Problem Set 13 – Homogeneous Functions

1. Determine which of the following functions are homogeneous, and if homogeneous, find the order of homogeneity:

   A. $f(x_1, x_2) = a x_1 x_2$

   $f(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$

   B.

   $f(x_1, x_2) = a x_1^{\alpha} x_2^{\beta}$

   C. $f(x_1, x_2) = \dfrac{x_1^2 - x_2}{x_1 + x_2}$

   D. $f(x_1, x_2) = \dfrac{\ln x_1 + \ln x_2}{\sqrt{x_1 \cdot x_2}}$

   E.

2. Let f and g be functions with n variables, with f homogeneous of degree r_1 and g homogeneous of degree r_2. For each of the following functions, determine whether it is homogeneous, and to which order of homogeneity:
   a. $f \cdot g$
   b. f/g
   c. $f + g$

3. Let a function f(x,y) be homogeneous of degree 2, where functions x = g(s,t) and y = h(s,t) are homogeneous of degree 3. Define w(s,t) = f(g(s,t),h(s,t). Determine whether w is a homogeneous function and to what order of homogeneity.

4. Let f(x,y) be a homogeneous function of degree 1. Prove that the following holds:
   $x^2 \cdot f_{xx} + 2xy \cdot f_{xy} + y^2 \cdot f_{yy} \equiv 0$

5. Suppose f(x,y) is homogeneous of degree 0. Prove that the following holds at each point:
   $$\frac{f_x}{f_y} = -\frac{y}{x}$$

6. Given the function $w(x,y,z) = (\frac{x-y+z}{x+y-z})^n$, use the properties of homogeneous functions in order to prove the following:

   a. $$x \cdot \frac{\partial w}{\partial x} + y \cdot \frac{\partial w}{\partial y} + z \cdot \frac{\partial w}{\partial z} \equiv 0$$

b. $$x^2 \cdot \frac{\partial^2 w}{\partial x^2} + y^2 \cdot \frac{\partial^2 w}{\partial y^2} + z^2 \cdot \frac{\partial^2 w}{\partial z^2} + 2xy \cdot \frac{\partial^2 w}{\partial x \partial y} + 2xz \frac{\partial^2 w}{\partial x \partial z} + 2yz \frac{\partial^2 w}{\partial y \partial z} \equiv 0$$

7. Let f(x,y) be homogeneous of degree 3 and satisfy f(6,9) = 54. Let g(t) be a function with one variable which is equal at every point to $f(0.5t^2, 0.25t^3 + 1)$. Calculate g'(2).

8. Let f(x,y) be homogeneous of degree 3 and satisfy f(2,-1) = 4.
Calculate $f_x(8,-4) - 2 \cdot f_y(4,-2)$.

9. Let g be a function with one variable, and let f be a function with two variables which satisfies:
$$f(x,y) \equiv \ln 2^x \cdot g(\frac{\sqrt{x \cdot y}}{x+y})$$
   a. Prove that f is homogeneous and determine the order of homogeneity.
   b. Given that. $g(\frac{1}{2}) = 3$, calculate $f_x(1,1) + f_y(1,1)$

10. Let f(x,y) be a homogeneous function that satisfies: $f_x(-2,1) = 30$, f(4,-2) = 120, f(-2,1) = 15. Calculate the order of homogeneity of f and of $f_y(4,-2)$.

11. Let f(x,y) be a homogeneous function of degree r.
   a. Is ln f(s,y) a homogeneous function?
   b. Prove that at the point (x,0), the following is satisfies: $\frac{\partial}{\partial x}(\ln f(x,y)) = \frac{r}{x}$