Problem Set 11 – Derivatives and Differentials in Multivariable Functions

1. Calculate all of the partial first and second order derivatives in the following functions:

   A. $f(x,y) = \frac{x^2}{y} + \frac{y^2}{x}$

   B. $f(x,y) = e^{2x+3y}$

   C. $f(x,y) = x^y$

   D. $f(x,y,z) = 3 \cdot x^2 \cdot y \cdot z + 4 \cdot x \cdot e^{z \cdot y}$

   E. $f(x,y) = 3x^{\frac{2}{3}} \cdot y^{\frac{1}{3}}$

2. Prove that the function $z = y \cdot \ln(x^2 - y^2)$ fulfils: $\frac{1}{x} \cdot \frac{\partial z}{\partial x} + \frac{1}{y} \cdot \frac{\partial z}{\partial y} = \frac{z}{y^2}$

3. Given the function f(x,y), where $\begin{cases} x = s^2 - t^2 \\ y = t^2 - s^2 \end{cases}$, prove that $t \cdot \frac{\partial f}{\partial s} + s \cdot \frac{\partial f}{\partial t} \equiv 0$ exists.

4. Make approximate calculations of the following expressions, using a differential:
   a. $\sqrt{3.01^3 + 3 \cdot 2.98}$
   b. $(1.01^2 \cdot 0.98)^{1/15}$

5. Given the implicit function $x^2y - xy^2 + x^2 + y^2 = 0$, calculate dy/dx and dx/dy.

6. Given the curve $5x - 2y + y^3 - x^2y = 0$, calculate the equation of the straight line that is tangential to the curve at the origin.

7. Given a box whose base is a square with a side x and height y, calculate dx/dy when the area remains unchanged.

8. Calculate $\partial z/\partial x$ and $\partial z/\partial y$ when it is given that $e^{x \cdot y} + e^{y \cdot z} + e^{x \cdot z} = 20$.

9. Prove that the following holds for every function $\Phi(x,y) = 0$:

$$\frac{d^2y}{dx^2} = \frac{-\Phi_y^{\,2}\cdot\Phi_{xx} + 2\cdot\Phi_x\cdot\Phi_y\cdot\Phi_{xy} - \Phi_x^{\,2}\cdot\Phi_{yy}}{\Phi_y^{\,3}}$$

10. In the implicit function $z^3 - xy - y = 0$, prove that the following exists:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{9z^2}$$