GRAPH VALUE FOR COOPERATIVE GAMES

ZIV HELLMAN AND RON PERETZ

ABSTRACT. We suppose that players in a cooperative game are located within a graph structure, such as a social network or supply route, that limits coalition formation to coalitions along connected paths within the graph. This leads to a generalisation of the Shapley value that is studied here from an axiomatic perspective. The resulting 'graph value' is endogenously asymmetric, with the automorphism group of the graph playing a crucial role in determining the relative values of players.

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1. INTRODUCTION

The standard interpretation of the Shapley value, as a measure of the average marginal contribution of a player to each and every possible coalition, may strain credulity if taken too literally in a great many social situations. This holds particularly when players may, due to affinity, consanguinity or other factors, have clear preferences for joining certain coalitions as opposed to others. Consider, for just one example, a job market. Is it not more likely that a potential hire will join a company if he knows someone within the company? How likely is it for a job seeker to join a company if she does not share a common language with any of its current employees?

Cases in which many theoretically possible coalitions will not realistically be formed are not limited to social situations alone. If one is studying cooperative coalitions amongst players connected via supply routes, computer networks or web links, there are clear structural reasons for entirely excluding some coalitions that would otherwise play a role in the calculation of the classic Shapley value and including in consideration instead only coalitions that are connected along the underlying network.

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Department of Economics, Bar Ilan University; Department of Mathematics, London School of Economics *email:* ziv.hellman@gmail.com, ronprtz@gmail.com
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Networks, for obvious reasons, have increasingly been a focus of study in several disciplines over the past two decades.[1] What we propose here is introducing network or graph structures directly into the study of coalitional game theory, by limiting consideration of potential coalitions solely to coalitions that are connected along the graph. Doing so, in the tradition of measuring average marginal contributions, yields different values that we argue may be more appropriate for assessing the values of players in many situations than the classic Shapley value.

The potential applications of a graph value are many. A partial list may include: coalition formation in complex political situations; studying power relations and cost sharing in situations with geographic constraints such as supply routes along roads or rivers; coalition formation in social networks; and perhaps even cooperation between neighbouring genes inside chromosomes.

This requires departing in some ways from the classic model of transferable utility games, which associates a certain worth to *every* coalition. That model implicitly assumes that the only force that drives the formation of coalitions is the worth they generate. The model we introduce here takes into account a proximity relation between players represented as edges of an undirected graph (a symmetric binary relation). It is assumed that a player only joins a coalition if he is connected to one of its members. As a result the only admissible coalitions are the connected subgraphs.

For our axioms we conservatively adopt the standard Shapley axioms (plus monotonicity), with minor adjustments to fit them for our model. The most significant difference this imposes is on symmetry (which is usually regarded as the least controversial of the Shapley value axioms). Classic symmetry cannot be carried over to our setting because the graph structure, and the relative positioning of players along the graph structure, is in itself an asymmetry. This leads to a weaker form of symmetry with respect only to automorphisms of the underlying graph.

By hewing closely to most of the standard Shapley axioms, we are able to carry out a step-by-step development of concepts that are directly analogous to those associated with the standard Shapley value, such as probabilistic values and random values. The price of using a weaker symmetry axiom, however, is that it leads to a graph value that is not uniquely determined by the axioms; we instead derive a convex set of possible values. Specification of a unique graph value, it turns out, will in most cases require specifying a particular random ordering, intuitively corresponding to agreement amongst

[1] Perhaps a contemporary canonical example would be an on-line social network, with coalitions naturally growing in size by way of adding at each stage friends of current members.

the players as to how coalitions are likely to be formed along the connected paths of the underlying graph.

On the other hand, the value we derive *is* a generalisation of the Shapley value, because when the underlying graph is the complete graph the set of admissible coalitions is again the full power set of the set of players. In that case, there is a unique graph value that is exactly the classic Shapley value. Conversely, we show that there are graphs for which the graph value is unique and yet it is different from the Shapley value.

It should be noted that we do not depart from the classic assumption of complete information. The study of coalitional games of incomplete information is important in its own right. See for example Forges and Serrano (2011).

1.1. **Related Literature.** Our main inspiration, and the paper that is most similar in approach to this one, is Álvarez, Hellman and Winter (2013), which proposes a way to measure the relative power of political parties in a parliament by explicitly taking into account a political spectrum. That paper notes that it is highly unlikely for a left-wing party to form a coalition with a party holding strongly diametrical right-wing views unless there are other parties in the coalition that can 'bridge' the ideological differences. In more general terms, a political party will tend to join a pre-existing coalition only if the coalition contains at least one other party that is ideologically close to it. To formalise this idea, Álvarez, Hellman and Winter (2013) postulates that parties can be ordered along a political spectrum (i.e., a strict linear ordering), from right to left, and a coalition will form only if it consists of a consecutive range of ideological views along this spectrum.[2]

One possible shortcoming of that approach is that it may be artificial to ascribe all ideological differences to positioning along a single linear ordering. In practice, ideologies are often multidimensional, relating to several issues. That observation led to the model presented in this paper, which is a generalisation of the model in Álvarez, Hellman and Winter (2013). As an added benefit, by extending the underlying topology of the connections between players to any graph, the model here is potentially applicable to a very wide range of cooperative situations, including but by no means restricted to political-coalitional settings.

Weakening the axiom of symmetry for the sake of considering variations on the Shapley value is a very old idea. Weighted Shapley values were

[2] As here, Álvarez, Hellman and Winter (2013) work with a weak version of symmetry and hence do not derive a unique value from the standard Shapley axioms alone. In that paper, an axiom reminiscent of various balanced contributions axioms, relating to unanimity games, needs to be added to attain uniqueness of the value.

proposed by Lloyd Shapley himself in his seminal PhD thesis (Shapley (1953b)). Each weighted Shapley value associates a positive weight with each player. These weights are the proportions of the players' shares in unanimity games. The symmetric Shapley value is the special case in which all weights are the same. This concept was studied axiomatically in Kalai and Samet (1987).

The weights in these models, however, are imposed exogenously, representing some pre-existing measure of the relative strengths of the players which is then used for calculating weighted Shapley values. In contrast, in the approach here asymmetries arise endogenously from the positioning of the players along the underlying graph structure.

This paper is also far from the first to study situations in which not every coalition is feasible or equally likely. The issue is usually tackled by considering some structure on the set of players that circumscribes the way players can form coalitions. Games with these kind of structures are usually denoted games with restricted cooperation.

Among the earliest efforts in this direction, the beginnings of a large literature, are Aumann and Drèze (1975) and Owen (1977). These start from the supposition that cooperative games are endowed with a coalitional structure, an exogenously given partition of the players. When coalitions are formed, the players interact at two levels: first, bargaining takes place among the unions and then bargaining takes place inside each union. Within each union, however, every possible coalition is admissible.

Edelman (1997) and Bilbao and Edelman (2000) take an approach similar to the one adopted in the present paper, using geometric constraints to dictate which coalitions may be formed and which are deemed impossible. They, however, use the theory of convex geometries as the basis for their research, as opposed to the more restricted model of connected graphs used here. We believe that much of the theory developed here can be extended to the more general model of convex geometries. The modification that would be required for this would apply to admissible coalitions, in that any lattice with maximal chains of the size of the entire set of players would be an acceptable admissible coalition, as opposed to connected subgraphs alone. The symmetry axiom would then need to be modified to relate to automorphisms of lattices. That said, we do wish to note that our axiomatic treatment of symmetry more closely resembles Shpaley's axiomatics as opposed to the more descriptive approach of Bilbao and Edelman (2000).

Graphs appear explicitly in Myerson (1977), but in a very different role from the one they have in this paper. There, an undirected graph describes communication possibilities between the players. A modification of the

Shapley value is then proposed under the assumption that coalitions that are not connected in this graph are split into connected components. In that model too, within components all possible coalitions are admissible.

Myerson's model implicitly assumes superadditivity by granting (disconnected) coalitions the sums of the worths of their connected components. In our model disconnected coalitions are simply impossible, hence they do not assume *any* worth and we need not assume superadditivity.

Myerson (1977) assumes a fixed coalitional function while letting network structures vary, with axioms focussed on how allocation rules are related as the network structure changes. We consider the network as given, with our axioms focussed on how allocation rules are related as coalition functions vary, in the tradition of Shapley (1953a). An interesting future extension to our research here might concentrate on relationships between values attached to different graphs. Perhaps such an attempt can lead to a unique specification of a value.

One may propose modelling impossible coalitions by setting their worths to zero while all other coalitions have positive worths. However, the choice of zero as the worth of impossible coalitions would be rather arbitrary and unjustified, as it makes the model variant under conditions of strategic equivalence.

A situation in which the above is particularly problematic is cost sharing models (Megiddo, 1978; Granot and Huberman, 1981; Young, 1985). Consider, for example, organisations attempting to establish a communication network or supply route between themselves. Setting up a link between two organisations induces a cost. Due to physical constraints or geographic barrier, not every pair of organizations can be linked directly, while indirect connections via a sequence of links requires the active cooperation of all the organisations along these links. Examples may include a network of monetary transactions between banks or commodity flows between countries. The cost associated with a connected (admissible) coalition is the minimal total cost of links that connect the members of that coalition (minimal spanning tree). The cost of setting up the entire network has to be divided amongst the players (organisations).

The Shapley value, with its axioms interpreted as describing acceptable requirements for 'fair' cost allocation, has been proposed as the solution for cost sharing problems (the literature on this is vast, going back at least as far as Shubik (1962)). When some coalitions are deemed impossible, one could be tempted to compute an appropriate Shapley value by associating an extremely large cost with each impossible coalition. This approach does not work, since very large costs dictate heavy the costs in the resultant

Shapley value. Some players will pay very large costs whereas others will receive very large payments (negative cost). As the costs associated with impossible coalitions grow, the actual costs of the links become negligible. The present paper proposes a solution that generalises the Shapley value to situations in which some coalitions are impossible while avoiding these potential conceptual pitfalls.

Jackson (2005) considers network games in which players can influence the structure of the network to serve their interests. Our model is different in that it exogenously imposes a fixed network structure.

1.2. **Content.** Section 2 defines the model and the basic concepts of coalitional games with an underlying graph. Section 3 provides an axiomatic definition for graph values and related solution concepts. Section 4 investigates a few special cases. Section 5 discusses the necessity of the axioms as well as a few questions for future research.

2. Graphs and Values

2.1. **Definitions.**

A finite set of *players* N, of cardinality $n = |N|$, will be assumed fixed throughout. We denote the set of all permutations over N, meaning bijective mappings $\pi : N \to N$, by Π_N. The i-th element of a permutation $\pi \in \Pi_N$ will be denoted by π_i, and we will also denote $\pi^{\swarrow i} := \{\pi_j \mid j < \pi^{-1}(i)\}$, i.e. the predecessors of i in the list $\pi_1, \pi_2, \ldots, \pi_n$.

With tolerable abuse of notation, given a permutation $\pi \in \Pi_N$ we will also consider π to be a mapping $\pi : 2^N \to 2^N$ by defining $\pi(\{i_1, i_2, \ldots, i_k\}) = \{\pi(i_1), \pi(i_2), \ldots, \pi(i_k)\}$. We will also abuse notation by sometimes writing i instead of the singleton set $\{i\}$ when no confusion is possible, for the sake of readability.

The set of *coalitions* is the set of subsets of N. Conventionally, a *coalitional game over* N is given by a *characteristic function* v which is a real-valued function over the set of all coalitions, i.e., $v : 2^N = \{S : S \subseteq N\} \to \mathbb{R}$ with the convention that $v(\emptyset) = 0$. Denote the set of all coalitional games by $\mathcal{K}$.

A *value for player* i on $\mathcal{K}$ is a function $\varphi_i : \mathcal{K} \to \mathbb{R}$. A *(group) value* on $\mathcal{K}$, $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$, associates a vector in $\mathbb{R}^N$ with each game.

We next suppose that there is additional structure on N making it a graph. An *undirected graph* G over N is an ordered pair (N, E), where N, the set of players, is now considered to be a set of vertices and E, the set of edges, is a set of pairs of distinct elements in N. A path is a sequence of edges connecting a sequence of vertices. Two players $i, j \in N$ are connected if G contains a path from i to j.

We will assume that every graph $G = (N, E)$ in this paper is connected, meaning that every pair of players are connected by some path.

The set of connected sub-graphs of a graph G (including the empty set) will be denoted by $\mathcal{A}(G)$. Clearly, since $\mathcal{A}(G) \subseteq 2^N$, each element of $\mathcal{A}(G)$ is in particular a coalition. We will term $\mathcal{A}(G)$ the set of *admissible coalitions*.

For each player $i \in N$, denote

$$\mathcal{A}(G)^{-i} := \{S \in \mathcal{A}(G) \mid i \notin S \text{ and } S \cup \{i\} \in \mathcal{A}(G)\}. \tag{1}$$

$\mathcal{A}(G)^{-i}$ is always non-empty, because at minimum it contains the empty set. In addition, given an admissible coalition $T \in \mathcal{A}(G)$, denote

$$T^+ := \{i \in N \setminus T \mid T \cup \{i\} \in \mathcal{A}(G)\} \tag{2}$$

(hence in particular $N^+ = \emptyset$), and

$$T^- := \{i \in T \mid T \setminus i \in \mathcal{A}(G)\} \tag{3}$$

Definition 2.1. A characteristic function v over the set of admissible coalitions, i.e., $v : \mathcal{A}(G) \to \mathbb{R}$, with the convention that $v(\emptyset) = 0$, is a *coalitional game over* G.

Denote the family of all coalitional games over a fixed set of players N by $\mathcal{G}(N)$. We will frequently write simply $\mathcal{G}$ when N is clear by context.

Definition 2.2. A sequence of distinct admissible coalitions

$$S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_k$$

ordered by containment is a *chain* over G.

A maximally ordered sequence of admissible coalitions

$$\emptyset = S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_{|N|} = N$$

is a *maximal chain* over G.

The set of all maximal chains over G will be denoted $\mathcal{C}(G)$. Note that for every pair of successive integers k and $k+1$ in a maximal chain, by maximality $S_{k+1} \setminus S_k$ is a singleton. This leads to the following concept.

Definition 2.3. For each maximal chain $c \in \mathcal{C}(G)$ there is an *admissible permutation* of the elements of N, given by the mapping $d : \mathcal{C}(G) \to \Pi_N$ defined by:

$$d(c) = (S_1 \setminus S_0, S_2 \setminus S_1, \ldots, S_n \setminus S_{n-1}) \tag{4}$$

The mapping d is obviously bijective. We henceforth denote $\mathcal{D}(G) := d(\mathcal{C}(G))$.

We adapt the following standard concepts from the literature on coalitional games. A game v is *simple* if for every admissible coalition S, either $v(S) = 1$ or $v(S) = 0$. A game v is *monotonic* if $v(S) \geq v(T)$ for all $S, T \in \mathcal{A}(G)$ satisfying $S \supseteq T$.

Definition 2.4. The *unanimity game* with *carrier* $T \in \mathcal{A}(G)$ is the monotonic simple game U_T satisfying the condition that $U_T(S) = 1$ if and only if $T \subseteq S$.

Relatedly, as in Weber (1988), define $\widehat{U}_T$ to be the monotonic simple game satisfying the condition that $\widehat{U}_T(S) = 1$ if and only if $T \subsetneq S$. ♦

Following the lines of many standard proofs in the theory of coalitional games, it is easy to show that $\mathcal{G}(N)$ is a vector space of dimension $|\mathcal{A}(G)|$.

We also introduce here a non-standard concept:

Definition 2.5. Let $\Psi \subseteq \mathcal{A}(G)$. The *non-monotonic simple game* W_Ψ with *multi-carrier* Ψ is

$$W_\Psi(S) = \begin{cases} 1 & \text{if } S \in \Psi \\ 0 & \text{otherwise.} \end{cases}$$

♦

Definition 2.6. Over the family of games $\mathcal{K}$, it is standard to define a probabilistic value for player i to be a value satisfying $\varphi_i(v) = \sum_{S \subseteq N \setminus i} p^i_S(v(S \cup i) - v(S))$ for a probability distribution $\{p^i_S\}_{S \subseteq N \setminus i}$. Over $\mathcal{G}$ the analogous expression for a *probabilistic value* is

$$\varphi_i(v) = \sum_{T \in \mathcal{A}(G)^{-i}} p^i_T(v(T \cup i) - v(T))$$

for a probability distribution $\{p^i_T\}$ over the set $\{T \in \mathcal{A}(G)^{-i}\}$. We say that φ_i is a *pre-probabilistic value*, if we only require that $\{p^i_T\}$ is a signed measure of total measure 1. That is, $\sum_{T \in \mathcal{A}(G)^{-i}} p^i_T = 1$, and $p^i_T \in \mathbb{R}$ may be negative. ♦

For a fixed game v, a player $i \in N$ is a *null player* if $v(S \cup \{i\}) = v(S)$ for all $S \in \mathcal{A}(G)^{-i}$. A player i is a dummy player if $v(S \cup i) = v(S) + v(\{i\})$ for all $S \in \mathcal{A}(G)^{-i}$. A null player is a dummy player with $v(\{i\}) = 0$.

Let $\pi \in \Pi_N$ be a permutation. For every chain $c = (\emptyset \subsetneq S_1 \subsetneq S_2 \subsetneq \ldots \subsetneq G)$, the image of c under π trivially satisfies the condition that $\emptyset \subsetneq \pi(S_1) \subsetneq \pi(S_2) \subsetneq \ldots \subsetneq G$. There is no guarantee, however, that $\pi(S_k)$ is an admissible coalition for any particular $k < |N|$ the coalition (in the terminology introduced in Dubey and Weber (1977), the class of games $\mathcal{G}$ is not symmetric under all possible permutations). We will want to note when a permutation of a graph preserves admissible coalitions.

Definition 2.7. A permutation $\pi \in \Pi_N$ is an *automorphism* of G if $\pi(S) \in \mathcal{A}(G)$ for all $S \in \mathcal{A}(G)$. ⧫

Denote the set of automorphisms of G by $Aut(G)$. Automorphisms are exactly what they are supposed to be, namely permutations of the graph structure:

Lemma 2.8. *A permutation π is an* automorphism *of $G = (N, E)$ if and only if for every pair $i, j \in N$, $(i, j) \in E$ implies that $(\pi(i), \pi(j)) \in E$.*

Proof. In one direction, suppose that σ is an automorphism and let $S = \{i, j\}$ be an admissible coalition of size two, which can only hold if $(i, j) \in E$. Then $\pi(S) = \{\pi(i), \pi(j)\}$ is also an admissible coalition. But that can only be true if $\pi(i)$ and $\pi(j)$ are connected, i.e., $(\pi(i), \pi(j)) \in E$.

In the other direction, first note that every permutation π trivially maps the empty set and singleton sets to admissible coalitions. Suppose that $(i, j) \in E$ implies that $(\pi(i), \pi(j)) \in E$. Then all admissible coalitions of size two are mapped to admissible coalitions. From here proceed by induction: is S is an admissible coalition of size k, then the assumption implies that every admissible coalition $S \cup i$ is mapped to an admissible coalition $\pi(S) \cup \pi(i)$. ∎

It follows immediately that for any automorphism $\pi \in Aut(G)$, for all players i, $\pi(S) \in \mathcal{A}(G)^{-\pi(i)}$ for each $S \in \mathcal{A}(G)^{-i}$ and $\pi(S^+) = \pi(S)^+$ for all $S \in \mathcal{A}(G)$ such that $S \neq N$. Furthermore, for every chain $\emptyset \subsetneq S_1 \subsetneq \ldots \subsetneq G$ the image $\emptyset \subsetneq \pi(S_1) \subsetneq \ldots \subsetneq G$ is also a chain in $\mathcal{C}(G)$. In the sequel we will consider $Aut(G)$ to be a group acting on $\mathcal{C}(G)$ or $\mathcal{D}(G)$.

Example 2.9. Let the set of edges E be the set of all pairs of elements in N, i.e. the resulting graph $G = (N, E)$ is a complete graph. Then trivially every subset of N is an admissible coalition of G and every permutation is an automorphism. The standard Shapley value is a value (in fact, the unique value) on the set of games over complete graphs. ⧫

Example 2.10. Enumerate the members of N as $1, \ldots, n$. Define the set of edges to be $E = \{(k, k+1) \mid 1 \leq k \leq n-1\}$. Call the resulting graph $G = (N, E)$ a spectrum graph. In this case the set of automorphisms contains only two elements: the identity mapping and the mapping that reverses the ordering of the players (so that player 1 is mapped to player n, player 2 to player $n-1$ and so on).

This structure and a related value over it is studied in Álvarez, Hellman and Winter (2013) ⧫

Strictly speaking, we need to distinguish between values for player i on $\mathcal{K}$ and values on $\mathcal{G}$, because games on $\mathcal{K}$ are distinct from $\mathcal{G}$ (their domains are different, because they admit different admissible coalitions), but we will usually refer simply to values without specifying the domain when the intended meaning is clear.

3. Axiomatics

3.1. Axioms and n-Linear Values.

Additivity Axiom. A value φ_i for i satisfies additivity if for every pair of games $v, w \in \mathcal{G}$,

$$\varphi_i(v + w) = \varphi_i(v) + \varphi_i(w).$$

A group value $\varphi = (\varphi_1, \ldots, \varphi_n)$ satisfies linearity if each of its individual constituent values does.

Linearity Axiom. A value φ_i for i satisfies linearity if it is a linear function, i.e., for every pair of games $v, w \in \mathcal{G}$ and $\alpha \in \mathbb{R}$

$$\varphi_i(v + \alpha w) = \varphi_i(v) + \alpha\varphi_i(w).$$

A group value $\varphi = (\varphi_1, \ldots, \varphi_n)$ satisfies linearity if each of its individual constituent values does.

Null Player Axiom. A value φ_i satisfies the null player axiom if $\varphi_i(v) = 0$ whenever i is a null player in any $v \in \mathcal{G}$. A group value $\varphi = (\varphi_1, \ldots, \varphi_n)$ satisfies the null player axiom if each of its individual constituent values does.

Dummy Axiom. A value φ_i satisfies the dummy axiom if $\varphi_i(v) = v(i)$ whenever i is a dummy player in any $v \in \mathcal{G}$. A group value $\varphi = (\varphi_1, \ldots, \varphi_n)$ satisfies the dummy axiom if each of its individual constituent values does.

Monotonicity Axiom. A value φ_i satisfies the monotonicity axiom if $\varphi_i(v) \geq 0$ for every monotonic game $v \in \mathcal{G}$. A group value $\varphi = (\varphi_1, \ldots, \varphi_n)$ satisfies the monotonicity axiom if each of its individual constituent values does.

Lemma 3.1. *Let φ_i be a value for i on $\mathcal{G}$ satisfying linearity. Then there is a collection of constants $\{a_T\}_{T \in \mathcal{A}(G)}$ such that for all $v \in \mathcal{G}$*

$$\varphi_i(v) = \sum_{T \in \mathcal{A}(G)} a_T v(T).$$

Furthermore, if φ_i satisfies the null-player axiom and $i \notin T^+ \cup T^-$ then $a_T = 0$, and if $i \in T^+ \cup T^-$ then $a_{T \cup i} + a_{T \setminus i} = 0$.

Proof. Consider the game $W_{\{T\}}$ that assigns 1 to the coalition T and 0 to all other coalitions. Any game v can be written as $v = \sum_{T\in\mathcal{A}(G)} v(T)W_{\{T\}}$ and by linearity $\varphi_i(v) = \sum_{T\in\mathcal{A}(G)} \varphi_i(W_{\{T\}})v(T)$. The proof is concluded by setting $a_T = \varphi_i(W_{\{T\}})$ and noting that $i \notin T^+ \cup T^-$ implies that i is a null player of $W_{\{T\}}$, and if $i \in T^+ \cup T^-$ then i is a null player in $W_{\{T\cup i, T\setminus i\}} = W_{\{T\cup i\}} + W_{\{T\setminus i\}}$. ∎

Corollary 3.2. *The values satisfying the linearity and null-player axioms are exactly all the values of the form*

$$\varphi_i(v) = \sum_{T\in\mathcal{A}^{-i}} p^i_T[v(T\cup i) - v(T)],$$

where $\{p^i_T : T \in \mathcal{A}(G), i \in T^+\}$ *are arbitrary real numbers.*

We call values that satisfy the linearity and null-player axioms *n-linear values*.

In the sequel, we will gradually introduce further axioms and examine the constraints that these axioms impose on the values of $\{p^i_T\}$. The picture one ought to have in mind is the lattice of all admissible coalitions (Figure 1). It can be described as an acyclic directed graph whose vertices are the connected coalitions $\mathcal{A}(G)$ and whose edges are pairs of the form $(T, T \cup i)$. Values satisfying (at least) the axioms of linearity and null-player correspond to assignments of weights to the edges of the graph, with $\varphi_i(W_{\{T\cup i\}}) = p^i_T$ being the weight assigned to the edge $(T, T \cup i)$.

Lemma 3.3. *An n-linear value* φ_i *for* i *on* $\mathcal{G}$ *satisfies the dummy axiom if and only if it is a pre-probabilistic value.*

Proof. In one direction we assume that φ_i satisfies the dummy axiom. For each $T \in \mathcal{A}(G)^{-i}$ define $p^i_T = a_{T\cup i} = -a_T = \varphi_i(W_{\{T\cup i\}})$. By Lemma 3.1,

$$\varphi_i(v) = \sum_{T\in\mathcal{A}(G)^{-i}} p^i_T[v(T\cup i) - v(T)].$$

Since i is a dummy player in the game $W_{\mathcal{A}(G)^{+i}} = \sum_{T\in\mathcal{A}(G)^{-i}} W_{\{T\cup i\}}$, we have

$$\sum_{T\in\mathcal{A}(G)^{-i}} p^i_T = \sum_{T\in\mathcal{A}(G)^{-i}} \varphi_i(W_{\{T\cup i\}}) = \varphi_i(W_{\mathcal{A}(G)^{+i}}) = W_{\mathcal{A}(G)^{-i}}(i) = 1.$$

Conversely, assume $\sum_{T\in\mathcal{A}(G)^{-i}} p^i_T = 1$ and let v be a game in which i is a dummy player. By Lemma 3.1,

$$\varphi_i(v) = \sum_{T\in\mathcal{A}(G)^{-i}} p^i_T[v(T\cup i) - v(T)] = \sum_{T\in\mathcal{A}(G)^{-i}} p^i_T v(i) = v(i).$$

∎

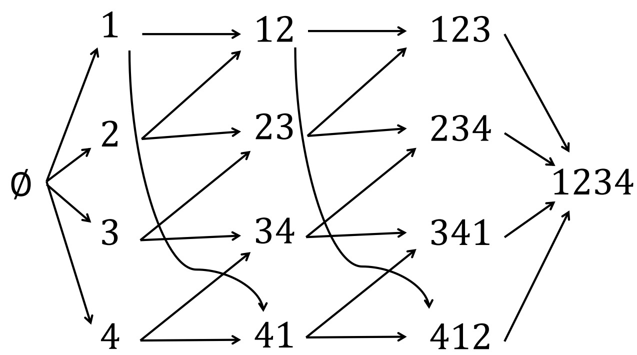

FIGURE 1. The lattice of coalitions for the 4-cycle C_4

Lemma 3.4. *An n-linear value φ_i for i on $\mathcal{G}$ satisfies monotonicity if and only if the associated weights $p^i_T = \varphi_i(W_{\{T\cup i\}})$ are non-negative, for every $T \in \mathcal{A}^{-i}(G)$.*

Proof. In one direction, we assume that φ_i is monotonic and show that the weights $p^i_T = \varphi_i(W_{\{T\cup i\}}$ are non-negative, for all $T \in \mathcal{A}(G)^{-i}$. For each T, consider the unanimity game $U_{T\cup i}$. We can write $U_{T\cup i}$ as $W_{\{T\cup i\}} + \widehat{U}_{T\cup i}$. Since i is a null player in $\widehat{U}_{T\cup i}$ and $U_{T\cup i}$ is monotonic, we get

$$P^i_T = \varphi_i(W_{\{T\cup i\}}) = \varphi_i(U_{T\cup i}) \geq 0.$$

Conversely, assume that the weights $p^i_T = \varphi_i(W_{\{T\cup i\}}$ are non-negative, and let v be a monotonic game. By Lemma 3.1 and since the marginal contribution of i to any coalition is non-negative,

$$\varphi_i(v) = \sum_{T\in\mathcal{A}(G)^{-i}} p^i_T[v(T\cup i) - v(T)] \geq 0.$$

∎

Corollary 3.5. *An n-linear value φ_i for i on $\mathcal{G}$ satisfies the dummy and monotonicity axioms if and only if it is a probabilistic value.*

Efficiency Axiom. A group value $\varphi = (\varphi_1, \dots, \varphi_n)$ satisfies efficiency if for every $v \in \mathcal{G}$

$$\sum_{i\in N} \varphi_i(v) = v(N).$$

Definition 3.6. Let T^+ and T^- be as defined in Equations (2) and (3). An assignment of weights $(p^i_T)_{i\in N, T\in\mathcal{A}^{-i}(G)}$ constitutes a *unit pre-flow* (on the coalitional lattice of G) if

$$\sum_{i\in T^-} p^i_{T\setminus i} = \sum_{j\in T^+} p^j_T \quad \forall T\in\mathcal{A}(G)\setminus\{\emptyset, N\},$$
$$\sum_{i\in N^-} p^i_{N\setminus i} = 1,$$

It is a *unit flow* if in addition p^i_T is non-negative for every $i\in N$ and $T\in\mathcal{A}^{-i}(G)$.

Lemma 3.7. *An n-linear value group φ on G satisfies efficiency if and only if the associated weights $\{p^i_T = \varphi_i(W_{\{T\cup i\}}) : T\in\mathcal{A}^{-i}(G)\}$ constitute a unit pre-flow.*

Proof. Let φ be an efficient n-linear value. We work here with the game $W_{\{T\}}$ that assigns 1 to the coalition T and 0 to all other coalitions. Let $\varphi_N(v) = \sum_{i\in N}\varphi_i(v)$ for any $v\in\mathcal{G}$. It is straightforward to show that

$$\begin{aligned}\varphi_N(v) &= \sum_{i\in N}\sum_{T\in\mathcal{A}(G)^{-i}} p^i_T(v(T\cup i)-v(T))\\ &= \sum_{T\in\mathcal{A}(G)} v(T)\left[\sum_{i\in T^-} p^i_{T\setminus i} - \sum_{j\in T^+} p^j_T\right].\end{aligned}$$

It immediately follows that $W_{\{T\}}(N) = \sum_{i\in T^-} p^i_{T\setminus i} - \sum_{j\in T^+} p^j_T$. But $W_{\{N\}}(N) = 1$, hence $\sum_{i\in N^-} p^i_{N\setminus i} = 1$, while $W_{\{T\}}(N) = 0$ for all $T\in\mathcal{A}(G)\setminus\{N\}$, hence $\sum_{i\in T^-} p^i_{T\setminus i} = \sum_{j\in T^+} p^j_T$.

Conversely, assume the weights $\{p^i_T = \varphi_i(W_{\{T\cup i\}}) : T\in\mathcal{A}^{-i}(G)\}$ constitute a unit pre-flow and let v be a game.

Since

$$\sum_{i\in T^-} p^i_{T\setminus i} - \sum_{j\in T^+} p^j_T = \begin{cases} 1 & \text{if } T = N \text{ and}\\ 0 & \text{if } \emptyset\subsetneq T\subsetneq N,\end{cases}$$

we have, by Lemma 3.1,

$$\begin{aligned}v(N) &= \sum_{T\in\mathcal{A}(G)} v(T)\left[\sum_{i\in T^-} p^i_{T\setminus i} - \sum_{j\in T^+} p^j_T\right]\\ &= \sum_{i\in N}\sum_{T\in\mathcal{A}(G)^{-i}} p^i_T(v(T\cup i)-v(T)) = \sum_{i\in N}\varphi_i(v).\end{aligned}$$

∎

Corollary 3.8. *An n-linear group value φ on G satisfies efficiency and monotonicity if and only if the associated weights $\{p^i_T = \varphi_i(W_{\{T\cup i\}}) : T \in \mathcal{A}^{-i}(G)\}$ constitute a unit flow.*

A cut is a collection of coalitions $\mathcal{C}$, such that $\emptyset \in \mathcal{C}$ and $N \notin \mathcal{C}$. Unit pre-flows have the feature that given any cut the difference between the weight that leaves and enters the cut is always 1. For example, $\mathcal{A}(G) \setminus \{N\}$ is a cut. It has no incoming edges and the weight of the outgoing edges $\sum_{i\in N^-} p^i_{N\setminus i}$ is indeed 1, for any unit pre-flow $\{p^i_T\}$. Considering the cuts $\mathcal{C}_i = \{S \in A(G) : i \notin S\}$, for any player i, and applying Lemma 3.3 we get the following corollary.

Corollary 3.9. *Let φ be an n-linear group value on G. If φ satisfies efficiency, then φ_i satisfies the dummy axiom for every player i.*

Let $\{r_\omega\}_{\omega\in\Pi_N}$ be a probability distribution over Π_N. For $\mathcal{K}$, a random order group value $\zeta = (\zeta_1, \ldots, \zeta_n)$ is defined by

$$\zeta_i(v) = \sum_{\omega\in\Pi_N} r_\omega(v(\omega^{\swarrow i} \cup i) - v(\omega^{\swarrow i})),$$

for all $i \in N$ and $v \in \mathcal{K}$.

The usual interpretation of this definition is that each permutation represents an ordered queue of the players, who enter a room one by one according to their number in the queue. Each permutation defines a dynamic way of forming a coalition, which grows by one player at a time, thus enabling us to measure the contribution of each player to the coalition formed by the players who preceded him or her in entering the room.

The corresponding notion here is that not every queue of entering players if possible: only those in which the next player to enter the room is 'connected' to at least one player who is already in the room are admissible. Hence we limit consideration only to admissible permutations, i.e. in the set $\mathcal{D}(G)$. In particular, now letting $\{r_\pi\}_{\pi\in\mathcal{D}(G)}$ be a probability distribution over $\mathcal{D}(G)$, a *random order (group) value* $\zeta = (\zeta_1, \ldots, \zeta_n)$ over $\mathcal{G}$ is defined by

$$\zeta_i(v) = \sum_{\pi\in\mathcal{D}(G)} r_\pi(v(\pi^{\swarrow i} \cup i) - v(\pi^{\swarrow i})),$$

for all $i \in N$ and $v \in \mathcal{G}$. Similarly, if we only require that $\{r_\pi\}_{\pi\in\mathcal{D}(G)}$ is a signed measure of total weight 1, then such a ζ is called a *pre-random order value*.

The most intuitive way to construct a random order value is to suppose that given an admissible coalition S one has a conditional distribution over the players in S^+ that represents the probability of choosing the next player to join. This induces a probability distribution over all admissible permutations from which the weights of a random order value can be derived. Conversely, it is easy to calculate the conditional probability of a player joining an already-formed coalition from the weights of a random order value. Lemma 3.10 shows that probabilistic values can be derived from random order values. Lemma 3.11 shows that if the linearity, efficiency, monotonicity and dummy axioms are assumed then essentially every random order value is derived from a conditional probability measuring how likely a player is to join an already-formed coalition, with the random order values and probabilistic values derivable each from the other.

Lemma 3.10. *Let (r_π) be a probability distribution over $\mathcal{D}(G)$.*[3] *Let $\zeta = (\zeta_1, \ldots, \zeta_n)$ be the associated random order value. Then there is a collection of probabilistic values $\varphi = (\varphi_1, \ldots, \varphi_n)$ such that $\varphi_i(v) = \zeta_i(v)$ for all $i \in N$ and all $v \in \mathcal{G}$.*

Proof. For $i \in N$ and $v \in \mathcal{G}$

$$\begin{aligned} \zeta_i(v) &= \sum_{\pi \in \mathcal{D}(G)} r_\pi (v(\pi^{\swarrow i} \cup i) - v(\pi^{\swarrow i})) \\ &= \sum_{T \in \mathcal{A}(G)^{-i}} \left(\sum_{\{\pi \in \mathcal{D}(G) | \pi^i = T\}} r_\pi \right) (v(T \cup i) - v(T)) \end{aligned}$$

Setting $p_T^i = \sum_{\{\pi \in \mathcal{D}(G) | \pi^{\swarrow i} = T\}} r_\pi$ for all $i \in N$ and $T \in \mathcal{A}(G)^{-i}$ and using that to construct a collection of probabilistic values suffices to complete the proof. ∎

Note that unlike the weights p_T^i of probabilistic values, the weights r_π of random order values do not uniquely determine a value. This is due to the fact that the number of admissible orderings $|\mathcal{D}(G)|$ may be larger than the number of pairs $\{(i, T) : i \in N, T \in \mathcal{A}(G)^{-i}\}$. E.g., in the graph whose n vertexes are connected as a simple cycle, $|\mathcal{D}(G)|$ is exponential in n, whereas $|\{(i, T) : i \in N, T \in \mathcal{A}(G)^{-i}\}|$ is only polynomial.

Lemma 3.11. *An n-linear group value φ satisfies efficiency if and only if it is a pre-random order value, and φ further satisfies monotonicity if and only if it is a random order value.*

Proof. Consider the set of unit flows on the coalitional lattice of G. This is a convex compact polytope in $\mathbb{R}^{\{(T, T \cup i) : i \in \mathcal{A}(G), \in T^+\}}$, hence it is the convex hull of its extreme points. The extreme points are unit flows supported on a single path. Furthermore, the set of unit pre-flows on the coalitional lattice of G is the affine span of the set of unit flows.

[3] Recalling that $\mathcal{D}(G)$ is the set $d(\mathcal{C}(G))$, the admissible permutations.

The proof is concluded by Lemma 3.7, Corollary 3.9, and Corollary 3.5, since unit flows supported on a single path correspond to random order values supported on a single admissible permutation. ∎

Symmetry Axiom. A group value $\varphi = (\varphi_1, \ldots, \varphi_n)$ satisfies symmetry if

$$\varphi_i(v) = \varphi_{\pi(i)}(\pi \circ v)$$

for all $v \in \mathcal{G}$, all $\pi \in Aut(G)$ and all $i \in N$, where $\pi \circ v(T) := v(\pi^{-1}(T))$.

3.2. **Characterisation of the Graph Value.**

Definition 3.12. A group value $\varphi = (\varphi_1, \ldots, \varphi_n)$ over a graph $G = (N, E)$ is a *graph value* if it satisfies linearity and the dummy, monotonicity, efficiency and symmetry axioms.

Recall that the linear space of games on G is denoted $\mathcal{G}$. The space $\mathbb{R}^N$ can be viewed as a subspace of $\mathcal{G}$, by letting $x(S) = \sum_{i \in S} x_i$, for $x \in \mathbb{R}^N$. The space of linear values $Hom(\mathcal{G}, \mathbb{R}^N)$, is by itself a (finite dimensional) linear space (over $\mathbb{R}$). The group $Aut(G)$ has right and left linear actions on $Hom(\mathcal{G}, \mathbb{R}^N)$ defined naturally by

$$(\pi \circ \varphi)(v) = \pi \circ (\varphi(v)),$$
$$(\varphi \circ \pi)(v) = \varphi(\pi \circ v),$$

for $\varphi \in Hom(\mathcal{G}, \mathbb{R}^N)$, $\pi \in Aut(G)$, and $v \in \mathcal{G}$. With this notation φ is symmetric if and only if $\varphi = \pi^{-1} \circ \varphi \circ \pi$.

It is also standard to define a linear action of $Aut(G)$ on the space of signed measures on $\mathcal{C}(G)$ by

$$(\pi \circ \mu)(A) = \mu(\pi^{-1}(A)),$$

for any $A \subset \mathcal{C}(G)$, $\pi \in Aut(G)$, and μ a signed measure on $\mathcal{C}(G)$.

Definition 3.13. A signed measure μ on $\mathcal{C}(G)$ is $Aut(G)$ *invariant* if $\mu = \pi \circ \mu$ for all $\pi \in Aut(G)$.

Given a signed measure μ, denote the induced pre-random order value by ζ_μ. It is straightforward to check that

$$\zeta_\mu \circ \pi = \pi \circ \zeta_{\pi \circ \mu},$$

for any $\pi \in Aut(G)$. If in addition ζ_μ is symmetric, we have

$$\zeta_\mu = \pi^{-1} \circ \zeta_\mu \circ \pi = \zeta_{\pi \circ \mu}, \quad \forall \pi \in Aut(G).$$

Since the mapping $\mu \mapsto \zeta_\mu$ is linear, taking

$$\hat{\mu} = \frac{1}{|Aut(G)|} \sum_{\pi \in Aut(G)} \pi \circ \mu$$

gives an $Aut(G)$-invariant measure such that $\zeta_\mu = \zeta_{\hat{\mu}}$.

This leads to a characterisation of the graph values.

Definition 3.14. A *symmetric pre-random order value* over G is a pre-random order value whose weights are $Aut(G)$-invariant. A *symmetric random order value* over G is a random order value whose weights are $Aut(G)$-invariant.

Theorem 3.15. *A group value $\varphi = (\varphi_1, \dots, \varphi_n)$ over G is a graph value if and only if it is a symmetric random over value.*

Proof. In one direction, let φ be a random order value whose weights are $Aut(G)$ invariant. It is easy to check that all the axioms are satisfied by such a value.

In the other direction, suppose that φ satisfies the above axioms. By Corollary 3.5, linearity and the dummy and monotonicity axioms justify using probabilistic values, while Lemma 3.11 shows that adding efficiency implies that $\varphi = \zeta_\mu$ is a random order value induced by a random order μ. By symmetry, the $Aut(G)$-invariant probability measure

$$\hat{\mu} = \frac{1}{|Aut(G)|} \sum_{\pi \in Aut(G)} \pi \circ \mu$$

induces φ. ∎

Since $Aut(G)$ is a group acting on the set $\mathcal{D}(G)$ (equivalently, on $\mathcal{C}(G)$) we can consider the set $\mathcal{O}(G)$ of orbits of $Aut(G)$. The set $\mathcal{O}(G)$ partitions $\mathcal{D}(G)$. Hence we can choose a representative element from each orbit $\omega \in \mathcal{O}(G)$. The condition of $Aut(G)$ invariance of random order weights immediately implies the next two corollaries.

Corollary 3.16. *A group value over G is a graph value if and only if it is a random order value whose weights $(r_\pi)_{\pi \in \mathcal{D}(G)}$ satisfy the condition that there exists a collection of non-negative weights $\{\rho_\omega\}_{\omega \in \mathcal{O}(G)}$ with $\sum_{\omega \in \mathcal{O}(G)} \rho_\omega = 1$ such that $r_\pi = |Aut(G)|^{-1} \rho_\omega$ for each $\omega \in \mathcal{O}(G)$ and each $\pi \in \omega$.*

Corollary 3.17. *For each orbit $\omega \in \mathcal{O}(G)$ denote by $U(\omega)$ the uniform probability distribution over $\{\pi\}_{\pi \in \omega}$. A group value over G is a graph value if and only if the system of weights of the associated random order value is contained in the convex hull of $\{U(\omega)\}_{\omega \in \mathcal{O}(G)}$.*

3.3. **Summary of the Axiomatics of the Graph Value.**

Recall that we defined a group value over G to be an n-linear value if it satisfies the linearity and null-player axioms. The results of this section are then succinctly summarised in Table 1.

n-Linear Group Value +:	**Equivalent to:**
dummy	pre-probabilistic value
dummy, monotonicity	probabilistic value
efficiency	pre-random order value
efficiency, monotonicity	random order value
dummy, efficiency, symmetry	$Aut(G)$-invariant pre-r.o.v.
dummy, efficiency, monotonicity, symmetry	$Aut(G)$-invariant r. o. v.

TABLE 1. Axiomatics summary.

4. SHAPLEY VALUE VS GRAPH VALUE

Let G be the complete graph over N, as in Example 2.9. Since $Aut(G) = \Pi_N$ in this case, there is only one orbit, and Corollary 3.16 implies that there exists a unique graph value. This unique graph value is precisely the Shapley value. It is, of course, a celebrated result of Shapley (1953a) that the Shapley value is unique, but we see this emerging from our discussion here from the perspective of graph values.

In contrast to the Shapley value, the graph value in general is not unique, because there may be several orbits. A graph is *entirely anti-symmetric* if $|\mathcal{O}(G)| = |\mathcal{D}(G)|$. This occurs, for example, if $Aut(G)$ consists solely of the identity permutation; there are many well-known examples of such graphs. If G is a entirely anti-symmetric graph then any probability distribution over $\mathcal{D}(G)$ defines the weights of a random-order value that is a graph value for G. It follows from this that there are graphs whose set of graph values contains more than one point.

From previous results it is clear that if $|\mathcal{O}(G)| = 1$ then there is only one graph value, namely the one random-order value that assigns uniform weight to each element of $\mathcal{D}(G)$. This, however does not mean that $|\mathcal{O}(G)| = 1$ is a necessary condition for the existence of a unique graph value, as the next result shows.

The n-*cycle*, for $n \geq 3$ is the graph whose vertex set is $\{1, \dots, n\}$ with edge set $E = \{\{1,2\}, \{2,3\}, \dots, \{n-1, 1\}\}$. There are two orbits for each n-cycle, which may be termed the 'cycle-structure order preserving orbit' and the 'non-order preserving orbit'.

Claim 4.1. *The graph value over the n-cycle is unique for all n. The graph value over the 3-cycle is the Shapley value, but for all $n \geq 4$ the graph value differs from the Shapley value.*

Proof. Let G be an n-cycle . If $n = 3$ the n-cycle is the complete graph over 3 elements and therefore there is a unique graph value that is the Shapley value. We concentrate henceforth on the case $n \geq 4$ and and let φ be any graph value over G.

By construction, for each player i there are exactly two players j and k that are connected to i in G. Let $T \subset G$ be a connected coalition of players in G of size $1 < |T| < n$. Define i to be an internal vertex of T if each of the two players j and k connected to i are also in T. Consider the unanimity game U_T. If i is an internal vertex of T then i is pivotal with respect to a given admissible permutation π of N iff i is the last player in the ordering defined by π. By symmetry, each internal player has an equal probability of being last; it follows that $\varphi_i(U_T) = 1/n$ for all internal vertices i.

The two players on the boundary of T are symmetric and they must therefore receive the same value by the symmetry axiom. By efficiency,

$$\varphi_j(U_T) = \frac{1}{2}\left(1 - \frac{|T| - 2}{n}\right).$$

for each player j on the boundary of T.

This is sufficient to show that the graph value is unique over the n-cycle and that it differs from the Shapley value, which would give each player equal value over U_T, not distinguishing between interal and boundary players. ∎

Finally, we consider one more example of a graph with an interesting graph value.

Example 4.2. The *n-star graph* is defined over the vertex set $\{0, i_1, i_2, \ldots, i_n\}$ with edges $\{\{0, i_1\}, \{0, i_2\}, \ldots, \{0, i_n\}\}$. Consider the simple majority game v and any graph value ψ over the n-star graph. Then straightforward combinatorial calculations show that

$$\psi_0(v) = 0$$
$$\psi_{i_1}(v) = \psi_{i_2}(v) = \ldots = \psi_{i_n}(v) = \frac{1}{n}$$

♦

The result in Example 4.2 is again very different from the Shapley value, because the internal vertex receives a zero value under all circumstances. This is because the graph value essentially counts the number of times each player is a pivot player among all admissible permutations. In the simple majority game over the star graph, the internal node can never be the pivot player in any admissible coalition.

This may at first seem surprising, since one natural representation of the internal node of a star graph is a market maker through whom everyone else needs to go to conduct trade, or similarly a hub for resource distribution.

One might think this would grant the internal player a great deal of power, yet the axioms that we assumed, which are almost verbatim adaptations of the standard Shapley axioms for our setting in which only connected coalitions may be formed, end up giving that player zero value.

One explanation for this phenomenon is as follows. In the standard Shapley value approach, measuring the average marginal gain a player causes by joining coalitions is entirely equivalent to measuring the average marginal loss he causes by leaving coalitions. In the graph value setting, this equivalence no longer obtains. Since only connected coalitions may be formed, leaving a coalition is only possible if the remaining coalition is connected. Another way to put this idea is that if a market maker disconnects from the other players, then no coalition of more than one player can be formed. A market maker who quits therefore cannot improve his own payoff.

5. Redundancy of Axioms and Solution Uniqueness

The original Shapley axioms are: additivity, null player, efficiency and symmetry. In the axiomatics of the graph value in the above section, the additivity and null player axioms are replaced by the stronger assumptions of linearity and the dummy axiom, and the monotonicity axiom is added. Naturally, we would like to know if the (seemingly) weaker set of axioms implies the stronger set of axioms.

It is not too hard to show that the dummy axiom is implied by the conjunctions of null-player, efficiency and additivity axioms (Corollary 3.9). Additivity is equivalent to linearity over $\mathbb{Q}$, and in conjunction with monotonicity[4] it implies linearity. Thus, the graph value axioms are equivalent to the Shapley axioms + Monotonicity.

Our question becomes: what are the graphs for which any solution concept that satisfies the Shapley axioms is monotonic? It turns out that the answer is related to the uniqueness of the graph value.

Theorem 5.1. *If G is a graph on which the graph value is unique, then any solution concept on G satisfying the Shapley axioms is the unique graph value.*

Theorem 5.2. *Let G be a connected graph on which the graph value is not unique. There are solution concepts on G satisfying*

(1) *the Shapley axioms but not linearity,*
(2) *the Shapley axioms + linearity but not monotonicity.*

[4] Continuity, a weaker assumption than monotonicity, is sufficient.

Examples of graphs on which the graph value is unique are the complete graph K_n (on which the graph value is the classic Shapley value) and the cycle C_n.

Question 5.3. Are the complete graph K_n and the cycle C_n the only graphs on which the graph value is unique.

Now we turn to proving Theorems 5.1 and 5.2. Theorem 5.2(2) follows from the next lemma.

Lemma 5.4. *The values that satisfy the Shapley axioms + Linearity are exactly the affine span of the graph values.*

Proof. We first fix some notation. Denote the following sets of values

RO	random order,
PRO	pre-random order,
SRO	symmetric random order (= graph values by Theorem 3.15),
SPRO	symmetric pre-random order (= Shapley axioms + Linearity axioms by Lemma 3.11).

PRO is an affine space that contains RO. Lemma 3.11 asserts that PRO is *exactly* the affine span of RO. The mapping

$$\Lambda : \varphi \mapsto \frac{1}{|Aut(G)|} \sum_{\pi \in Aut(G)} \pi^{-1} \circ \varphi \circ \pi$$

is a linear projection from the space of linear values to the subspace of symmetric linear values. The sets PRO and RO are convex $Aut(G)$-invariant sets; therefore they are Λ invariant. It follows that Λ(PRO)=SPRO is the affine span of Λ(RO)=SRO. ∎

Lemma 5.4 implies that if the graph value is unique then it is the only value that satisfies the Shapley axioms + linearity. Next we examine the effect of weakening linearity to additivity.

Proof of Theorem 5.1. Additivity is equivalent to linearity over $\mathbb{Q}$. Denote the $\mathbb{Q}$-linear space of all rational games by

$$\mathcal{G}_1 = \{v \in \mathcal{G} : v(S) \in \mathbb{Q},\ \forall S\}.$$

For a real number $x \neq 0$ let

$$\mathcal{G}_x = x\mathcal{G}_1 = \{v \in \mathcal{G} : x^{-1}v(S) \in \mathbb{Q},\ \forall S\}.$$

For any value that satisfies the Shapley axioms ψ and any $x \in \mathbb{R} \setminus \{0\}$, let ψ_x be the restriction of ψ to $\mathcal{G}_x$ and φ_x the $\mathbb{R}$-linear extension of ψ_x to $\mathcal{G}$. Note that φ_x is a value that satisfies the Shapley axioms + linearity. If there is only one value satisfying the Shapley axioms + linearity, then ψ is uniquely defined on every $\mathcal{G}_x$, and since $\mathcal{G} = \sum_{x\in\mathbb{R}\setminus\{0\}} \mathcal{G}_x$, ψ is uniquely defined on $\mathcal{G}$. This completes the proof of Theorem 5.1. ∎

Proof of Theorem 5.2(1). Let Φ be the set of all graph values. Let $\mathcal{B}$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$. Any function $h : \mathcal{B} \to \Phi$ defines a distinct $\mathbb{Q}$-linear value ψ^h, by $\psi^h_b = h(b)_b$, for all $b \in \mathcal{B}$, proving Theorem 5.2(1). ∎

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