# Mathematics for Economists (66110)

## Lecture Notes 1

### Sets

Definition: A set is a collection of distinct items. The items in a set are called 'elements' of the set.

Examples:

1) The set of integers greater than or equal to 1 and less than or equal to 5
2) The set of students in a particular class
3) The set of polygons with less than five edges

We will usually denote sets using majuscule (capital) letters, such as A, B, C, and so forth.

A set is identified by its elements, and written in the form:

A={1,2,3,4,5}

Comma separated elements list

Curly brackets at start and end of elements list

B= {Abraham ,Sarah}

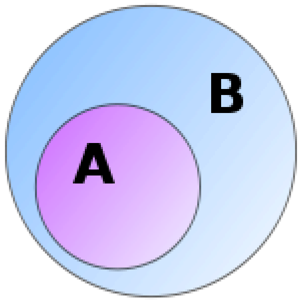
C= {Triangle ,Rectangle}

Two sets are equal if they have exactly the same elements, with the ordering of those elements immaterial. For example, {1,2,3} = {3,2,1}.

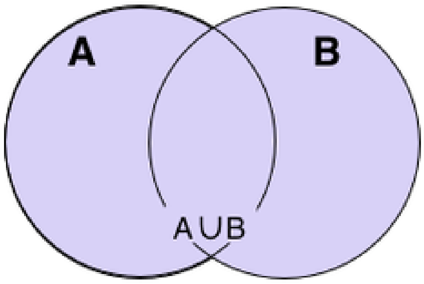
<u>Set Theoretic Concepts</u>

1) <u>Membership</u>. An element that belongs to a set is called a member of that set. Set membership is denoted by $\in$, for example, $1 \in \{1,2,3,4,5\}$, or $1 \in A$. The denotation $\notin$ stands for non-membership, as in $6 \notin \{1,2,3,4,5\}$ or $6 \notin A$.
2) The <u>empty set</u> is the set containing no elements. It is denoted by $\emptyset$ .
3) <u>Containment</u>. Given two sets A and B, we say that A is contained in B, or that A is a subset of B, denoted $A \subset B$ , if every element of A is also an element of B, that is, $x \in A \rightarrow x \in B$.
   Example: $\{1,2,3,4,5\} \subset \{1,2,\ldots,10\}$. Note that when $A \subset B$ it might be true that there are elements of B that are not elements of A, such as 10 in this example.

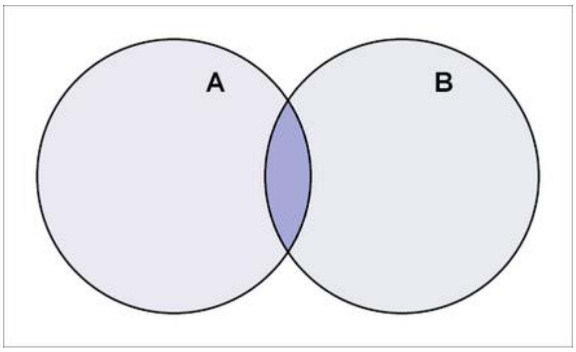
Venn diagram of set containment:




4) Set Union: The union of two sets A and B is the set of all elements that are members of A *or* of B (or of both). Denotation: $\mathrm{A} \cup \mathrm{B}$. Example: $\{3,5\} \cup \{1,2,4,5\} = \{1,2,3,4,5\}$.
How are A, B and $A \cup B$ related in terms of set containment? Answer: $\mathrm{A}, \mathrm{B} \subset \mathrm{A} \cup \mathrm{B}$.
By definition, $\mathrm{A} \cup \mathrm{B} = \mathrm{B} \cup \mathrm{A}$ .
Venn diagram of set union:




5) Set Intersection: The intersection of two sets A and B is the set of all elements that are members of A *and* of B. Denotation: $\mathrm{A} \cap \mathrm{B}$. Example: $\{3,5\} \cap \{1,2,4,5\} = \{5\}$.
How are A, B and$\mathrm{A} \cap \mathrm{B}$ related in terms of set containment? Answer: $\mathrm{A} \cap \mathrm{B} \subset A, B$.
By definition, $\mathrm{A} \cap \mathrm{B} = B \cap A$ .
Venn diagram of set intersection:



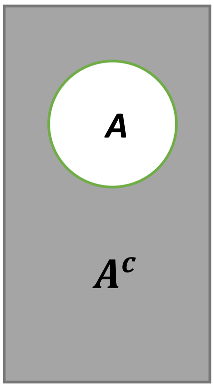

6) Set Complementation: We often speak of a set within the context of a larger space, meaning that we imagine it existing in relation to a larger possible set that contains it.

For example, consider the set of all positive integers and within that set the set of all positive even integers. If we regard the space here to be the set of all positive integers, then we can intelligibly speak about the complement of the set of positive even integers, which would be the set of positive odd integers, since that is the set of all positive integers that are not positive even integers. If we do not have an overall space then we cannot speak of the complement of a set in an unambiguous manner. In this example, if we did not specify the space to be the set of all positive integers, then it would be unclear if the complement of the set of positive even integers refers to the set of positive odd integers or perhaps instead to the all the odd integers, positive and negative.

If we have specified a space within which a set A exists, then we can define the set complement of A, denoted $A^C$, or A' or $\bar{A}$, to be the set of all elements of the space that are not elements of A. In notation, $\boldsymbol{x \in A^C \Leftrightarrow x \notin A}$.

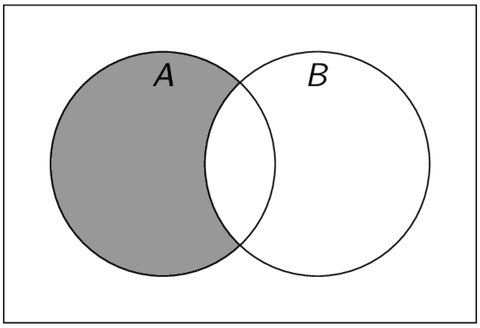
Note that $(A^c)^c = A$.

Venn diagram of set complementation:




7) Set Minus: For a pair of sets A and B, the set $A\backslash B$ is defined to be $A \cap B^C$, i.e., it is the set of all elements that are members of A but not members of B. For example, $\{1,2,4,5\} \setminus \{3,5\} = \{1,2,4\}$.

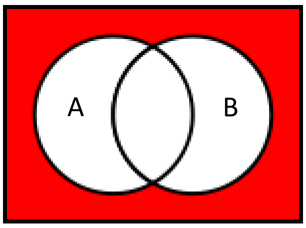
Venn diagram of set minus:




The De Morgan Laws:

$$(A \cup B)^C = A^C \cap B^C$$

$$(A \cap B)^C = A^C \cup B^C$$




$(A \cup B)^C$

Proof of the first De Morgan law:

We need to show:

$$x \in (A \cup B)^C \Leftrightarrow x \in A^C \cap B^C$$

Which means showing that these two sets contain exactly the same elements.

First we will show that $x \in (A \cup B)^C \Longrightarrow x \in A^C \cap B^C$ or equivalently that

$(A \cup B)^C \subset A^C \cap B^C$. In the second step we will show that

$x \in A^C \cap B^C \Longrightarrow x \in (A \cup B)^C$ or equivalently that

$\underline{A^C \cap B^C \subset (A \cup B)^C}$.

Step one: suppose that $x \in (A \cup B)^C$, i.e. that $x \notin A \cup B$. Recall that $x \in A \cup B \Leftrightarrow x \in A \text{ or } x \in B$ and therefore $x \notin A \cup B$ does not hold when $x \in A \text{ or } x \in B$, hence both $x \notin A$ and $x \notin B$, hence both $x \in B^C$ and $x \in A^C$. Taken together, we have proven that

$$\underline{A^C \cap B^C \supset (A \cup B)^C}$$

In the other direction, suppose that $x \in A^C \cap B^C$, i.e. that $x \notin B$ and $x \notin A$, hence " $x \in A$ *or* $x \in B$" does *not* hold, hence $x \in A \cup B$, hence $x \notin A \cup B$ and therefore $x \in (A \cup B)^C$ , which is what we needed to show.

QED for the first De Morgan Law.

For the proof of the second De Morgan Law, we can make use of $A^C, B^C$ in place of A and B in the first law (since the law holds for all sets, hence in particular for these sets), and therefore $(A^C \cup B^C)^C = (A^C)^C \cap (B^C)^C = A \cap B$.

In other words, $x \in A \cap B \Leftrightarrow x \in (A^C \cup B^C)^C$ , and therefore

$x \notin A \cap B \Leftrightarrow x \notin (A^C \cup B^C)^C$, which is the same as $x \in (A \cup B)^C \Leftrightarrow x \in A^C \cup B^C$, which implies that $(A \cap B)^C = A^C \cup B^C$.

Some more general laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Exercises:

1) Let A be the set of all integers divisible by 3 and let B be the set of all integers divisible by 9. Find B\A, $A \backslash B, A \cap B, A \cup B$ .
   $A \cup B$ = the set of all integers divisible by 3 or by 9 = the set of all integers divisible by 3.
   $A \cap B$ = the set of all integers divisibly by 3 and by 9 = the set of all integers divisible by 9.
   A\B $= A \cap B^C$ = the set of all integers divisible by 3 but not divisible by 9.
   $B \backslash A = B \cap A^C$ = the set of all integers divisible by 9 but not divisible by 3 = Φ .

Question: what is the relationship between A and B? Answer: $B \subset A$.

When $B \subset A$ it can be shown that it is always the case that

$$\mathrm{B} \backslash \mathrm{A} = \Phi \qquad A \cap B = B \qquad A \cup B = A$$

2) Define $\mathrm{A} = \{ \{1\}, \{2\}, \{3\} \}$ and $B = \{ \{1\}, 2, \{2,3\}\}$ .
   a. Is $1 \in \mathrm{A} \cap \mathrm{B}$ ?
      No. 1 is not an element that is a member of either set. What is true is that $A \cap B = \{ \{1\} \}$ .
      Note that {3} is not an element of the intersection because the set containing 3, i.e. {3}, is not an element of B.
   b. What is A ∪ B? Answer: $\mathrm{A} \cup \mathrm{B} = \{ \{1\}, \{2\}, 2, \{3\}, \{2,3\} \ \}$