Problem Set 12 – Maxima and Minima in Multivariable Functions

1. Calculate extremal points for the following functions:

   A. $f(x_1,x_2) = 2x_1 + x_2 - x_1^2 + x_1x_2 - x_2^2$

   $f(x_1,x_2) = x_1^3 + x_2^2 - 6x_1x_2 - 12x_1 + 12x_2 + 20$

   B.

   $f(x_1,x_2) = x_1^3 + 3x_1x_2^2 - 3x_1^2 - 3x_2^2 + 14$

   C.

   $f(x,y) = 8\ln x - xy + y^2$

   D.

   $f(x,y) = 4x^2 - x\cdot y + \frac{y^3}{3} - 2y + 9x + 1$

   E.

2. Prove that the function $f(x,y) = e^x + x \cdot y + e^y$ has no extremal points.

3. A company produces two types of chocolates. The production costs are NIS 5 per kg of the first type, and NIS 6 per kg of the second type. If the chocolates are sold at prices P_1 and P_2 per kg, respectively, then the amounts sold in a week are: $x_1 = 5(P_2 - P_1)$ and $x_2 = 40 + 5P_1 - 10P_2$.
   Determine P_1 and P_2 such that the company's profit will be maximised.

4. Given the objective function , $f(x_1,x_2) = x_1^2 + x_2^2$, subject to the constraint:
   $x_1^2 + x_2^2 - 4x_1 - 2x_2 - 15 = 0$, find points that are suspected of being extrema.

5. Given the objective function $f(x,y) = x^2 + y^2$, subject to the constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 > b^2$, prove that the values of the function that are suspected of being extrema are a^2 and b^2.

6. Find points suspected of being extrema for the following functions (where s.t. stands for 'such that':

   A. $\begin{cases} f(x,y) = x + 2y \\ s.t. x^2 + y^2 = 1 \end{cases}$

   $\begin{cases} f(x,y,z) = x \cdot y \cdot z \\ s.t. x^2 + y^2 + z^2 = 3 \end{cases}$

   B.

7. Calculate points suspected of being extrema of z for:
   $2x^2 + 3y^2 + 3z^2 - 12xy + 4xz = 85$

8. A rectangular box that is open at the top must have a volume of 32 cm^3.
What must its dimensions be so that its total area will be minimal? Solve the question in two ways:
   a. Using Lagrange multipliers.
   b. By substituting the constraint in the objective function.

9. Reuben and Shimon consume two products. If x and y are the quantities of the products (respectively), then Reuben's utility function is: $U(x,y) = \ln x + 2 \ln y$; and Shimon's is $U(x,y) = x \cdot y^2$. The prices of the products per unit are: $P_x = 5$ and $P_y = 2$. Reuben and Shimon have identical incomes, of NIS 90 each. Find the optimal consumed quantity from each of the two products, for each of Reuben and Shimon.