

## Problem Set 8 – Maxima, Minima and Function Sketching – Selected Solutions

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1. Investigate and sketch the following functions:

- |    |                        |
|----|------------------------|
| A. | $\frac{\ln x}{x}$      |
| B. | $2x^3 - 5x^2 - 4x$     |
| C. | $x \cdot e^{-x}$       |
| D. | $\frac{1}{e^x - 1}$    |
| E. | $\frac{x^3}{x^2 - 12}$ |

2. Using the following definitions for various costs of producing  $q$  products, with respect to the total cost function  $TC(q)$ :

Fixed cost	$TFC = TC(0)$
Variable cost	$TVC = TC(q) - TC(0)$
Average total cost	$ATC = \frac{TC(q)}{q}$
Average fixed cost	$AFC = \frac{TFC(q)}{q}$
Average variable cost	$AVC = \frac{TVC(q)}{q}$
Marginal cost	$MC = TC'$

Given a total cost function,  $TC(q) = q^3 - 3q^2 + 10q + 5$ ,

a. Find the functions: MC, AVC, AFC, TVC, TFC.

$$\mathbf{TFC = 5; TVC = q^3 - 3q^2 + 10q; AFC = 5/q; AVC = q^2 - 3q + 10; MC = 3q^2 - 6q + 10}$$

b. Find the ranges in which the functions you obtained in the previous paragraph increase and decrease.

**TFC is constant, hence neither increases nor decreases. TVC monotonically increases since its first derivative is always positive (use the discriminant). AFC monotonically**

decreases. AVC decreases up to the point  $q=1.5$ , after which it increases. MC increases up to the point  $q=1$ , after which it decreases.

3. Prove:
  - a. If  $f(x)$  and  $g(x)$  are (strictly) increasing functions then  $f + g$  is also (strictly) increasing.
  - b. Prove that if  $f$  is strictly increasing and  $g$  is strictly decreasing, then the function  $f - g$  is strictly increasing.
4. Give an example of two functions  $f(x)$  and  $g(x)$  that are strictly decreasing such that the function  $h(x) = f(x)g(x)$  is strictly increasing.

Let  $f(x) = -x$  for all positive  $x$ . Let  $g(x) = f(x)$ . Then their product is  $(x) = x^2$ , which is strictly increasing over the domain of the positive numbers.

5. Prove that if  $f$  and  $g$  are (strictly) increasing, then the compound function  $g[f(x)]$  is also (strictly) increasing.
6. A production possibility curve  $y = f(x)$ , where  $x$  is the quantity of one product and  $y$  is the quantity of another product, must satisfy the following properties:
  - a.  $f(x) \geq 0$
  - b.  $f'(x) \leq 0$
  - c.  $f''(x) \leq 0$

Which of the following functions can possibly be production possibility curves?

$$0 \leq x \leq 5 \quad f(x) = 25 - x^2 \quad (1)$$

$$0 \leq x \leq 5 \quad f(x) = \sqrt{25 - x^2} \quad (2)$$

$$1 \leq x \leq 10 \quad f(x) = \frac{x-1}{x} \quad (3)$$

$$0 \leq x \leq 100 \quad f(x) = \frac{100}{x} \quad (4)$$

**Answers: 1, 2**

7. a. Prove that the sum of two convex functions is a convex function.  
b. Prove that the sum of two concave functions is a concave function
8. The elasticity of demand of a function  $f(x)$  at each point  $x$  is labelled  $\eta(x)$  and defined as  $\frac{xf'(x)}{f(x)}$ . This measures the ratio between the relative change in  $f(x)$  and the relative change in  $x$ .

The price elasticity, in contrast, is defined as  $\eta = \frac{\Delta Q / Q}{\Delta P / P}$ .

There are 100 items in the market, where the demand curve for each item is  $D(p): q = 4 - 0.2p$ .

- Find the elasticity of demand for each item at price  $p = 3$ . **-0.176**
- Find the elasticity of demand for each item at price  $p = 10$ . **-1**
- Find the demand function of the entire market.  **$q = 400 - 20p$**
- Find the elasticity of the demand of the market at price  $p = 3$ . **0.176**

9. We will denote by  $\eta_f, \eta_g, \eta_h$  the demand elasticities of functions  $f, g$  and  $h$ . Express  $\eta_h$  using  $\eta_f$  and  $\eta_g$  for the following cases:

A.  $h(x) = c \cdot f(x)$  c is a constant

B.  $h(x) = g(x) \cdot f(x)$

C.  $h(x) = \frac{g(x)}{f(x)}$

A.  $\eta_h = \eta_f$

B.  $\eta_h = \eta_f + \eta_g$

C.  $\eta_h = \eta_g - \eta_f$

10. The demand function for heaters is:  $q = 42 - 3p$

- Find the quantity  $q$  at which the producer will attain the maximal revenue.  **$q = 21$**
- What is the producer's maximal revenue if the cost function is  $c = 2q^2$ ? **The maximal revenue is attained at  $q = 3$ , where the revenue function is  $\pi = 14 - \frac{14}{3}q$**

11. Prove that the demand function  $p = \frac{a}{q+b} - c$  (where  $a, b$  and  $c$  are constant and positive) is decreasing and convex. Does the marginal revenue (MR) function have the same properties?

12. A radio producer sells  $x$  radios per week at a price of  $p$  per radio. The demand function for one week is:  $x = 75 - \frac{3}{5}p$ . The cost of producing  $x$  radios is:

$$500 + 15x + \frac{1}{6}x^2$$

- What is the optimal quantity of radios that it is worthwhile for the producer to sell per week? **30 radios**
- Assume that the government levies a tax  $t$  on each radio that the producer sells. The producer adds the tax to the production costs. What will be the optimal production amount now?  **$30 - \frac{3}{11}t$** . What should the value of  $t$  be to maximise the state's revenue?  **$t = 55$**

13. The total production cost of  $x$  radios per day is  $0.25x^2 + 35x + 25$ , and the price of one radio when it sells is  $50 - 0.5x$ .

- What should be the scope of daily production for maximal total profit?  **$x = 10$**
- Show that at the optimal point for total profit the cost of producing one radio is minimal. Is this rule always true? **No**