to start or finish their equilibrium route in s or in t. The cost of this edge for each user has to be set sufficiently low so that, in both s and t, it is less than (i) the difference between the cost of the equilibrium route and that of any other route (since the equilibria are strict, this difference is greater than zero) and (ii) the difference between the user's equilibrium costs in s and t. ■

Proof of Proposition 1. By Lemma 1, for every network *G*, at least one of the two conditions in the proposition holds. By Lemmas 2 and 3, condition (i) implies that *G* has the uniqueness property. By Lemmas 4 and 5, condition (ii) implies that there is an assignment of cost functions for *G* with two strict Nash equilibria such that each user's equilibrium cost in one equilibrium is different from that in the other. This clearly implies that *G* does not have the uniqueness property (since this property would imply that the cost of each route for each user is the same in both equilibria). Therefore, conditions (i) and (ii) cannot both hold for *G*. ■

Proof of Proposition 2. Suppose that the users' cost functions are identical up to additive constants, and let i_0 be one of the users. Then,

$$c_a^i(x) - c_a^i(y) = c_a^{i_0}(x) - c_a^{i_0}(y) \qquad (2)$$

for all users i, arcs a, and $0 \le x, y \le 1$. Let s and $\hat{s}$ be two Nash equilibria, and, for each arc a, let f_a and $\hat{f}_a$ be the flow through a in s and in $\hat{s}$, respectively. For every user i and route r, define $s_r(i)$ as 1 or 0 according to whether r is equal to or different from $s(i)$, respectively. Define $\hat{s}_r(i)$ in a similar manner. Since s is a Nash equilibrium, it follows from (1) that, for all users i,

$$\sum_{r \in \mathcal{R}} \Big[\sum_{\substack{a \in \mathcal{A} \\ a \text{ is part of } r}} c_a^i(f_a) \Big] (s_r(i) - \hat{s}_r(i)) \le 0.$$

Since $\hat{s}$ is a Nash equilibrium, similar inequalities hold with s and $\hat{s}$ interchanged and f_a replaced by $\hat{f}_a$. Therefore, for all users i,

$$\sum_{r \in \mathcal{R}} \Big[\sum_{\substack{a \in \mathcal{A} \\ a \text{ is part of } r}} (c_a^i(f_a) - c_a^i(\hat{f}_a)) \Big] (s_r(i) - \hat{s}_r(i)) \le 0.$$

Changing the order of summation and using (2) gives

$$\sum_{a \in \mathcal{A}} \Big[\sum_{\substack{r \in \mathcal{R} \\ a \text{ is part of } r}} (s_r(i) - \hat{s}_r(i)) \Big] (c_a^{i_0}(f_a) - c_a^{i_0}(\hat{f}_a)) \le 0,$$

for all users i. Integration over i now gives

$$\sum_{a \in \mathcal{A}} (f_a - \hat{f}_a)\, (c_a^{i_0}(f_a) - c_a^{i_0}(\hat{f}_a)) \le 0.$$

By strict monotonicity of the cost functions, each term in the last sum is nonnegative, and is moreover positive if $f_a \ne \hat{f}_a$. Therefore, all terms must be zero, and $f_a = \hat{f}_a$ for all arcs a. This implies that the cost of each route for each user is the same in s and $\hat{s}$, and, therefore, the equilibrium costs are also the same. ■

Proof of Proposition 3. It is shown in the proof of Proposition 1 that, for every network G, either the uniqueness property holds for G, in which case the equilibrium costs are unique for any assignment of cost functions; or the uniqueness property does not hold for G and, moreover, there is an assignment of cost functions with two strict Nash equilibria such that each user's equilibrium cost in one equilibrium is different from that in the other. ■

Proof of Theorem 1. The proof is contained in that of Proposition 1. ■

Proof of Theorem 2. Suppose that condition (i) does not hold. By the same argument used in the proof of Proposition 1, there is an assignment of cost functions for G with two strict Nash equilibria such that, for each user, one equilibrium cost is different from the other, and the two equilibria are therefore not equivalent. Moreover, inspection of the proofs of Lemmas 4 and 5 shows that there is a partition of the population into three user classes such that this assignment is in (the corresponding) $\mathcal{G}$. Since the two equilibria are strict, there is some $e > 0$ such that each of them is also an equilibrium in every assignment of cost functions in $\mathcal{G}$ that is less than a distance e from the original one. This shows that the set of all assignments of cost functions for which all Nash equilibria are equivalent is not dense in $\mathcal{G}$, and thus proves that condition (ii) of the theorem implies (i). It remains to prove the reverse implication.

Suppose that the network G satisfies (i). Fix some partition of the population, with user classes $I_1, I_2, \dots, I_n$, and consider the corresponding space of assignments of cost functions $\mathcal{G}$. For each element of $\mathcal{G}$ and each user class I_m, the number of least cost routes for the users in I_m, which will be denoted by l_m, is the same in all Nash equilibria. This is because the cost for each user of each route is determined by the arc flows, which, by Theorem 1, are the same in all Nash equilibria. Therefore, the mean number of least cost routes, $l = \sum_{m=1}^{n} m(I_m)\, l_m$, is also the same in all equilibria, and thus defines a real-valued function on $\mathcal{G}$.

CLAIM 1. *The function* $l : \mathcal{G} \to \mathbb{R}$ *is upper semicontinuous and has a finite range.*

The proof of the first part of the claim is very similar to that of [7, Lemma 3.4], and will be omitted. The second part follows from the fact that the cardinality of the range of l does not exceed the number of routes in G times the number of user classes.

CLAIM 2. *For every assignment of cost functions in* $\mathcal{G}$ *that is a point of continuity of* l, *all Nash equilibria are equivalent.*

To prove this claim, consider an assignment of cost functions in $\mathcal{G}$ with two non-equivalent Nash equilibria s and $\hat{s}$. It has to be proved that l has a discontinuity at this assignment. For each user class I_m and arc a, denote by f_a^m the measure of the set of all users $i \in I_m$ such that a is part of $s(i)$, and by $\hat{f}_a^m$ the corresponding quantity for $\hat{s}$. Since the two equilibria are not equivalent, there is some user class I_{m_0} and some arc a_0 such that $f^{m_0}_{a_0} \neq \hat{f}^{m_0}_{a_0}$. By assumption, G is nearly parallel or it consists of several nearly parallel networks connected in series. The arc a_0 is in one of these networks, G'. Without loss of generality, it may be assumed that, for each user i, the routes $s(i)$ and $\hat{s}(i)$ coincide outside G'. (If this is not so, $\hat{s}$ can be replaced by another Nash equilibrium, in which the users' routes agree with their routes in s outside G' and with their routes in $\hat{s}$ inside G'.) For every route r' in G' and every user class I_m, let $f^m_{r'}$ be the measure of the set of all users $i \in I_m$ such that r' is part of (or coincides with) $s(i)$, and $\hat{f}^m_{r'}$ the corresponding quantity for $\hat{s}$. For some such route and user class, $f^m_{r'} \neq \hat{f}^m_{r'}$. Therefore, some real number α exists such that the affine combination $\tilde{f}^m_{r'} \overset{\text{def}}{=} \alpha\, f^m_{r'} + (1-\alpha)\, \hat{f}^m_{r'}$ is nonnegative for all routes r' in G' and all user classes I_m, and zero for some such route r'_1 and user class I_{m_1} with $f^{m_1}_{r'_1} \neq \hat{f}^{m_1}_{r'_1}$. Since the measure m is nonatomic, a strategy profile $\tilde{s}$ exists such that, for all routes r' in G' and all user classes I_m, the measure of the set of all users i in I_m such that r' is part of (or coincides with) $\tilde{s}(i)$ is equal to $\tilde{f}^m_{r'}$. Since, by the uniqueness property, $\sum_{m=1}^{n} f^m_a = \sum_{m=1}^{n} \hat{f}^m_a$ for all arcs a, the flow through each arc in $\tilde{s}$ is the same as in s and $\hat{s}$. For each route r' in G' and each user class I_m, $\tilde{f}^m_{r'} > 0$ only if $f^m_{r'} > 0$ or $\hat{f}^m_{r'} > 0$, and, hence, only if r' is part of some least cost route for user class I_m. Therefore, $\tilde{s}$ is a Nash equilibrium.

CLAIM 3. *There is some arc a_1 in G', which is part of r'_1, such that $\tilde{f}^{m_1}_{a_1} = 0$ but $f^{m_1}_{a_1} > 0$ or $\hat{f}^{m_1}_{a_1} > 0$.*

This is easily shown if G' is homeomorphic to one of the networks in Figures 2(a)–2(d). In each of these networks, each route r' contains an arc a that is not part of any other route. Since, by construction, $\tilde{f}^{m_1}_{r'_1} = 0$ and $f^{m_1}_{r'_1} \neq \hat{f}^{m_1}_{r'_1}$, this implies that similar equality and inequality must with the route r'_1 replaced by one of its arcs a_1. If G' is homeomorphic to the network in Figure 2(e), there are two cases to consider. If r'_1 contains both u and v (the two non-terminal vertices with degree three of more), then there is an arc that is part of this route and no other route in G'. Therefore, this case is similar to the one considered above. If r'_1 contains only u or only v, then denote by a_2 and a_3 the first and last arcs in r'_1, respectively. If either $\tilde{f}^{m_1}_{a_2}$ or $\tilde{f}^{m_1}_{a_3}$ is greater than zero, the other must be zero. This is because, since $\tilde{f}^{m_1}_{r'_1} = 0$, if $\tilde{f}^{m_1}_{a_2}$, $\tilde{f}^{m_1}_{a_3} > 0$, then $\tilde{f}^{m_1}_{r'_2} > 0$ for some route r'_2 in G' containing a_2 but not a_3, and $\tilde{f}^{m_1}_{r'_3} > 0$ for some route r'_3 containing

a_3 but not a_2. Since $\tilde{s}$ is an equilibrium and the cost functions are nonnegative and strictly increasing, one of the last two inequalities implies that the cost to user class I_{m_1} of the path starting at u and terminating at d (and does not pass through v) is strictly greater than the cost of the path starting at v and terminating at d (and does not pass through u), while the other inequality implies that the second cost is strictly greater than the first. This contradiction proves that $\tilde{f}^{m_1}_{a_2} = 0$ or $\tilde{f}^{m_1}_{a_3} = 0$. Since $f^{m_1}_{a_2} + \hat{f}^{m_1}_{a_2} \geq f^{m_1}_{r'_1} + \hat{f}^{m_1}_{r'_1} > 0$, $f^{m_1}_{a_2}$ and $\hat{f}^{m_1}_{a_2}$ cannot both be zero. Similarly, $f^{m_1}_{a_3}$ and $\hat{f}^{m_1}_{a_3}$ cannot both be zero. This completes the proof of Claim 3.

The proof of Claim 2 can now be completed. It follows from Claim 3 that, for every $e > 0$, $\tilde{s}$ is an equilibrium in the assignment of cost functions obtained from that considered above by adding e to each cost function $c^i_a(x)$ with $i \in I_{m_1}$ and $a = a_1$ and leaving all the other cost functions unchanged. The distance between this assignment and the old one is e, and therefore can be chosen arbitrarily small. In the new assignment, the set of least cost routes for each user i is a subset of the old set, since the cost of each route r is either equal to the old cost or exceeds it by e. The latter possibility holds if and only if $i \in I_{m_1}$ and a_1 is part of r. By Claim 2, $f^{m_1}_{a_1} > 0$ or $\hat{f}^{m_1}_{a_1} > 0$, which implies that, in the old assignment of cost functions, a_1 is part of a least cost route for user class I_{m_1}. Since this is not so in the new assignment, the value of l_{m_1} for this assignment is smaller by at least unity than for the old one. This proves that l has a discontinuity at the old assignment. This completes the proof of Claim 2.

Together with Claims 1 and 2, the following claim completes the proof of the theorem.

CLAIM 4. *In every metric space $\mathcal{X}$, the set of all points of continuity of an upper semicontinuous function $g : \mathcal{X} \to \mathbb{R}$ with a finite range is open and dense.*

Every point of continuity of g has an open neighborhood in which g is constant, and hence continuous. This proves that the set of all points of continuity is open. To prove that it is dense in $\mathcal{X}$, consider any open set U. Let $x_0 \in U$ be such that $g(x_0) = \min_{x \in U} g(x)$. By upper semicontinuity of g, there is a neighborhood V of x_0 such that $g(x) \leq g(x_0)$ for all $x \in V$. Clearly, in $U \cap V$, $g(x) = g(x_0)$ for all x, and g is therefore continuous at x_0. ■

REFERENCES

1. H. Z. Aashtiani, T. L. Magnanti, Equilibria on a congested transportation network, *SIAM Journal on Algebraic and Discrete Methods* **2** (1981), 213–226.

2. E. Altman, H. Kameda, "Equilibria for Multiclass Routing in Multi-Agent Networks," *Proceedings of the 40th IEEE Conference on Decision and Control*, Orlando, pp. 604–609, December 2001.

3. M. Beckmann, C. B. McGuire, and C. B. Winsten, Studies in the Economics of Transportation, Yale Univ. Press, New Haven, CT, 1956.

4. M. G. H. Bell, Y. Iida, Transportation Network Analysis, Wiley, Chichester, UK, 1997.

5. H. Konishi, Uniqueness of user equilibrium in transportation networks with heterogeneous commuters, forthcoming in *Transportation Science*.

6. A. Mas-Colell, The Theory of General Economic Equilibrium: A Differentiable Approach, Cambridge Univ. Press, Cambridge, UK, 1985.

7. I. Milchtaich, Generic uniqueness of equilibrium in large crowding games, *Mathematics of Operations Research* **25** (2000), 349–364.

8. I. Milchtaich, Network Topology and the Efficiency of Equilibrium," Bar-Ilan University, Department of Economics Working Paper 12-01, June 2001.

9. A. Nagurney, Network Economics: A Variational Inequality Approach, 2nd ed., Kluwer, Boston, 1999.

10. G. F. Newell, Traffic Flow on Transportation Networks, MIT Press, Cambridge, MA, 1980.

11. K. P. Rath, A direct proof of the existence of pure strategy equilibria in games with a continuum of players, *Economic Theory* **2** (1992), 427–433.

12. D. Schmeidler, Equilibrium points of nonatomic games, *Journal of Statistical Physics* **7** (1970), 295–300.

13. Y. Sheffi, Urban Transportation Networks, Prentice-Hall, Englewood Cliffs, NJ, 1985.

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