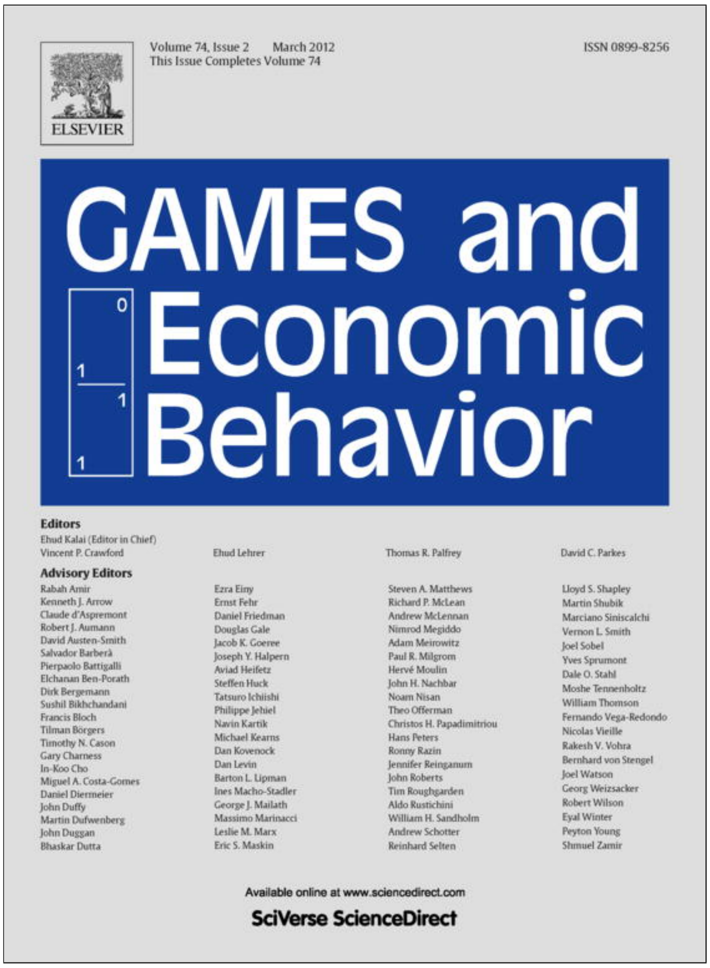
Volume 74, Issue 2 March 2012
This Issue Completes Volume 74
ISSN 0899-8256
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Games and Economic Behavior 74 (2012) 517–525

Contents lists available at SciVerse ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

How common are common priors? ☆

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ARTICLE INFO

Article history:
Received 30 November 2007
Available online 31 August 2011

JEL classification:
D82
D83

Keywords:
Common prior
Common knowledge
Knowledge
Belief

ABSTRACT

To answer the question in the title we vary agents' beliefs against the background of a fixed knowledge space, that is, a state space with a partition for each agent. Beliefs are the posterior probabilities of agents, which we call type profiles. We then ask what is the topological size of the set of *consistent* type profiles, those that are derived from a common prior (or a common improper prior in the case of an infinite state space). The answer depends on what we term the tightness of the partition profile. A partition profile is *tight* if in some state it is common knowledge that any increase of any single agent's knowledge results in an increase in common knowledge. We show that for partition profiles that are tight the set of consistent type profiles is topologically large, while for partition profiles that are not tight this set is topologically small.

1. Introduction

Ever since the introduction of games with incomplete information by Harsanyi (1967–1968), the assumption that players' posterior beliefs in models of differential information are derived from a common prior has been ubiquitous in the literature. It plays an essential role in the no agreements theorem of Aumann (1976) and in the no trade theorems that followed. It is also a basic building block of the solution concept of correlated equilibrium which was interpreted by Aumann (1987) as the expression of common knowledge of rationality. As pointed out in that paper, the assumption of a common prior, also known as the Harsanyi doctrine, is pervasively "explicit or implicit in the vast majority of the differential information literature in economics and game theory". Despite its pervasiveness, the justification and the use of the common prior assumption was, and still is, debated and challenged (see Gul, 1998 and Aumann, 1998).

The special interest in the common prior assumption leads naturally to the question how restrictive an assumption it is, or equivalently, how common common priors are. We study this question in a general model of differential information that has two parts, a *knowledge space* and the agents' *posterior beliefs*. The first is given by a finite or countably infinite state space with a partition profile of the state space, one for each agent, which define the agents' knowledge. An agent's posterior beliefs are given by a *type function* which assigns to each element in the agent's partition a probability function on this element. A type profile—one type function for each agent—is *consistent* if all the type functions are derived from one probability on the state space—the *common prior*—by conditioning on the partitions' elements. In the countably infinite state space, we consider a type profile to be consistent if it can be derived from a *common improper prior* via conditioning.

☆ The research of the first author has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007–2013)/ERC grant agreement No. 249159. The research of the second author was partially financed by the Henry Crown Institute of Business Research in Israel.

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0899-8256/$ – see front matter
doi:10.1016/j.geb.2011.08.008

Against the background of a fixed knowledge space, we vary the type profiles and study the topological size of the set of consistent type profiles. As we show, this size depends on the partition profile of the knowledge space, through its common knowledge structure. We say that knowledge is *tight* at a state when any increase of the agents' knowledge in this state results in increasing common knowledge. We say that the partition profile is *tight* if at some state it is common knowledge that knowledge is tight. We show that when the partition profile is tight the set of consistent type profiles is topologically large, and when it is not tight this set is topologically small. The characterization of tightness in the finite case is particularly simple: the tightness of a type profile, the meet of which consists of one element, can be determined solely by the total number of elements in the partition profile. In the proofs we use another characterization of tightness, which holds for both the finite and the infinite case, in terms of chains which are defined in turn in terms of the accessibility relation on states. The existence of a prior for a given type profile can be characterized by a condition on the relation between chains and the type profile. Such a condition was used in Harsanyi (1967–1968) for Harsanyi type spaces, and was extended by Rodrigues-Neto (2009) for general knowledge spaces.

The precise meaning of large and small depends on whether the state space is finite or countably infinite. For finite knowledge spaces, when the partition profile is tight *each* type profile is consistent, and when it is not tight, the set of consistent type profiles is nowhere dense. For countably infinite knowledge spaces, we endow the set of all type profiles with a topology that makes it a Baire space. When the partition profile is tight the set of consistent type profiles is big, as its complement is of first category, which in a Baire space is a small set. When it is not tight, the set of consistent type profiles is small, being of first category.

2. Preliminaries

2.1. Knowledge spaces

A knowledge space for a nonempty finite set of **agents** I, is a couple $(\Omega, \boldsymbol{\Pi})$, where Ω is a nonempty set called a **state space**, and $\boldsymbol{\Pi} = (\Pi_i)_{i\in I}$ is a **partition profile**, where for each i, Π_i is a partition of Ω. The knowledge space is called finite or countably infinite when Ω is finite of countably infinite, correspondingly. An **event** is a subset of Ω. For a partition Π of Ω and a state ω, $\Pi(\omega)$ is the element of Π that contains ω. We say that agent i **knows** an event E at ω if $\Pi_i(\omega) \subseteq E$. We define for each i a **knowledge operator** $K_i: 2^\Omega \to 2^\Omega$, by $K_i(E) = \{\omega \mid \Pi_i(\omega) \subseteq E\}$. Thus, $K_i(E)$ is the event that i knows E.

For a pair of partitions Π and Π' and state ω, we write $\Pi' \succcurlyeq_\omega \Pi$ when $\Pi'(\omega) \subseteq \Pi(\omega)$. For the partition profiles $\boldsymbol{\Pi}$ and $\boldsymbol{\Pi}'$, $\boldsymbol{\Pi}' \succcurlyeq_\omega \boldsymbol{\Pi}$ means that for each i, $\Pi'_i \succcurlyeq_\omega \Pi_i$. The partition Π' is a **refinement** of Π, denoted $\Pi' \succcurlyeq \Pi$, when $\Pi' \succcurlyeq_\omega \Pi$ for each state ω. The partition profile $\boldsymbol{\Pi}'$ is a **refinement** of $\boldsymbol{\Pi}$, denoted $\boldsymbol{\Pi}' \succcurlyeq \boldsymbol{\Pi}$, if for each i, $\Pi'_i \succcurlyeq \Pi_i$. For each of these four relations, a corresponding relation with $\succ$ instead of $\succcurlyeq$ is obtained by discarding the reflexive part of the relation $\succcurlyeq$. The two irreflexive relations describe an increase of knowledge, while the two reflexive relations describe a weak increase of knowledge. Thus, for example, if $\Pi' \succ_\omega \Pi$, and K and K' are the knowledge operators associated with Π and Π' respectively, then for each event E, if $\omega \in K(E)$, then $\omega \in K'(E)$, but for some events, for example $E = \Pi'(\omega)$, $\omega \in K'(E)$ but $\omega \notin K(E)$.

The **meet** of $\boldsymbol{\Pi}$, denoted $\wedge\boldsymbol{\Pi}$, is the partition which is the finest among all the partitions Π that satisfy $\Pi \succcurlyeq \Pi_i$ for each i. The knowledge operator K_c defined by the meet partition is called the **common knowledge** operator (Aumann, 1976). It can be described in terms of the knowledge operator K_i as follows. Denote by $K(E)$ the event that all agents know E. That is, $K(E) = \bigcap_{i\in I} K_i(E)$. Then $K_c(E) = \bigcap_{n=1}^{\infty} K^n(E)$. For $M \in \wedge\boldsymbol{\Pi}$, the elements of Π_i contained in M form a partition of M. Thus, $(M, \boldsymbol{\Pi}_M)$, where $\boldsymbol{\Pi}_M$ is the restriction of $\boldsymbol{\Pi}$ to M, is a knowledge space.

2.2. Beliefs

The beliefs of an agent in a given state are described by a probability distribution over the state space. These beliefs are related to the agent's knowledge as follows. Denote by $\Delta(\Omega)$ the set of all probability functions on Ω. A **type function** for Π_i is a function $t_i: \Omega \times \Omega \to \mathbb{R}$ that satisfies:

(a) for each ω, $t_i(\omega, \cdot) \in \Delta(\Omega)$,
(b) for each i and $\pi \in \Pi_i$, if $\{\omega, \omega'\} \subseteq \pi$, then $t_i(\omega', \cdot) = t_i(\omega, \cdot)$,
(c) for each i, $\pi \in \Pi_i$, and $\omega \in \pi$, the support of $t_i(\omega, \cdot)$ is π, i.e., $t_i(\omega, \pi) = 1$.

We say that $t_i(\omega, \cdot)$ is i's **type** at ω. By condition (b), the type of i is measurable with respect to Π_i, i.e., the type of i is the same in all states in π which means that i knows her type, or equivalently, knows her beliefs. In light of (b) we sometimes write for i and $\pi \in \Pi_i$, $t_i(\pi, \cdot)$ for the type of i in all the states in π. Condition (c) implies that whenever i knows E at ω she assigns probability 1 to it, i.e., whatever she knows she is certain of.[1]

A **type profile** for $\boldsymbol{\Pi}$ is a vector of type functions, $\mathbf{t} = (t_i)_{i\in I}$, where for each i, t_i is a type function for Π_i. Denote by $\mathcal{T}(\boldsymbol{\Pi})$ the set of all type profiles for $\boldsymbol{\Pi}$. A type profile assigns for each i and ω an element $t_i(\omega, \cdot)$ in $\Delta(\Omega)$. Thus, we may

[1] Conditions (b) and (c) are part of the definition of the space of knowledge and belief in Aumann (1976). The meaning given to them here are expressed as two axioms on the relation between knowledge and belief in Hintikka (1962).

consider $\mathcal{T}(\boldsymbol{\Pi})$ as a subset of $\Delta(\Omega)^{\Omega\times I}$. In particular, for a finite state space we consider $\mathcal{T}(\boldsymbol{\Pi})$ as a topological space with the topology induced by the standard topology of the Euclidean space in which $\mathcal{T}(\boldsymbol{\Pi})$ is embedded.

A **prior** for a type function t_i is a probability function $p \in \Delta(\Omega)$ such that for each $\pi \in \Pi_i$, $p(\pi)t_i(\pi,\omega) = p(\omega)$ for all $\omega \in \pi$. A **common prior** (*cp*) for the type profile **t** is a probability function $p \in \Delta(\Omega)$ which is a prior for each agent i.[2] A type profile **t** is **consistent** when it has a common prior.

The model of knowledge space with beliefs used here is the same as the model in Aumann (1976), except that in the latter the assumption is made that there exists a common prior. Our model is also the discrete case of the abstract S-based belief space in Mertens and Zamir (1985), where S is Ω. Although knowledge is not introduced explicitly in their work, the partitions of the space into agents' types makes it a partition model.

3. Main results

3.1. Tightness

The commonness of consistent type profiles for a given knowledge space depends on a property of the knowledge space we call tightness. In both the finite and the infinite case, when the partition profile is not tight, the set of consistent type profiles is topologically small. When the partition profile is tight this set is topologically large.

We say that knowledge is tight at a state if increasing any agents' knowledge at this state must result in increasing common knowledge. A partition profile is tight if at some state there is common knowledge that knowledge is tight. Formally,

Definition 1. For a partition profile $\boldsymbol{\Pi}$, knowledge is **tight at** ω, when for each $\boldsymbol{\Pi}' \succ \boldsymbol{\Pi}$, if $\boldsymbol{\Pi}' \succ_\omega \boldsymbol{\Pi}$ then $\wedge\boldsymbol{\Pi}' \succ \wedge\boldsymbol{\Pi}$. Let T be the event that knowledge is tight. We say that $\boldsymbol{\Pi}$ is **tight**, if $K_c(T) \neq \emptyset$.

In the following example we illustrate the notions of tight knowledge and tight partition profiles.

Example 1. Let $\Omega = \{\omega_1,\omega_2,\omega_3,\omega_4\}$ and $I = \{1,2\}$. Consider the partition profile $\boldsymbol{\Pi} = (\Pi_1,\Pi_2)$, where $\Pi_1 = \{\{\omega_1,\omega_2\}, \{\omega_3,\omega_4\}\}$ and $\Pi_2 = \{\{\omega_1,\omega_2,\omega_3\},\{\omega_4\}\}$. Obviously, $\wedge\boldsymbol{\Pi} = \{\Omega\}$. Suppose that $\boldsymbol{\Pi}' \succ \boldsymbol{\Pi}$, and $\boldsymbol{\Pi}' \succ_{\omega_4} \boldsymbol{\Pi}$. The only way that knowledge can increase at ω_4 is by splitting the partition element $\{\omega_3,\omega_4\}$. Thus, $\Pi'_1(\omega_4) = \{\omega_4\}$. Therefore $\{\omega_4\} \in \wedge\boldsymbol{\Pi}'$ which means that $\wedge\boldsymbol{\Pi}' \succ \wedge\boldsymbol{\Pi}$. We conclude that knowledge at ω_4 is tight. Consider now the partition profile $\boldsymbol{\Pi}'$ where $\Pi'_1 = \Pi_1$ and $\Pi'_2 = \{\{\omega_1\},\{\omega_2,\omega_3\},\{\omega_4\}\}$. Then $\boldsymbol{\Pi}' \succ \boldsymbol{\Pi}$, and $\boldsymbol{\Pi}' \succ_{\omega_3} \boldsymbol{\Pi}$. Yet, $\wedge\boldsymbol{\Pi}' = \wedge\boldsymbol{\Pi}$, which shows that knowledge at ω_3 is not tight. Therefore $T \neq \Omega$, and hence $K_c(T) = \emptyset$. We conclude that $\boldsymbol{\Pi}$ is not tight. It is easy to check that for the last partition profile $\boldsymbol{\Pi}'$ knowledge is tight at each state and therefore $\boldsymbol{\Pi}'$ is tight.

The tightness of a partition profile can be expressed without explicit reference to common knowledge, as follows. We say that $\boldsymbol{\Pi}$ is **connected** when $\wedge\boldsymbol{\Pi} = \{\Omega\}$. For each $M \in \wedge\boldsymbol{\Pi}$, $\boldsymbol{\Pi}_M$, the restriction of $\boldsymbol{\Pi}$ to M, is connected.

Proposition 1.

(a) *A connected partition profile $\boldsymbol{\Pi}$ is tight if and only if for any $\boldsymbol{\Pi}' \succ \boldsymbol{\Pi}$, $\boldsymbol{\Pi}'$ is not connected.*
(b) *A partition profile $\boldsymbol{\Pi}$ is tight if and only if there exists $M \in \wedge\boldsymbol{\Pi}$ such that $\boldsymbol{\Pi}_M$ is tight.*

A third characterization of tightness, in terms of chains, is given in Proposition 5 below. In the finite case, there exists yet another simple characterization of the tightness of a connected type profile, in terms of the total number of partition elements.

Proposition 2. *Let Ω be a finite state space and $\boldsymbol{\Pi}$ a connected partition profile. Then $\sum_{i\in I}|\Pi_i| \leqslant (|I|-1)|\Omega|+1$ and equality holds if and only if $\boldsymbol{\Pi}$ is tight.*

Observe, that the dimension of the set of types $\mathcal{T}(\boldsymbol{\Pi})$ is $\sum_{i\in I}(|\Omega|-|\Pi_i|) = |I||\Omega| - \sum_{i\in I}|\Pi_i|$ and the dimension of the set of priors $\Delta(\Omega)$ is $|\Omega|-1$. Thus, Proposition 2 characterizes the connected tight partition profiles $\boldsymbol{\Pi}$ as the ones with minimal dimension of $\mathcal{T}(\boldsymbol{\Pi})$ which equals the dimension of $\Delta(\Omega)$.[3]

[2] Contrasting a prior for t_i with the types $t_i(\omega,\cdot)$, the latter are referred to as the posterior probabilities of i.

[3] This observation suggests a proof for the smallness of the set of consistent type profiles, for partition profiles which are not tight, based on dimensional considerations. We have elected instead to implement an elementary combinatorial proof, which can be applied equally well for the infinite case. Nyarko (2010) states that in a finite Harsanyi type space the set of consistent posteriors has measure zero. The proof requires differential geometry arguments based on dimensionality considerations.

3.2. The size of the set of consistent type profiles

Theorem 1. *Let $(\Omega, \boldsymbol{\Pi})$ be a finite knowledge space.*

1. *If $\boldsymbol{\Pi}$ is tight then each type profile is consistent.*
2. *If $\boldsymbol{\Pi}$ is not tight then the set of consistent type profiles is nowhere dense.*[4]

In order to prove results similar to those of Theorem 1 for countable state spaces, we need to generalize the notion of a common prior. A **common improper prior** (*cip*) for a type profile **t** is a non-negative and non-zero function $p : \Omega \to \mathbb{R}$ such that for each i and $\pi \in \Pi_i$, $p(\pi) < \infty$ and $p(\pi)t_i(\pi, \omega) = p(\omega)$ for all $\omega \in \pi$. Note that although for any $\pi \in \Pi_i$, $p(\pi) < \infty$, the possibility that $p(\Omega) = \infty$ is not ruled out, so that p may not be normalizable. Obviously, a *cp* is in particular a *cip*. Note also that if p is a *cip*, then for any constant $\gamma > 0$, γp is also a *cip*. In particular, if p is a *cip* and $p(\Omega) < \infty$ then $p(\Omega)^{-1} p$ is a common prior. Thus, for a finite space, a profile type has a common prior if and only if it has a common improper prior. In light of this the following definition of consistency for countable spaces generalizes the one given for finite spaces. A type profile **t** is **consistent** when it has a common improper prior and **inconsistent** otherwise.

To measure the topological size of sets in the countable case we use the notion of a set of first category (called also a meager set), namely, a set which is a countable union of nowhere dense sets. A topological space is a Baire space if every set of first category has an empty interior. Therefore, in a Baire space, sets of first category are considered small. We now proceed to define a topology on $\mathcal{T}(\boldsymbol{\Pi})$ for which it is a Baire space.

Consider the complete normed vector space $l^1(\Omega)$ of absolutely summable functions $x : \Omega \to \mathbb{R}$, with the norm $\|x\| = \sum_{\omega \in \Omega} |x(\omega)|$. The set $\Delta(\Omega)$ is closed in $l^1(\Omega)$.[5] Therefore, $\Delta(\Omega)$ with the metric induced on it from $l^1(\Omega)$ is a complete metric space. Hence, the product space $\Delta(\Omega)^{\Omega \times I}$ is a completely metrizable topological space (see Munkres, 1975). Finally, the equalities in the definition of a type guarantee that $\mathcal{T}(\boldsymbol{\Pi})$ is closed in $\Delta(\Omega)^{\Omega \times I}$ and therefore $\mathcal{T}(\boldsymbol{\Pi})$ is a completely metrizable topological space. This implies that $\mathcal{T}(\boldsymbol{\Pi})$ is a Baire space. Obviously, in the finite case the topology just described is the standard topology on finite dimensional Euclidean spaces.

Theorem 2. *Let $(\Omega, \boldsymbol{\Pi})$ be a countable knowledge space.*

1. *If $\boldsymbol{\Pi}$ is tight then the set of inconsistent type profiles is of first category.*
2. *If $\boldsymbol{\Pi}$ is not tight then the set of consistent type profiles is of first category.*

In contrast with the finite case, here the set of inconsistent type profiles of a tight partition profile need not be empty. Example 2 in the next section shows that we cannot even strengthen this part by changing "of first category" to "nowhere dense". Example 3 shows that similar strengthening is also impossible in the second part of the theorem.

3.3. Harsanyi type spaces

Of special interest are Harsanyi type spaces. In such a space $\Omega = \times_{i \in I} T_i$, where for each i, T_i is a set of types of player i. With each player i we associate the natural partition of Ω, Π_i, into i's types. It is easy to see, using Proposition 1, that the partition profile of a non-trivial Harsanyi type space (one that has more than one state and more than one agent) is connected and not tight. Therefore, if we vary the posterior beliefs of the types on such a finite or countably infinite space, while keeping the sets of types fixed, the set of consistent posterior beliefs is small. The lack of tightness of non-trivial finite Harsanyi type spaces can be also checked using Proposition 2. Obviously, for such a space $|\Pi_i| = |T_i|$, and $|\Omega| = \times_{i \in I} |\Pi_i|$. It is easy to prove that $\sum_{i \in I} |\Pi_i| < (|I| - 1) \times_{i \in I} |\Pi_i| + 1 = (|I| - 1)|\Omega| + 1$.

4. Proofs and examples

4.1. Proof of Proposition 1

(a) Let $\boldsymbol{\Pi}$ be a connected partition profile. Then Ω is the only event E such that $K_c(E) \neq \emptyset$. Thus, if $\boldsymbol{\Pi}$ is tight then $T = \Omega$. If $\boldsymbol{\Pi}' \succ \boldsymbol{\Pi}$, then for some ω, $\boldsymbol{\Pi}' \succ_\omega \boldsymbol{\Pi}$, and by tightness, $\wedge\boldsymbol{\Pi}' \succ \wedge\boldsymbol{\Pi}$. Conversely, if $\boldsymbol{\Pi}' \succ \boldsymbol{\Pi}$ implies $\wedge\boldsymbol{\Pi}' \succ \wedge\boldsymbol{\Pi}$, then obviously, each ω is in T, and hence $T = \Omega$, and $K_c(T) = \Omega$.

(b) The partition profile $\boldsymbol{\Pi}$ is tight iff there exists $M \in \wedge\boldsymbol{\Pi}$ such that $M \subseteq T$. It is easy to see that $M \subseteq T$ iff for the knowledge space $(M, \boldsymbol{\Pi}_M)$, knowledge is tight at each $\omega \in M$, which is a necessary and sufficient condition for $\boldsymbol{\Pi}_M$ to be tight. □

[4] A set is nowhere dense if its closure has an empty interior. Such a set is considered topological small.

[5] To see this, consider the linear functional on $l^1(\Omega)$ defined by $f(x) = \sum_{\omega \in \Omega} x(\omega)$. Since $|f(x)| \leqslant \|x\|$, f is continuous. Now, $\Delta(\Omega)$ is the intersection of two closed sets: $f^{-1}(1)$ and the non-negative orthant of $l^1(\Omega)$.

4.2. Chains

We define a **chain** of length $n \geqslant 0$, for the partition profile $\boldsymbol{\Pi}$, from one state to another by induction on n. A state ω_0 is a chain of length 0 from ω_0 to ω_0. A chain of length $n+1$, from ω_0 to ω, is a sequence $c \xrightarrow{i} \omega$, where c is a chain of length n from ω_0 to ω', and $\omega \in \Pi_i(\omega')$. Thus, a chain of positive length n is a sequence $c = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} \omega_n$, such that for $s = 0, \dots, n-1$, $\omega_{s+1} \in \Pi_{i_s}(\omega_s)$.

Obviously, for each i, chains of length 1, $\omega \xrightarrow{i} \omega'$, define an equivalence binary relation and Π_i is the partition of Ω into its equivalence classes. We write $\omega \to \omega'$ when there is a chain from ω to ω'. The binary relation $\to$ is the transitive closure of the union of the relations $\xrightarrow{i}$, and it is an equivalence relation. We say that ω and ω' are **connected** for $\boldsymbol{\Pi}$, if there is a chain for $\boldsymbol{\Pi}$ from ω to ω'.

Claim 1. *The meet of $\boldsymbol{\Pi}$ is the partition of Ω into the equivalence classes of $\to$.*

To see this, denote by Π_{con} the partition of Ω into equivalence classes of $\to$. Since each of the partitions Π_i is finer than $\wedge\boldsymbol{\Pi}$, it follows by induction on the length of chains that if $\omega \to \omega'$ then $\omega' \in \wedge\boldsymbol{\Pi}(\omega)$. Thus, for each ω, $\Pi_{\text{con}}(\omega) \subseteq \wedge\boldsymbol{\Pi}(\omega)$, i.e., Π_{con} is finer than $\wedge\boldsymbol{\Pi}$. Also, if $\omega' \in \Pi_{\text{con}}(\omega)$ then for all i and $\omega'' \in \Pi_i(\omega')$, $\omega'' \in \Pi_{\text{con}}(\omega)$, i.e., $\Pi_i(\omega') \subseteq \Pi_{\text{con}}(\omega)$. Thus, each of the partitions in Π is finer than Π_{con}. As $\wedge\boldsymbol{\Pi}$ is the finest partition with this property it follows that $\Pi_{\text{con}} = \wedge\boldsymbol{\Pi}$.

Thus, we conclude:

Claim 2. *A partition profile $\boldsymbol{\Pi}$ is connected if and only if every two states are connected.*

We say that a type profile $\mathbf{t}$ is **positive** if for each i, $\pi \in \Pi_i$, and $\omega \in \pi$, $t_i(\pi, \omega) > 0$. Let $\mathbf{t}$ be a positive type profile and (ω_1, ω_2) an ordered pair of states in $\pi \in \Pi_i$. The **type ratio** of (ω_1, ω_2) given i is $\text{tr}^i_t(\omega_1, \omega_2) = t_i(\pi, \omega_2)/t_i(\pi, \omega_1)$. The **type ratio** of a chain $c = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} \omega_n$ of length $n > 0$ is $\text{tr}_{\mathbf{t}}(c) = \bigtimes_{k=0}^{n-1} \text{tr}^{i_k}_{\mathbf{t}}(\omega_k, \omega_{k+1})$. For a chain c of length 0, $\text{tr}_{\mathbf{t}}(c) = 1$. Thus, if $c = c' \xrightarrow{i} \omega$ where c' is a chain from ω_0 to ω', $\text{tr}_{\mathbf{t}}(c) = \text{tr}_{\mathbf{t}}(c')\,\text{tr}^i_{\mathbf{t}}(\omega', \omega)$. When we discuss only one type profile we omit the subscript $\mathbf{t}$ in $\text{tr}_{\mathbf{t}}$.

Proposition 3. *If a positive type profile over a connected partition profile has a common improper prior p, then all its common improper priors are of the form γp for some constant $\gamma > 0$. A type profile over a connected partition profile can therefore have at most one common prior.*

Proof. If p is a *cip* for a positive $\mathbf{t}$, then $\text{tr}^i_{\mathbf{t}}(\omega_1, \omega_2) = p(\omega_2)/p(\omega_1)$. Substituting the right-hand side for the left-hand side in the definition of the type ratio of chains, we conclude that for any chain c from ω_0 to ω, $\text{tr}_{\mathbf{t}}(c) = p(\omega)/p(\omega_0)$. Thus, for any *cip*'s for $\mathbf{t}$, p and p', and for any two states ω_0 and ω, $p(\omega)/p(\omega_0) = p'(\omega)/p'(\omega_0)$. □

Proposition 3 was proved in Harsanyi (1967–1968) for Harsanyi type spaces. Samet (1998) noted that for finite spaces the uniqueness of a common prior can be interpreted as the uniqueness of an invariant probability function for an ergodic Markov chain. The simple proof here, for countable spaces, is an extension of the proof in Harsanyi (1967–1968) to general knowledge spaces.

The following proposition is close in its content to the main result in Rodrigues-Neto (2009).

Proposition 4. *Let $\mathbf{t}$ be a positive type profile over a connected partition profile. Then there exists a common improper prior for $\mathbf{t}$ iff for each ω_0 and ω, and chains c and c' from ω_0 to ω, $\text{tr}_{\mathbf{t}}(c) = \text{tr}_{\mathbf{t}}(c')$.*

Proof. As we have shown before, if there exists a common improper prior p for t, then all chains c connecting ω_0 and ω satisfy $\text{tr}(c) = p(\omega)/p(\omega_0)$. Conversely, suppose that for each ω_0 and ω, all the chains from ω_0 to ω have the same type ratio. Fix ω_0 and for each ω let $p(\omega) = \text{tr}(c)$ for some c from ω_0 to ω. To see that p is a *cip* consider $\pi \in \Pi_i$ and $\omega \in \pi$. Let c be a chain from ω_0 to ω. For $\omega' \in \pi$, consider the chain $c' = c \xrightarrow{i} \omega'$. Then, by the definitions of tr and p, $p(\omega') = \text{tr}(c') = \text{tr}(c)\,\text{tr}^i(\omega, \omega') = p(\omega) t_i(\pi, \omega')/t_i(\pi, \omega)$. Thus, $p(\pi) = \sum_{\omega' \in \pi} p(\omega') = [p(\omega)/t_i(\pi, \omega)] \sum_{\omega' \in \pi} t_i(\pi, \omega') = p(\omega)/t_i(\pi, \omega) < \infty$, and $p(\omega) = p(\pi) t_i(\pi, \omega)$. □

4.3. Proof of the second parts of Theorems 1 and 2

We first prove our claims for a connected partition profile $\boldsymbol{\Pi}$. Let P be the set of positive types in $\mathcal{T}(\boldsymbol{\Pi})$ and C the set of consistent type profiles in $\mathcal{T}(\boldsymbol{\Pi})$.

We show in the following two results that $C \cap P$ is nowhere dense, that is, that the complement of its closure is dense.

Lemma 1. *If $\boldsymbol{\Pi}$ is connected, then* $\mathrm{cl}(C \cap P) \subseteq (C \cap P) \cup P^c$.

Proof. We need to show that if a sequence of type profiles $\mathbf{t}^n$ in $C \cap P$ converges to $\mathbf{t} \in P$, then $\mathbf{t} \in C$. Let c and c' be chains from ω_0 to ω. By Proposition 4, $\mathrm{tr}_{\mathbf{t}^n}(c) = \mathrm{tr}_{\mathbf{t}^n}(c')$ for each n. Since each chain involves only finitely many states, it follows by continuity that $\mathrm{tr}_{\mathbf{t}}(c) = \mathrm{tr}_{\mathbf{t}}(c')$. Again, by Proposition 4, this implies that $t \in C$. □

Thus, $[\mathrm{cl}(C \cap P)]^c \supseteq [(C \cap P) \cup P^c]^c = C^c \cap P$, and it is enough to show that $C^c \cap P$ is dense.

Lemma 2. *If $\boldsymbol{\Pi}$ is connected and not tight then $C^c \cap P$ is dense in $\mathcal{T}(\boldsymbol{\Pi})$.*

Proof. We show that $C \cap P \subseteq \mathrm{cl}(C^c \cap P)$. Thus, $P \subseteq \mathrm{cl}(C^c \cap P)$, and as P is dense, the claim of the proposition follows.

Since $\boldsymbol{\Pi}$ is connected but not tight, there exists, by Proposition 1 a connected partition profile $\boldsymbol{\Pi}'$ which properly refines $\boldsymbol{\Pi}$. We may assume that $\boldsymbol{\Pi}'$ is obtained from $\boldsymbol{\Pi}$ by splitting one partition element $\pi \in \Pi_i$, for some i, into π^1 and π^2.

For $\mathbf{t} \in P$, define a type profile $\hat{\mathbf{t}}$ for $\boldsymbol{\Pi}$ which agrees with $\mathbf{t}$ except on π. Formally, for each $j \neq i$, $\hat{t}_j = t_j$. For each $\bar{\pi} \neq \pi$ in Π_i, $\hat{t}_i(\bar{\pi}, \cdot) = t_i(\bar{\pi}, \cdot)$. For $\omega \in \pi^1$, $\hat{t}_i(\pi, \omega) = (1 + \varepsilon)t_i(\pi, \omega)/c$, and for $\omega \in \pi^2$, $\hat{t}_i(\pi, \omega) = (1 - \varepsilon)t_i(\pi, \omega)/c$, where $c = 1 + \varepsilon[t_i(\pi, \pi^1) - t_i(\pi, \pi^2)]$ and $\varepsilon \neq 0$ between -1 and 1. By choosing ε close enough to 0, $\hat{\mathbf{t}}$ can be made arbitrarily close to $\mathbf{t}$.

For any type profile $\mathbf{t}$ in P, let $\mathbf{t}'$ be the type profile for $\boldsymbol{\Pi}'$ which is naturally induced by $\mathbf{t}$ as follows. For each $j \neq i$, $t'_j = t_j$. For each $\bar{\pi} \neq \pi$ in Π_i, $t'_i(\bar{\pi}, \cdot) = t_i(\bar{\pi}, \cdot)$. Finally, for $k = 1, 2$, $t'_i(\pi^k, \cdot) = t_i(\pi, \cdot)/t_i(\pi, \pi^k)$.

Now, let $\mathbf{t} \in P \cap C$ have a *cip* p. We show that $\hat{\mathbf{t}} \in C^c \cap P$. Obviously, p is also a *cip* for $\mathbf{t}'$. Suppose that $\hat{\mathbf{t}}$ has a *cip*, and denote it by $\hat{p}$. Then $\hat{p}$ is also a *cip* for $\hat{\mathbf{t}}'$. But $\hat{\mathbf{t}}' = \mathbf{t}'$, and as $\boldsymbol{\Pi}'$ is connected, it follows, by Proposition 3, that p and $\hat{p}$ differ by a multiplicative constant. Thus, p is a *cip* for $\hat{\mathbf{t}}$ as well. Hence p must satisfy $t_i(\pi, \pi^1) = p(\pi^1)/p(\pi) = \hat{t}_i(\pi, \pi^1)$. But this does not hold as $\hat{t}_i(\pi, \pi^1) = (1 + \varepsilon)t_i(\pi, \pi^1)/c$ and $(1 + \varepsilon)/c \neq 1$. □

Assume throughout the rest of this section that $\boldsymbol{\Pi}$ is not tight.

One has that $C = (C \cap P) \cup (C \cap P^c) \subseteq (C \cap P) \cup P^c$, and we have shown that $(C \cap P)$ is nowhere dense. In the finite case, P is an open dense set and thus P^c is nowhere dense, so that $(C \cap P) \cup P^c$ is nowhere dense as a finite union of nowhere dense set, and C is nowhere dense as a subset of a nowhere dense set.

For the infinite case, it suffices to show that P^c is of first category. This is indeed the case, because the set T^i_ω of type profiles t for which $t_i(\Pi_i(\omega), \omega) = 0$ is closed and has an empty interior, as its complement contains P which is dense. Thus, T^i_ω is nowhere dense. Finally, $P^c = \bigcup_i \bigcup_\omega T^i_\omega$.

Consider next a partition profile $\boldsymbol{\Pi}$ that is not connected and not tight. For $M \in \wedge\boldsymbol{\Pi}$, denote by $\mathcal{T}_M(\boldsymbol{\Pi}_M)$ the set of type profiles over the knowledge space $(M, \boldsymbol{\Pi}_M)$, and let C_M be the set of consistent type profiles in $\mathcal{T}_M(\boldsymbol{\Pi}_M)$. We can obviously identify $\mathcal{T}(\boldsymbol{\Pi})$ with $\bigtimes_{M \in \wedge\boldsymbol{\Pi}} \mathcal{T}_M(\boldsymbol{\Pi}_M)$.

A type profile $\mathbf{t}$ for $\boldsymbol{\Pi}$ has a *cip* if and only if there exists $M \in \wedge\boldsymbol{\Pi}$ for which $\mathbf{t}_M$, the restriction of $\mathbf{t}$ to $M \times M$, is in C_M, the set of consistent type profiles in $\mathcal{T}_M(\boldsymbol{\Pi}_M)$. Indeed, if $\mathbf{t}_M$ has a *cip* p_M, then the function p on Ω that agrees with p_M on M and vanishes outside M is a *cip* for $\mathbf{t}$. Conversely, if p is a *cip* for $\mathbf{t}$, then for some M, p is not identically 0 on M and thus the restriction of p to M is a *cip* for $\mathbf{t}_M$. We conclude that C, the set of consistent type profiles in $\mathcal{T}(\boldsymbol{\Pi})$, is $\bigcup_{M \in \wedge\boldsymbol{\Pi}}[C_M \times (\bigtimes_{M' \neq M} \mathcal{T}_{M'}(\boldsymbol{\Pi}_{M'}))]$.

Since $\boldsymbol{\Pi}$ is not tight, it follows by Proposition 1 that for each $M \in \wedge\boldsymbol{\Pi}$, $\boldsymbol{\Pi}_M$ is not tight.

In the finite case, this implies that C_M is nowhere dense in $\mathcal{T}_M(\boldsymbol{\Pi}_M)$ and therefore each of the sets in the union is nowhere dense in $\mathcal{T}(\boldsymbol{\Pi})$. Hence, C is nowhere dense as a finite union of nowhere dense sets.

In the infinite case, C_M is of first category and therefore each of the sets in the union is of first category in $\mathcal{T}(\boldsymbol{\Pi})$. Hence, C is of first category as a countable union of sets of first category. We have thus completed the proofs of the second parts of both Theorems 1 and 2. □

4.4. Proof of the first parts of Theorems 1 and 2

We say that a chain c is **alternating** if no two consecutive states, ω_s and ω_{s+1}, in c, are the same, and no two consecutive agents, i_s and i_{s+1}, in c, are the same. In particular, any chain of length 0 is alternating and any chain of length 1 from ω_0 to $\omega \neq \omega_0$ is alternating.

Given a connected partition profile $\boldsymbol{\Pi}$, define a distance function d on $\Omega \times \Omega$ such that for each ω and ω', $d(\omega, \omega')$ is the minimal length of a chain from ω to ω'. It is easy to see that d is a metric. A chain from ω_0 to ω of the minimal length $d(\omega, \omega_0)$ is called a **minimal chain**. It is easy to see that if $\omega_0 \ldots \omega_n$ is minimal then $\omega_0 \ldots \omega_s$ is a minimal chain for each $s = 0, \ldots, n$, and therefore $d(\omega_s, \omega_0) = s$. Moreover, the chain must be alternating, because if either $\omega_s = \omega_{s+1}$ or $i_s = i_{s+1}$ we get a shorter chain from ω_0 to ω_n by omitting $\xrightarrow{i_s} \omega_{s+1}$.

Clearly, if for some i, $\omega' \in \Pi_i(\omega)$, then $d(\omega, \omega') \leqslant 1$. Thus, by the triangle inequality, if for some i, $\omega' \in \Pi_i(\omega)$, then for any ω_0, $|d(\omega, \omega_0) - d(\omega', \omega_0)| \leqslant 1$. Thus, on each partition element π, $d(\cdot, \omega_0)$ can have at most two values. In particular, for any chain $c = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} \omega_n$, $|d(\omega_{s+1}, \omega_0) - d(\omega_s, \omega_0)| \leqslant 1$ for $s = 0, \ldots, n-1$.

Proposition 5. *A connected partition profile $\boldsymbol{\Pi}$ is tight if and only if for any states ω and ω' there exists a unique alternating chain for $\boldsymbol{\Pi}$ from ω to ω'.*

Proof. Assume that $\boldsymbol{\Pi}$ is not tight. Then, there exists a connected partition profile $\boldsymbol{\Pi}'$ such that $\boldsymbol{\Pi}' \succ \boldsymbol{\Pi}$. Let ω, ω' and i be such that $\omega' \in \Pi_i(\omega)$ but $\omega' \notin \Pi'_i(\omega)$. Since $\boldsymbol{\Pi}'$ is connected, there exists a minimal chain c for $\boldsymbol{\Pi}'$ from ω to ω', which, as we have shown, is alternating. Since $\boldsymbol{\Pi}'$ is a refinement of $\boldsymbol{\Pi}$, c is also a chain for $\boldsymbol{\Pi}$ and it is alternating. But as $\omega \neq \omega'$, $c' = \omega \xrightarrow{i} \omega'$ is also an alternating chain for $\boldsymbol{\Pi}$ which is different from c, since $\omega' \notin \Pi'_i(\omega)$.

Assume now that $\boldsymbol{\Pi}$ is tight. To show that the condition in the proposition holds we use the following two lemmas.

Lemma 3. *If $\boldsymbol{\Pi}$ is tight then for each ω_0 and ω there exists a unique minimal chain from ω_0 to ω.*

We show that if there are two distinct minimal chains from one state to another then $\boldsymbol{\Pi}$ is not tight. Let $c = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} \omega_n$ and $c' = \omega_0 \xrightarrow{i'_0} \omega'_1 \xrightarrow{i'_1} \cdots \xrightarrow{i'_{n-1}} \omega_n$ be distinct minimal chains, and assume that n is the minimal number for which such a pair exists. Obviously, $n > 0$. It is impossible that both $i_0 = i'_0$ and $\omega_1 = \omega'_1$, because either $n = 1$ in which case $c = c'$, or else $n > 1$ in which case $\omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} \omega_n$ and $\omega'_1 \xrightarrow{i'_1} \cdots \xrightarrow{i'_{n-1}} \omega_n$ are distinct minimal chains of length $n-1$ contrary to the minimality of n. Thus, either $i_0 \neq i'_0$ or $\omega_1 \neq \omega'_1$.

Consider the refinement $\boldsymbol{\Pi}'$ of $\boldsymbol{\Pi}$ obtained by splitting $\Pi_i(\omega_0)$ into $\{\omega_1\}$ and $\Pi_i(\omega_0) \setminus \{\omega_1\}$. The latter set is not empty since being minimal, c is alternating and thus, $\omega_0 \neq \omega_1$. We will prove that $\boldsymbol{\Pi}$ is not tight by showing that $\boldsymbol{\Pi}'$ is connected. To do so, it suffices to prove that every $\hat{\omega} \in \Pi_i(\omega_0) \setminus \{\omega_1\}$ is connected to ω_1 for $\boldsymbol{\Pi}'$. As $\hat{\omega}$ is connected to ω_0 for $\boldsymbol{\Pi}'$, it suffices to show that ω_0 is connected to ω_1 for $\boldsymbol{\Pi}'$.

Assume first that $\omega_1 = \omega'_1$. Thus, $i_0 \neq i'_0$ and therefore $\omega_0 \xrightarrow{i'_0} \omega_1$, is a chain for $\boldsymbol{\Pi}'$. Now assume that $\omega_1 \neq \omega'_1$, which implies that $n > 1$. Note that all states in $\Pi_i(\omega_0)$ are of distance not greater than 1 from ω_0 and thus each of the states $\omega_2, \ldots, \omega_{n-1}, \omega_n$ and $\omega'_2, \ldots, \omega'_{n-1}, \omega_n$ are not in this set, as their distance from ω_0 is greater than 1. Thus, $\omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{m-1}} \omega_m$ and $\omega'_1 \xrightarrow{i'_1} \cdots \xrightarrow{i'_{m-1}} \omega_m$ are chains for $\boldsymbol{\Pi}'$ too. Also, because $\omega_1 \neq \omega'_1$, $\omega_0 \xrightarrow{i'_0} \omega'_1$ is a chain for $\boldsymbol{\Pi}'$ (even if $i_0 = i'_0$). Thus, we have shown that the following relations hold for $\boldsymbol{\Pi}'$: $\omega_0 \to \omega'_1$ $\omega'_1 \to \omega_n$, and $\omega_n \to \omega_1$, which amounts to saying that ω_0 and ω_1 are connected in $\boldsymbol{\Pi}'$.

Lemma 4. *If $\boldsymbol{\Pi}$ is tight then every alternating chain for $\boldsymbol{\Pi}$ is minimal.*

The proof is by induction on n, the length of the chain. The claim is obvious for alternating chains of lengths $n = 0$ and $n = 1$. Suppose the claim holds for alternating chains of length $n = k \geqslant 1$, and assume that c is an alternating chain $c = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \cdots \to \omega_k \xrightarrow{i_k} \omega$ of length $k+1$. By the induction hypothesis the alternating chain $\omega_0 \ldots \omega_k$ is minimal and thus for all $s \leqslant k$, $d(\omega_s, \omega_0) = s$. In particular $d(\omega_k, \omega_0) = k$ and thus $d(\omega, \omega_0)$ is either $k+1$, or k, or $k-1$. We only need to prove that the last two values are impossible. Suppose that $d(\omega, \omega_0) < k+1$. Let $c' = \omega_0 \xrightarrow{i'_0} \omega'_1 \ldots \omega$ be a minimal chain from ω_0 to ω. Then it is of length $k-1$ or k.

Consider the refinement $\boldsymbol{\Pi}'$ obtained by splitting $\Pi_{i_k}(\omega_k)$ into $\{\omega_k\}$ and $\Pi_{i_k}(\omega_k) \setminus \{\omega_k\}$. The latter set is not empty, since by the alternation of c, $\omega_k \neq \omega$. We show that for each $\hat{\omega} \in \Pi_{i_k}(\omega_k) \setminus \{\omega_k\}$ there is a chain for $\boldsymbol{\Pi}'$ from ω_k to $\hat{\omega}$. Note, first, that the chain $\omega_0 \ldots \omega_k$ is a chain for $\boldsymbol{\Pi}'$. To see this, observe that by alternation $i_{k-1} \neq i_k$. Thus, $\omega_{k-1} \xrightarrow{i_{k-1}} \omega_k$ is a chain for $\boldsymbol{\Pi}'$. Also the states $\omega_0, \ldots, \omega_{k-2}$ are of distance less than $k-1$ from ω_0 while all states in $\Pi_{i_k}(\omega_k)$ are of distance $k-1$ at least. Thus, $\omega_0 \ldots \omega_{k-2}$ is also a chain for $\boldsymbol{\Pi}'$. We conclude that there is a chain for $\boldsymbol{\Pi}'$ from ω_k to ω_0. We end the proof by showing that there is a chain in $\boldsymbol{\Pi}'$ from $\hat{\omega}$ to ω_0. First, $\hat{\omega} \xrightarrow{i_k} \omega$ is a chain for $\boldsymbol{\Pi}'$ by the construction of the latter. Moreover, none of the states in c' is ω_k, which implies that c' is also a chain for $\boldsymbol{\Pi}'$. Indeed, $\omega \neq \omega_k$ by the alternation of c, and the distance from ω_0 to all of the states in c' that precede ω is less than k, while ω_k is of distance k from ω_0. □

The characterization of tightness in terms of alternating chains is used to prove the following.

Lemma 5. *If $\boldsymbol{\Pi}$ is connected and tight then every positive type profile has a common improper prior.*

Proof. We show that the condition in Proposition 4 holds. By Proposition 5 for any states ω_0 and ω there exists a unique alternating chain $c(\omega_0, \omega)$ from ω_0 to ω. We show that for any ω_0 and ω and any chain c from ω_0 to ω, $\mathrm{tr}(c) = \mathrm{tr}(c(\omega_0, \omega))$. The proof is by induction on the length of c. The claim trivially holds for chains of length 0. Assume that it holds for all chains of length $n-1$ for $n > 0$ and let $c = c' \xrightarrow{i} \omega$ be a chain from ω_0 to ω of length n, where c' is a chain from ω_0 to ω'. By the induction hypothesis, $\mathrm{tr}(c') = \mathrm{tr}(c(\omega_0, \omega'))$. Consider the chain $\hat{c} = c(\omega_0, \omega') \xrightarrow{i} \omega$. Then, $\mathrm{tr}(c) = \mathrm{tr}(c')\,\mathrm{tr}^i(\omega', \omega) = \mathrm{tr}(c(\omega_0, \omega'))\,\mathrm{tr}^i(\omega', \omega) = \mathrm{tr}(\hat{c})$, and therefore it suffices to show that $\mathrm{tr}(\hat{c}) = \mathrm{tr}(c(\omega_0, \omega))$. If $\hat{c}$ is alternating then $\hat{c} = c(\omega_0, \omega)$, and we are done. If $\hat{c}$ is not alternating, then, since $c(\omega_0, \omega')$ is alternating, one of the following two cases holds. Case 1: $\omega' = \omega$. In this case $\mathrm{tr}(\hat{c}) = \mathrm{tr}(c(\omega_0, \omega))\,\mathrm{tr}^i(\omega, \omega) = \mathrm{tr}(c(\omega_0, \omega))$. Case 2: $\omega \neq \omega'$, and for $c(\omega_0, \omega') = \omega_0 \ldots \omega_{n-2} \xrightarrow{i_{n-2}} \omega'$, $i_{n-2} = i$. Here, $\mathrm{tr}(\hat{c}) = \mathrm{tr}(\omega_0 \ldots \omega_{n-2})\,\mathrm{tr}^i(\omega_{n-2}, \omega')\,\mathrm{tr}^i(\omega', \omega) = \mathrm{tr}(\omega_0 \ldots \omega_{n-2})\,\mathrm{tr}^i(\omega_{n-2}, \omega)$. But the latter is the type ratio of the chain $\omega_0 \ldots \omega_{n-2} \xrightarrow{i} \omega$, which, being alternating is $c(\omega_0, \omega)$. □

The proof of part 1 in Theorems 1 and 2 follows readily in case $\boldsymbol{\Pi}$ is connected. As before, P is the set of positive type profiles and C the set of type profiles that have a *cip*. By Lemma 5, $C^c \subseteq P^c$.

In the infinite case, we have shown that P^c is of first category, and thus, C^c is of first category. In the finite case, C is closed. Indeed, let $\mathbf{t}^n$ be a sequence of type profiles in C that converges to $\mathbf{t}$. For each n, $\mathbf{t}^n$ has a common prior p^n that satisfies for each i and $\pi \in \Pi_i$, $p^n(\pi) t_i^n(\pi, \cdot) = p^n(\cdot)$. By the compactness of $\Delta(\Omega)$, a subsequence of p^n converges to a probability function p. By continuity, $p(\pi) t_i(\pi, \cdot) = p(\cdot)$ for each i and $\pi \in \Pi_i$. Thus, p is a common prior for $\mathbf{t}$ and $\mathbf{t} \in C$. By Lemma 5, $P \subset C$ and thus, $\mathcal{T}(\boldsymbol{\Pi}) = \mathrm{cl}(P) \subseteq \mathrm{cl}(C) = C$.

Suppose that $\boldsymbol{\Pi}$ is not connected, then $C^c = \bigtimes_{M \in \wedge \boldsymbol{\Pi}} C_M^c$. As $\boldsymbol{\Pi}$ is tight, there is an $\hat{M} \in \wedge \boldsymbol{\Pi}$ such that $\boldsymbol{\Pi}_{\hat{M}}$ is tight. In the finite case, $C_{\hat{M}}^c = \emptyset$ and therefore $C^c = \emptyset$. In the infinite case, $C_{\hat{M}}^c \subseteq P_{\hat{M}}^c$, and as $P_{\hat{M}}^c$ is of first category, so is C^c. □

4.5. Proof of Proposition 2

Let $\boldsymbol{\Pi}$ be a connected tight partition profile. The proof is by induction on the size on Ω. If Ω is a singleton the equality in the proposition is obvious. Suppose the equality is proved for all state spaces smaller than $n > 1$ and let $|\Omega| = n$. Since $n \geqslant 2$ and $\boldsymbol{\Pi}$ is connected, there must be i and ω_0 such that $\Pi_i(\omega_0)$ is not a singleton. Consider the refinement of $\boldsymbol{\Pi}$, $\hat{\boldsymbol{\Pi}}$, obtained by splitting $\Pi_i(\omega_0)$ into $\{\omega_0\}$ and $\Pi_i(\omega_0) \setminus \{\omega_0\}$. By the tightness of $\boldsymbol{\Pi}$, $\hat{\boldsymbol{\Pi}}$ is not connected.

Let Ω_0 consist of all states ω such that there is a chain for $\hat{\boldsymbol{\Pi}}$ from ω_0 to ω. Fix ω_1 in the set $\Pi_i(\omega_0) \setminus \{\omega_0\}$ and let Ω_1 be the set of all ω such that there is a chain for $\hat{\boldsymbol{\Pi}}$ from ω_1 to ω. Each of Ω_0 and Ω_1 is an element of the meet of $\hat{\boldsymbol{\Pi}}$. They are disjoint because if they shared a state then ω_0 would be connected to ω_1 which would make $\hat{\boldsymbol{\Pi}}$ connected. Each state ω is in either Ω_0 or Ω_1. Indeed, let c be a minimal chain for $\boldsymbol{\Pi}$ from ω_0 to an arbitrary ω. If c does not contain a state $\hat{\omega} \in \Pi_i(\omega_0) \setminus \{\omega_0\}$, then c is a chain for $\hat{\boldsymbol{\Pi}}$ and $\omega \in \Omega_0$. If c does contain such a $\hat{\omega}$, then no state that follows $\hat{\omega}$ in c is ω_0 (because the distance of each state in c to ω_0, other than ω_0 itself, is positive). Thus, there is a chain for $\hat{\boldsymbol{\Pi}}$ from $\hat{\omega}$ to ω, and trivially there is a chain for $\hat{\boldsymbol{\Pi}}$ from $\hat{\omega}$ to ω_1, so that $\omega \in \Omega_1$. Thus, the meet of $\hat{\boldsymbol{\Pi}}$ is exactly the set $\{\Omega_0, \Omega_1\}$, and each of $\hat{\boldsymbol{\Pi}}^0$ and $\hat{\boldsymbol{\Pi}}^1$, respectively the restriction of $\hat{\boldsymbol{\Pi}}$ to Ω_0 and Ω_1, is connected.

By the induction hypothesis, for $k = 0, 1$, $\sum_{i \in I} |\hat{\boldsymbol{\Pi}}_i^k| = (|I| - 1)|\Omega_k| + 1$. By adding the two equations and noting that $\sum_{i \in I} |\hat{\boldsymbol{\Pi}}_i^0| + \sum_{i \in I} |\hat{\boldsymbol{\Pi}}_i^1| = \sum_{i \in I} |\Pi_i| + 1$ we get the desired equality.

If $\boldsymbol{\Pi}$ is not tight, then it must have a refinement which is tight, and therefore it satisfies the inequality of the proposition. □

4.6. Examples

Example 2. We construct an infinite knowledge space with a tight partition profile, such that the set of inconsistent type profiles is dense. Therefore it is not nowhere dense, since the complement of a nowhere dense set contains a nonempty open set. To show that the set of inconsistent type profiles is dense, it is enough to show that it is dense in the set of positive type profiles, since the latter is dense.

Consider a knowledge space for two agents, where Ω is the set of integers $\mathbb{Z}$, and the partitions are $\Pi_1 = \{\pi_1^n \mid n \in \mathbb{Z}\}$, where $\pi_1^n = \{2n, 2n+1\}$, and $\Pi_2 = \{\pi_2^n \mid n \in \mathbb{Z}\}$, where $\pi_2^n = \{2n-1, 2n\}$. The partition profile $\boldsymbol{\Pi} = (\Pi_1, \Pi_2)$ is tight, since it is connected and any proper refinement of $\boldsymbol{\Pi}$ is not. Let $\mathbf{t} = (t_1, t_2)$ be a positive type profile over $\boldsymbol{\Pi}$. We construct a sequence of inconsistent type profiles $\mathbf{t}^k$ such that $\mathbf{t}^k$ converges to $\mathbf{t}$ as $k \to -\infty$. For $n \leqslant k$, $t_1^k(\pi_1^n, 2n) = 1$ and $t_2^k(\pi_2^n, 2n) = 0$. For $n > k$, $t_1^k(\pi_1^n, \cdot) = t_1(\pi_1^n, \cdot)$ and $t_2^k(\pi_2^n, \cdot) = t_2(\pi_2^n, \cdot)$. Obviously, $\mathbf{t}^k$ converges to $\mathbf{t}$ as $k \to -\infty$.

To show that $\mathbf{t}^k$ is inconsistent we prove that if p is a *cip* for $\mathbf{t}^k$, then it must be identically 0, which is impossible for a *cip*. By the definition of *cip* whenever for some i, $\pi \in \Pi_i$ and $\omega \in \pi$, $t_i(\pi, \omega) = 0$, then $p(\omega) = 0$. Now, $t_1^k(\pi_1^k, 2k+1) = 0$ and therefore $p(2k+1) = 0$. Also, for each $m \leqslant 2k$, either $t_1^k(\Pi_1(m), m) = 0$ or $t_2^k(\Pi_2(m), m) = 0$. Thus, $p(m) = 0$ for all $m \leqslant 2k+1$. We prove now by induction on m that $p(m) = 0$ for all $m \geqslant 2k+1$. This holds as we have shown for $m = 2k+1$. Suppose that for $m = 2n+1$, $p(m) = 0$. Since $t_2^k(\pi_2^{2n+1}, 2n+1) = t_2(\pi_2^{2n+1}, 2n+1) > 0$ it follows by the definition of *cip* that $p(\pi_2^{2n+1}) = 0$. This implies that $p(2n+2) = 0$. The induction step when $m = 2n$ is similar.

Example 3. We construct an infinite knowledge space for two agents with a partition profile which is not tight, such that the set of consistent type profiles is dense, which shows that it is not nowhere dense. To show this we prove that the set of consistent type profiles is dense in the set of positive type profiles.

Let Ω be the set $\mathbb{N} \times \mathbb{N}$. Player 1's partition consists of the rows and 2's the columns. That is, $\Pi_1 = \{\pi_1^i \mid i \in \mathbb{N}\}$, where $\pi_1^i = \{(i, j) \mid j \in \mathbb{N}\}$, and $\Pi_2 = \{\pi_2^j \mid j \in \mathbb{N}\}$, where $\pi_2^j = \{(i, j) \mid i \in \mathbb{N}\}$.

Let $\mathbf{t}$ be a positive type profile for this partition profile. We define a sequence of consistent type profiles $\mathbf{t}^n$ that converge to $\mathbf{t}$ as $n \to \infty$. Fix a consistent type profile $\hat{\mathbf{t}}$ with *cip* p. For each $i \leqslant n$ and $j \leqslant n$, let

$$t_1^n\big(\pi_1^i, (i, j)\big) = t_1\big(\pi_1^i, (i, j)\big) \Big/ \sum_{k=1}^{n} t_1\big(\pi_1^i, (i, k)\big)$$

and

$$t_2^n\big(\pi_2^j, (i, j)\big) = t_2\big(\pi_2^j, (i, j)\big) \Big/ \sum_{k=1}^{n} t_2\big(\pi_2^j, (k, j)\big).$$

For $i \geqslant n+1$ and $j \geqslant n+1$, $t_1^n(\pi_1^i, (i, j)) = \hat{t}_1(\pi_1^{i-n}, (i-n, j-n))$, and $t_2^n(\pi_2^j, (i, j)) = \hat{t}_2(\pi_2^{j-n}, (i-n, j-n))$. For $i \leqslant n$ and $j \geqslant n+1$, or $i \geqslant n = 1$ and $j \leqslant n$, $t_1^i(\pi_1^i, (i, j)) = t_2^j(\pi_2^j, (i, j)) = 0$.

It is easy to check that for each i, $\|t_1^n(\pi_1^i, \cdot) - t_1(\pi_1^i, \cdot)\| \to 0$ when $n \to \infty$, and a similar convergence holds for agent 2. Thus, $\mathbf{t}^n \to \mathbf{t}$. To see that $\mathbf{t}^n$ is consistent, we define p^n by $p^n(i, j) = p(i-n, j-n)$ for $i \geqslant n+1$ and $j \geqslant n+1$ and $p^n(i, j) = 0$ otherwise. It is easy to see that p^n is a *cip* for $\mathbf{t}^n$.

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