Problem Set 8 – Maxima, Minima and Function Sketching

1. Investigate and sketch the following functions:

   A. $\frac{\ln x}{x}$

   B. $2x^3 - 5x^2 - 4x$

   C. $x \cdot e^{-x}$

   D. $\frac{1}{e^x - 1}$

   E. $\frac{x^3}{x^2 - 12}$

2. Using the following definitions for various costs of producing q products, with respect to the total cost function TC(q):

Fixed cost	$TFC = TC(0)$
Variable cost	$TVC = TC(q) - TC(0)$
Average total cost	$ATC = \frac{TC(q)}{q}$
Average fixed cost	$AFC = \frac{TFC(q)}{q}$
Average variable cost	$AVC = \frac{TVC(q)}{q}$
Marginal cost	$MC = TC'$

Given a total cost function, $TC(q) = q^3 - 3q^2 + 10q + 5$,

   a. Find the functions: MC, AVC, AFC, TVC, TFC.
   b. Find the ranges in which the functions you obtained in the previous paragraph increase and decrease.

3. Prove:
   a. If f(x) and g(x) are (strictly) increasing functions then f + g is also (strictly) increasing.
   b. Prove that if f is strictly increasing and g is strictly decreasing, then the function f – g is strictly increasing.

4. Give an example of two functions f(x) and g(x) that are strictly decreasing such that the function $h(x) = f(x)g(x)$ is strictly increasing.

5. Prove that if f and g are (strictly) increasing, then the compound function g[f(x)] is also (strictly) increasing.

6. A production possibility curve y = f(x), where x is the quantity of one product and y is the quantity of another product, must satisfy the following properties:

   a. $f(x) \geq 0$
   b. $f'(x) \leq 0$
   c. $f''(x) \leq 0$

   Which of the following functions can possibly be production possibility curves?

$$0 \leq x \leq 5 \qquad f(x) = 25 - x^2 \qquad (1)$$

$$0 \leq x \leq 5 \qquad f(x) = \sqrt{25 - x^2} \qquad (2)$$

$$1 \leq x \leq 10 \qquad f(x) = \frac{x-1}{x} \qquad (3)$$

$$0 \leq x \leq 100 \qquad f(x) = \frac{100}{x} \qquad (\text{ד})$$

7. a. Prove that the sum of two convex functions is a convex function.
   b. Prove that the sum of two concave functions is a concave function

8. The elasticity of demand of a function f(x) at each point x is labelled $\eta(x)$ and defined as $\frac{xf'(x)}{f(x)}$. This measures the ratio between the relative change in f(x) and the relative change in x.

   The price elasticity, in contrast, is defined as $(\eta = \frac{\Delta Q / Q}{\Delta P / P}$.

   There are 100 items in the market, where the demand curve for each item is D(p): q = 4 – 0.2p.
   a. Find the elasticity of demand for each item at price p = 3.
   b. Find the elasticity of demand for each item at price p = 10.
   c. Find the demand function of the entire market.
   d. Find the elasticity of the demand of the market at price p = 3.

9. We will denote by η_f, η_g, η_h the demand elasticities of functions f, g and h. Express η_h using η_f and η_g for the following cases:
   A. $h(x) = c \bullet f(x)$ c is a constant
   B. $h(x) = g(x) \bullet f(x)$
   C. $h(x) = \frac{g(x)}{f(x)}$

10. The demand function for heaters is: $q = 42 - 3p$.
   a. Find the quantity q at which the producer will attain the maximal revenue.
   b. What is the producer's maximal revenue if the cost function is $c = 2q^2$?

11. Prove that the demand function $p = \frac{a}{q+b} - c$ (where a, b and c are constant and positive) is decreasing and convex. Does the marginal revenue (MR) function have the same properties?

12. A radio producer sells x radios per week at a price of p per radio. The demand function for one week is: $x = 75 - \frac{3}{5}p$. The cost of producing x radios is: $500 + 15x + \frac{1}{6}x^2$.
   a. What is the optimal quantity of radios that it is worthwhile for the producer to sell per week?
   b. Assume that the government levies a tax t on each radio that the producer sells. The producer adds the tax to the production costs. What will be the optimal production amount now? What should the value of t be to maximise the state's revenue?

13. The total production cost of x radios per day is $0.25x^2 + 35x + 25$, and the price of one radio when it sells is $50 - 0.5x$.
   a. What should be the scope of daily production for maximal total profit?
   b. Show that at the optimal point for total profit the cost of producing one radio is minimal. Is this rule always true?