Mathematics for Economists (66110)

Lecture Notes 4

Odd and Even Functions

Definition: A function $f(x)$ is <u>even</u> if $f(-x) = f(x)$ for all x in its domain.

Definition: A function $f(x)$ is <u>odd</u> if $f(-x) = -f(x)$ for all x in its domain.

Example: $f(x) = x^2$ is an even function, $f(x) = x$ is an odd function, $f(x) = 2^x$ is neither.

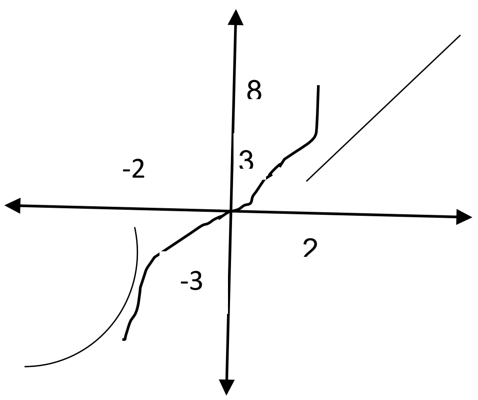
Exercise: Let

$$f(x) = \begin{cases} x - 1, x < -2 \\ x^3, -2 \leq -x \leq 2 \\ x + 1\,, x > 2 \end{cases}$$

A. Is this function one-to-one (injective)? Has it got an inverse?
B. Is this function even or odd (or neither)?

Solution:

A. A quick sketch shows that the function is not one-to-one and therefore cannot have an inverse.



B. Calculate $f(-x)$. If that is equal to $f(x)$, then the function is even. If that is equal to $-f(x)$, then the function is odd.
We find that:

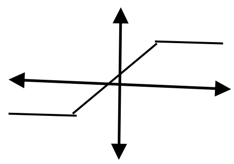
$$f(-x) = \begin{cases} -x - 1, -x < -2 \\ (-x)^3, -2 \leq -x \leq 2 \\ -x + 1\,, -x > 2 \end{cases} = \begin{cases} -(x + 1), x > 2 \\ -x^3, -2 \leq x \leq 2 \\ -(x - 1), x < -2 \end{cases} = -f(x)$$

to conclude that the function is odd.

Claim: If a function is even that it cannot be invertible.

Explanation: Since an even function satisfies the condition that $f(x) = f(-x)$ for all x, it follows immediately that there are (at least) two source points in the domain that have the same image in the range. Hence it is impossible for such a function to be injective.

An odd function may be injective (see the following graph), but may also not be injective.



Properties of even functions:

Let *f* and *g* be two even functions.

A. Let $h(x) = f(x) * g(x)$. Then $h(-x) = f(-x) * g(-x) = f(x) * g(x) = h(x)$, which means that the product of two even functions is always an even function.
B. Let $s(x) = f\big(g(x)\big)$. Then $s(-x) = f\big(g(-x)\big) = f\big(g(x)\big) = s(x)$, which means that the composition of two even functions is always an even function.
C. The sum $g(x) + g(x)$ and the difference $g(x) - g(x)$ of two even functions is always an even function.
D. The quotient $\frac{f(x)}{g(x)}$ of two even functions is always an even function, on condition, of course, that $g(x) \neq 0$.

Properties of odd functions:

Let *f* and *g* be two odd functions.

A. Let $h(x) = f(x) * g(x)$. Then $h(-x) = f(-x) - g(-x) = \big(-f(x)\big) * \big(-g(x)\big) = (x) * g(x) = h(x)$, which means that the product of two odd functions is always an *even* function.
B. Let $s(x) = f\big(g(x)\big)$. Then $s(-x) = f\big(g(-x)\big) = f\big(-g(x)\big) = -f\big(g(x)\big)$ which means that the composition of two odd functions is always an odd function.
C. The sum $f(x) + g(x)$ of two odd functions, is calculated as $f(-x) + g(-x) = -f(x) - g(x) = -\big(f(x) + g(x)\big)$, which means it is an odd function. The difference $f(x) - g(x)$ of two odd functions, is calculated as $f(-x) - g(-x) = -f(x) + g(x) = -\big(f(x) - g(x)\big)$, which means that its, too, is an odd function.
D. The quotient $\frac{f(x)}{g(x)}$ of two odd functions is always an *even* function, on condition, of course, that $g(x) \neq 0$.

Regarding B) above, what happens if *f* is even, *g* is odd, and we look at the composition of the two functions? Then $f\big(g(-x)\big) = f\big(-g(x)\big) = f(g(x))$, which means that the composition in this case is an even function.

Theorem: If $f(x)$ is an odd function and 0 is in the domain of *f*, then $f(0) = 0$. If a function *f* with 0 in its domain does not satisfy $f(0) = 0$, then *f cannot* be an odd function.

Proof: $-f(0) = f(-0) = f(0) \Rightarrow 0 = 2f(0) \Longrightarrow 0 = f(0)$.