

Problem Set 12 – Maxima and Minima in Multivariable Functions –

Selected Solutions

1. Calculate extremal points for the following functions:

- A. $f(x_1, x_2) = 2x_1 + x_2 - x_1^2 + x_1x_2 - x_2^2$
- B. $f(x_1, x_2) = x_1^3 + x_2^2 - 6x_1x_2 - 12x_1 + 12x_2 + 20$
- $f(x_1, x_2) = x_1^3 + 3x_1x_2^2 - 3x_1^2 - 3x_2^2 + 14$
- C. $f(x, y) = 8 \ln x - xy + y^2$
- D. $f(x, y) = 4x^2 - x \cdot y + \frac{y^3}{3} - 2y + 9x + 1$
- E.

A. $(\frac{5}{3}, \frac{4}{3})$ is a maximal point. B. $(2, 0)$ is a saddle point; $(4, 6)$ is a minimal point. C. $(0,0)$ is a maximal point; $(2,0)$ is a minimal point; $(1,1)$ and $(1,-1)$ are saddle points. D. $(4,2)$ and $(-4,-2)$ are saddle points. E. $(-1.234375, -0.875)$ is a saddle point; $(-1,1)$ is a minimal point.

2. Prove that the function $f(x,y) = e^x + x \cdot y + e^y$ has no extremal points.
3. A company produces two types of chocolates. The production costs are NIS 5 per kg of the first type, and NIS 6 per kg of the second type. If the chocolates are sold at prices P_1 and P_2 per kg, respectively, then the amounts sold in a week are: $x_1 = 5(P_2 - P_1)$ and $x_2 = 40 + 5P_1 - 10P_2$. Determine P_1 and P_2 such that the company's profit will be maximised.

$$P_1 = 6.5, P_2 = 7$$

4. Given the objective function $f(x_1, x_2) = x_1^2 + x_2^2$, subject to the constraint: $x_1^2 + x_2^2 - 4x_1 - 2x_2 - 15 = 0$, find points that are suspected of being extrema.

$(-2,-1)$ and $(6,3)$

5. Given the objective function $f(x,y) = x^2 + y^2$, subject to the constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$, prove that the values of the function that are suspected of being extrema are a^2 and b^2 .

6. Find points suspected of being extrema for the following functions (where s.t. stands for 'such that':

A.
$$\begin{cases} f(x, y) = x + 2y \\ \text{s.t. } x^2 + y^2 = 1 \end{cases}$$

B.
$$\begin{cases} f(x, y, z) = x \cdot y \cdot z \\ \text{s.t. } x^2 + y^2 + z^2 = 3 \end{cases}$$

- A. $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$ and $(\frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}})$ B. **There are a total of 14 such points here! (I won't list them all)**

7. Calculate points suspected of being extrema of z for:
 $2x^2 + 3y^2 + 3z^2 - 12xy + 4xz = 85$

(1,2,5) and (-1,-2,-5)

8. A rectangular box that is open at the top must have a volume of 32 cm^3 . What must its dimensions be so that its total area will be minimal? Solve the question in two ways:
 a. Using Lagrange multipliers.
 b. By substituting the constraint in the objective function.

X=Y=4, Z=2

9. Reuben and Shimon consume two products. If x and y are the quantities of the products (respectively), then Reuben's utility function is: $U(x,y) = \ln x + 2 \ln y$; and Shimon's is $U(x,y) = x \cdot y^2$. The prices of the products per unit are: $P_x = 5$ and $P_y = 2$. Reuben and Shimon have identical incomes, of NIS 90 each. Find the optimal consumed quantity from each of the two products, for each of Reuben and Shimon.

x=6, y=30