

Problem Set 5 – The Derivative (Selected Solutions)

1. Differentiate the following functions:

$$\frac{d}{dx} x^4 - 3x^2 - 6 = 4x^3 - 6x \quad \text{A.}$$

$$\frac{d}{dx} \frac{x^5}{a+b} - \frac{x^2}{a-b} - x = \frac{5x^4}{a+b} - \frac{2x}{a-b} - 1 \quad \text{B.}$$

$$\frac{d}{dx} (1+4x^3)(1+2x^2) = (1+4x^3)4x + (1+2x^2)12x^2 \quad \text{C.}$$

$$\frac{d}{dx} \frac{2x^4}{b^2 - x^2} = \frac{(b^2 - x^2)8x^3 - 2x^4(-2x)}{(b^2 - x^2)^2} \quad \text{D.}$$

$$\frac{d}{dx} \left(x^2 - \frac{1}{x}\right)^3 = 3\left(x^2 - \frac{1}{x}\right)^2 \left(2x + \frac{1}{x^2}\right) \quad \text{E.}$$

$$\frac{d}{dx} (5x-1)^4 (2-3x)^5 = 5(5x-1)^4 (2-3x)^4 (-3) + 20(2-3x)^5 (5x-1)^3 \quad \text{F.}$$

$$\frac{d}{dx} \left(\frac{2x-1}{x+2}\right)^3 = 3\left(\frac{2x-1}{x+2}\right)^2 \frac{2(x+2) - (2x-1)}{(x+2)^2} \quad \text{G.}$$

2. Calculate the derivative of the function $f(x) = |x|$. What can you say about the derivative at the point $x = 0$?

For $x > 0$, the $f'(x)$ equals 1. For $x < 0$, the $f'(x)$ equals -1. For $x=0$, the derivative is not defined.

3. Calculate the derivative of the function $f(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ stands for the nearest integer smaller than x .

$$f'(x) = \begin{cases} 0 & \text{if } x \text{ is not an integer} \\ \text{undefined} & \text{if } x \text{ is an integer} \end{cases}$$

4. Calculate the derivative of the function $y = \sqrt{1+2x}$ in three ways:

- a. Using the definition of the derivative.
- b. Using the inverse function.
- c. As a compound function.

5. Calculate the function of the line tangent to the graph of the function $\frac{x+3}{1-x}$ at the point $x = -1$.

$$y = x + 2$$

6. Find tangent lines to the function $y = x^3 + x - 2$ that are parallel to the line $y = 4x + 3$.

$$y = 4x - 4 ; y = 4x$$

7. Find a tangent line to the function $y = x^2 - 2x + 5$ that is parallel to the line segment between the points (3,8) and (1,4).

$$y = 2x + 1$$

8. Find the equation of the line passing through the point (5,0) and tangent to the graph of the function $y = x^2 - 6x + 9$ (note that the given point is not on the graph of the function).

$$y = 8x - 40$$

9. The total cost function of a firm is given by $TC(q) = q^2 + 27q + 16$.

a. Find the average cost function AC. $q + 27 + \frac{16}{q}$

b. Find the marginal cost function MC. $2q + 27$

c. Find the point of intersection of the two functions. $q=4$

10. The total cost function of a firm is given by $TC(q) = aq^2 + bq$, for $a > 0$. Show that for every q in the domain of definition of this function (i.e., for all $q > 0$), $MC > AC$.