Mathematics for Economists (66110)

Lecture Notes 2

Sets of Numbers

(1) The set of natural numbers: this is the set of positive integers, 1,2,3,4,… . It is an infinite set, denoted by $\mathbb{N}$.

(2) The set of integers: this is the union of the set of positive integers, the set of negative integers, and zero, …-2,-1,0,1,2… . The set of integers contains the set of natural numbers as a proper subset. The set of integers is denoted by $\mathbb{Z}$.

(3) The set of rational numbers: this is the set of numbers that can be expressed as fractions, i.e., $\frac{m}{n}$ where m,n are integer such that $n \neq 0$.

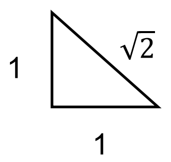
Note that the set of rational numbers contains the set of integers as a proper subset, for example, the number 1 is a rational number because it can be written as $\frac{1}{1}$. It can be proved that the decimal expansion of a rational number either contains a finite number of digits after the decimal point or the expansion is repeated after a certain point (for example, 1.123412341234). The set of rational numbers is denoted by $\mathbb{Q}$.

Claim: Between every pair of rational numbers there is another rational number.

Proof: We need to show that between any pair of rationals $\frac{m}{n}, \frac{E}{l}$ there is another rational number. Here is one such number: $\frac{ml+nk}{2nl} = \frac{\frac{m}{n}+\frac{k}{l}}{2}$. QED.

There exist numbers that are not rational, called irrational numbers.

Claim: $\sqrt{2}$ is irrational.



Proof: This is a proof by contradiction. We will suppose that $\sqrt{2}$ is rational and show that this leads to a contradiction.

If $\sqrt{2}$ is rational then there exist a pair of integers m,n such that $n \neq 0$, $\frac{m}{n} = \sqrt{2}$. Furthermore, we may assume that $\frac{m}{n}$ is already expressed in reduced form, i.e., there do not exist a,b such that $\frac{a}{b} = \sqrt{2}$ and $b < n, b \neq 0$.

If $\sqrt{2} = \frac{m}{n}$ then $m = \sqrt{2}n$, hence (by multiplying by $\sqrt{2}$) we get $\sqrt{2}m = 2n$, leading to

$$\sqrt{2} = \frac{m}{n} = \frac{m\,(\sqrt{2}-1)}{n\,(\sqrt{2}-1)} = \frac{2n-m}{m-n}$$

Note that $1 < \sqrt{2} < 2$ (because $1^2 = 1 < 2 < 2^2 = 4$) and hence $0 < \sqrt{2} - 1 < 1$ by multiplying by n:

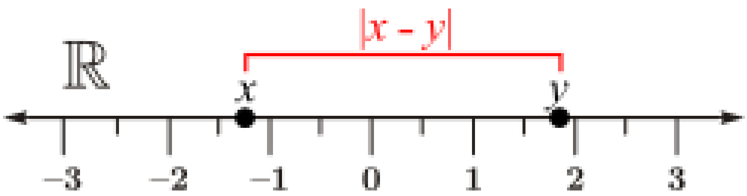
$0 < m - n = n(\sqrt{2} - 1) < n.$

We have concluded that $m - n \neq 0$, $\sqrt{2} = \frac{2n-m}{m-n}$ with $m - n < n$, contradicting the assumption that $\frac{m}{n}$ is in reduced form.

The contradiction implies that $\sqrt{2}$ must be rational. QED.

(4) The set of real numbers: this is the set of numbers that can be written as a decimal expansion, either finite or infinite. The set of real numbers is the union of the rational numbers and the irrational numbers.

The set of real numbers is often depicted as an infinite number line.



Each point on the real number line represents a single real number.

The set of real numbers is denoted by $\mathbb{R}$.

Sometimes we want to relate to the subset of the set of real numbers that contains only non-negative or positive numbers. We denote that by adding a "+" or "++" denotation, for example

$\mathbb{R}_+ = \{X \in \mathbb{R}: X \geq 0\}$ → **Non negative real numbers**
$\mathbb{R}_{++} = \{X \in \mathbb{R}: X > 0\}$ → **Positive real numbers**

Bounded Sets

A set of numbers (i.e., a subset of $A \subset \mathbb{R}$) is *bounded from above* if there exists a real number $x \in \mathbb{R}$ such that for each $a \in A$, we have that $a \leq x$. We say that $A \subset \mathbb{R}$ is bounded from below if there exists a real number $x \in \mathbb{R}$ such that for each $a \in A$, we have that $x \leq a$.

Examples:

1. The set of natural numbers is bounded from below (by the number zero) but not from above.
2. The set of negative integers is bounded from above but not from below.
3. The set of integers, the set of rational numbers, and the set of real numbers are neither bounded from below nor from above.

4. The sets $\mathbb{R}_{++}$,$\mathbb{R}_{+}$ are bounded from below but not from above.
5. Is the set $\{\frac{2n}{3n+3}:\ n \in \mathbb{N}\}$ bounded?

   To answer this, write $\frac{2n}{3n+3} = \frac{2}{3} \times \frac{n}{n+1}$. We see that for each natural number n, this expression is positive, that is, for each $n \in \mathbb{N}$ we have that $0 < \frac{2n}{3n+3}$, hence the set is bounded from below. Also, $\frac{2}{3} < 1$ for each natural number n, from which we conclude that the set if bounded from above.

Intervals

A particular type of subset of the real numbers is the interval. An interval is a subset of the real numbers satisfying the condition that it is bounded from above by a real number *b*, bounded from below by *a*, and it contains all the real numbers between *a* and *b*.

There are several types of intervals:

$\{x \in \mathbb{R}: a < x < b\} \ = \ (a, b)$ i.e., not including the endpoints *a* and *b*. This is called an *open interval*.

$\{x \in \mathbb{R}: a \leq x \leq b\} \ = \ [a, b]$ i.e., including both endpoints *a* and *b*. This is called a *closed interval*.

$\{x \in \mathbb{R}: a \leq x < b\} \ = \ [a, b)$ i.e. including *a* but not including *b*.

$\{x \in \mathbb{R}: a < x \leq b\} \ = \ (a, b]$ i.e. including *b* but not including *a*.

All these intervals are bounded, both from above and from below.

There are also unbounded intervals. Let a $\in \mathbb{R}$. Then

$\{x \in \mathbb{R}: x < a\} \ = (-\infty, a)$, $\quad \{x \in \mathbb{R}: x \leq a\} \ = (-\infty, a]$,
$\{x \in \mathbb{R}: x > a\} \ = (a, \infty)$, $\quad \{x \in \mathbb{R}: x \geq a\} \ = \ [a, \infty)$.