Problem Set 2 – Solutions to Selected Questions

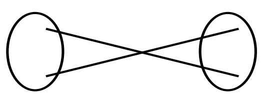
1. Let A be the set of all possible sentences in the English language. Associate each element of A with the number of letters it contains. Does this define a function, and if yes, is it injective (one-to-one)?

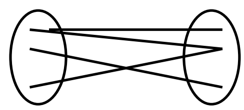
Yes it is a function, no it is not injective.

2. Let a function f: [0, 20] → **R** be defined by f(p) = 10,000 - 500p. Suppose that this function is the total amount of apples (in kilograms) demanded by consumers of a particular market when the price of apples is p shekels per kilogram.
   a. What is the amount demanded when the price p is 10 shekels per kilogram? **5000**
   b. Denote demand by Q=f(p). Calculate the price when the demand is Q=2000. **P=16 shekels**
   c. What is the range of the function f? $\{\boldsymbol{Q \mid 0 \leq Q \leq 10,000}\}$
   d. Is f an injective function? **Yes**
   e. Find f^{-1} **.** $\boldsymbol{f^{-1}(Q) = P = \frac{10000-Q}{500}}$
   f. What are the functions $f(f^{-1})$ and $f^{-1}(f)$? $\mathbf{f(f^{-1}(Q)) = Q}$ **,** $\mathbf{f^{-1}(f(P)) = P}$

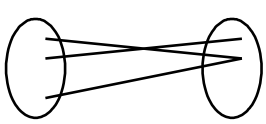
3.
   a. In each of the following diagrams, determine whether or not the diagram depicts a function from A to B.

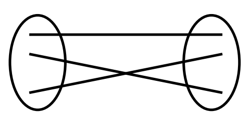
A **B** **2** **A** **B** **1**





4 **A** **B** **3**





Answers: 1. No 2. Yes 3. Yes 4. Yes

   b. Which of the functions you identified in part a) have inverse functions? **2 and 3**

4. Which of the following equations define Y as a function of x?
   a. $\sqrt{Y+1} = x-5$ **Yes**
   b. $(Y+1)^4 = x^3-2$ **No**

5. Determine the domain of definition of each of the following functions:

a. $y = x^3 + 2x^2 - x + 1$

b. $y = \sqrt{\dfrac{x-2}{x+1}}$

c. $y = \dfrac{x^2 - x + 1}{2x^2 + 3x - 5}$

d. $y = \dfrac{\sqrt{x-2}}{\sqrt{x+1}}$

e. $y = \dfrac{2x-1}{x^2 - x + 2}$

f. $y = \dfrac{1}{\sqrt{|x-4| - 3}}$

g. $y = \sqrt{-x^2 - 2}$

Answers: a. All of R b. $\left\{x \middle| \frac{x-2}{x+1} \geq 0\right\} = \{x | x < -1 \text{ or } x \geq 2\}$ **c.** $\{x | 2x^2 + 3x - 5 \neq 0\}$ **d.** $\{x | x \geq 2\}$ **e. All of R f.** $\{x | |x-4| - 3 \neq 0$ **g.** $\emptyset$

6. Which of the following functions have inverse functions? For those that do, explicitly write the inverse functions:

$y = 3x - 9 \rightarrow inverse: x = \frac{1}{3}y + 3$

$y = \dfrac{5}{x-10} \rightarrow inverse: x = \dfrac{5}{y} + 10$

$y = x^2 + 1 \rightarrow No$

$y = \dfrac{5}{(2x-1)^3} \rightarrow inverse: x = \dfrac{\sqrt[3]{\frac{5}{y}}}{2} + \dfrac{1}{2}$

$y = (x-1)^4 + 3 \rightarrow No$

$y = \dfrac{1}{\frac{1}{x} + 1} \rightarrow inverse: x = \dfrac{1}{\frac{1}{y} - 1}$

7. Let f(x) and g(x) be injective (one to one) functions.

a. Prove that f^{-1} is also injective.

b. Prove that if the function g[f(x)] is well-defined then it, too, is injective.

c. Use what you showed above to determine whether or not the following function is injective: $f(x) = \sqrt{3x+4}$

8. Determine the domain and range of the first two functions in Question 6. Furthermore, determine the domain and range of the composition of the first function with the second function and the composition of the second function with the first function.

First function: domain and range are both all of R.

Second function: domain $\{x|x \neq 10\}$, range $\{y|y \neq 0\}$.

9. Given the following functions:

$$f(x) = \frac{2x+2}{x+3}$$

$$g(x) = \frac{1}{4x+1}$$

$$h(x) = x+1$$

$$t(x) = \frac{1}{x}$$

Prove that $f[h(x)] = g[t(x)] + 1$ for all x not equal to 0.

10. Given the functions:

$$f(x) = \begin{cases} x+2 & x<2 \\ 3x & x \geq 2 \end{cases}$$

$$g(x) = \begin{cases} 2x & x<1 \\ x^2 & x \geq 1 \end{cases}$$

a. For each of the above functions, determine whether it is monotonic, and whether it is injective, and find its inverse function if such exists.

***f* is monotonic and one-to-one. *g* is neither monotonic nor one-to-one.**

b. Determine f(g(x)), g(f(x)) and f(f(x)).

11. What is the domain and range of the function $y = |3x - 6|$? Is it injective?

Domain is all of R. Range is $\{y | y \geq 0\}$. The function is not injective.

12. Which of the following functions are even/odd/neither?

$f(x) = x^2 \rightarrow Even$

$f(x) = \frac{1}{2}(a^{2x} + a^{-2x}) \rightarrow \text{Even}$

$f(x) = \frac{x^2 - 4}{x^2 + x} + x \cdot 10^{-x^2} \rightarrow Neither$

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