Int J Game Theory
DOI 10.1007/s00182-013-0390-x

Countable spaces and common priors

Ziv Hellman

Accepted: 2 July 2013

Abstract We show that the no betting characterisation of the existence of common priors over finite type spaces extends only partially to improper priors in the countably infinite state space context: the existence of a common prior implies the absence of a bounded agreeable bet, and the absence of a common improper prior implies the existence of a bounded agreeable bet. However, a type space that lacks a common prior but has a common improper prior may or may not have a bounded agreeable bet. As a side-benefit of the proofs here, we also obtain a constructive proof of the no betting characterisation in finite spaces.

Keywords Common priors · Improper priors · Agreeing to disagree · No betting and no trade · Knowledge and beliefs

JEL Classification C02 · C07 · D82 · D83

1 Introduction

The common prior assumption (as first introduced in Harsányi 1967–1968) is taken as an integral assumption in the vast majority of models of incomplete information. It asserts that the beliefs of individuals in different states of the world are the posteriors that they form, after each is given private information, from a prior that is common to them all.

Despite its pervasiveness, the common prior assumption was, and still is, debated and challenged (see Gul 1998; Aumann 1998). It has been noted that, in many cases

Z. Hellman (✉)
Department of Statistics and Operations Research, School of Mathematical Sciences,
Tel Aviv University, Tel Aviv, Israel
e-mail: ziv.hellman@gmail.com

Published online: 01 August 2013

of interest, all that observers have are profiles of posteriors, not priors, and that there are examples of posteriors that could not possibly have been derived from common priors.

Given the importance of the common prior assumption, intense interest has been focussed on fully characterising the existence of a common prior in terms of the posterior profiles—since we are interested in the players at the present time, it is desirable to express the assumption of a common prior in present-time terms only. Aumann (1976) in his agreement theorem, gave a necessary condition for the existence of a common prior in terms of present beliefs: if there is a common prior, then it is impossible to agree to disagree, i.e., to have common knowledge of differences in the beliefs of any given event. By extending the notion of disagreement to differences in the expectation of a general random variable, several researchers (Morris 1994; Feinberg 2000; Samet 1998) were able to show that the impossibility of there being common knowledge of disagreement is not only a necessary, but also a sufficient condition for the existence of a common prior. Since this characterisation is based on the criterion of whether or not there exists a bet such that the players take opposite sides of the bet, yet each player ascribes positive expected value to the bet at every state of the world (we will henceforth term such a bet an *agreeable bet*), it is often termed the 'no betting' characterisation. It has also been proved that this characterisation obtains for type spaces over compact, continuous state spaces (see Feinberg 2000; Heifetz 2006).

That left open the question of characterising the existence of common priors in type spaces over countable state spaces, a major lacuna given the many models of incomplete information in the game theory and economics literature that involve countable state spaces. That the no betting characterisation cannot be extended 'as is' to countable spaces was shown in Sect. 7 of Feinberg (2000), which presents an example of a type space over a countable state space that has no common prior, yet also admits no bounded agreeable bet (in fact, even no agreeable bet bounded from only above or from below).

Several researchers, however, noted that the counter-example in Feinberg (2000), and several other counter-examples (see, for example, Simon 2000; Heifetz 2006; Lehrer and Samet 2011) admit no common prior but satisfy the property of having a common *improper* prior. An improper prior for a player is a measure over the state space that may not be normalisable, i.e., the measure of the entire space may be infinite.

There has been an open conjecture for several years, due to Heifetz (2006), that the no betting characterisation or a close variant of it might obtain with respect to common improper priors over countable state spaces. In this paper, we directly address this conjecture, and prove that the absence of a common improper prior is a sufficient condition for the existence of a bounded agreeable bet among players. It is not, however, a necessary condition; we exhibit a simple example of a type space over a countable state space that has both a common improper prior and a bounded agreeable bet. We also show that the existence of a (proper) common prior is a sufficient condition for the absence of a bounded agreeable bet.

These results, along with the example in Feinberg (2000), indicate that the no betting criterion is rather weak in the countable state space case. In particular, the 'intermediate' case of a type space with no common prior but a common improper prior is consistent both with the existence of a bounded agreeable bet and the absence of a bounded agreeable bet. We can present this schematically as follows:

- Common prior $\Rightarrow$ No betting
  No betting $\not\Rightarrow$ Common prior
- No betting $\Rightarrow$ Common improper prior
  Common improper prior $\not\Rightarrow$ No betting

The study of common priors in the context of countable spaces involves a much richer set of concepts than the finite space context. Consistency in a countable space is a subtle issue. Unlike the finite case, in the infinite one the non-existence of a common prior can have different characteristics that have bearing on the question of consistency, and mapping the relationships between different concepts is an on-going effort as of this writing. Research studies in this field complementary to this paper include Lehrer and Samet (2013), who provide a sufficient condition for type spaces that have a common improper prior to admit an agreeable bet.

The appendix presents a constructive proof of the no betting characterisation for finite spaces (all previous proofs were ultimately based on one or another variant of the Separation Theorem for convex sets). Putting together the elements of the proof yields an algorithm which, given a finite type space, determines whether or not it has a common prior; if it does have a common prior, the algorithm then constructs the common prior; if it does not have a common prior, the algorithm constructs an agreeable bet, thus indicating a random variables about whose expected values the players 'agree to disagree'. Example 3 presents a calculation of an agreeable bet over a simple type space, illustrating the main idea behind this algorithm.

2 Preliminaries

2.1 Knowledge and belief

A *knowledge space* for a nonempty set of *players* I, is a pair (Ω, Π). In this context, Ω is a nonempty set called a *state space*, and $\Pi = (\Pi_i)_{i \in I}$ is a *partition profile*, where for each $i \in I$, Π_i is a partition of Ω into measurable sets with positive measure. We will assume throughout this paper that every state space Ω is either finite, or countably infinite, and that $|I| = m$, where $m \geq 2$ is a finite integer.

Denote by $\Delta(\Omega)$ the set of probability distributions over Ω. We will assume throughout that $\Delta(\Omega)$ is endowed with the standard weak topology of convergence of measures.

When working with a knowledge space (Ω, Π), an element $\omega \in \Omega$ is typically termed a *state*. For each $\omega \in \Omega$, we denote by $\Pi_i(\omega)$ the element of Π_i containing ω. Π_i is interpreted as the information available to player i; $\Pi_i(\omega)$ is the set of all states that are indistinguishable to i when ω occurs. Player i is said to *know* an event E at ω if $\Pi_i(\omega) \subseteq E$. We define for each i a *knowledge operator* $K_i : 2^\Omega \to 2^\Omega$, by $K_i(E) = \{\omega \mid \Pi_i(\omega) \subseteq E\}$. Thus, $K_i(E)$ is the event that i knows E.

A partition Π' is a *refinement* of Π if every element of Π' is a subset of an element of Π. Refinement intuitively describes an increase of knowledge. The *meet* of Π, denoted Π, is the partition that is the finest among the partitions that are simultaneously coarser than all the partitions Π_i. Π is called *connected* when $\Pi = \{\Omega\}$. (By abuse of notation, when Π is clear from context, we will sometimes say that Ω is connected when the intention is to say that Π is connected).

A *type function* for Π_i is a function $t_i : \Omega \to \Delta(\Omega)$ that associates with each state ω a distribution in $\Delta(\Omega)$, in which case the latter is termed the *type* of i at ω. Each type function t_i further satisfies the following two conditions:

(a) $t_i(\omega)(\Pi_i(\omega)) = 1$, for each $\omega \in \Omega$;
(b) t_i is constant over each element of Π_i.

A *type profile* for Π is an m-tuple of type functions, $\tau = (t_i)_{i \in I}$, where for each i, t_i is a type function for Π_i, which intuitively represents the player's beliefs. We call a knowledge space together with a type profile a *type space*. A type space τ is *positive* if $t_i(\omega)(\omega) > 0$ for each i, and each state ω.

By definition of a type function, abusing notation, we may write $t_i(\omega)$ as short-hand for $t_i(\omega)(\omega)$, with the distinction between the intended interpretation of $t_i(\omega)$ as an element of $\Delta(\Omega)$ or as an element of $\mathbb{R}$ clear from context. A *random variable* f over Ω is any element of $\mathbb{R}^\Omega$. Given a probability measure $\mu \in \Delta(\Omega)$ and a random variable f, the *expected value of* f with respect to μ, is

$$E_\mu f := \sum_{\omega \in \Omega} f(\omega)\mu(\omega). \tag{1}$$

For a random variable f, denote by $E_i f$ the element of $\mathbb{R}^\Omega$ defined by

$$E_i f(\omega) := \sum_{\omega' \in \Pi_i(\omega)} t_i(\omega') f(\omega'). \tag{2}$$

We will alternatively also sometimes write $E_i(f|\omega)$ in place of $E_i f(\omega)$, and call this the *interim expected value* player i ascribes to f at ω.

2.2 Cones

Let X be a vector space. A non-empty set $K \subseteq X$ that satisfies the condition $\lambda K \subseteq K$ for all $\lambda \geq 0$ is a *cone*. The intersection of a family of cones is a cone.

The cones $K = \{0\}$ and $K = X$ are called *trivial* cones. The intersection of a family of non-trivial cones is a non-trivial cone.

2.3 Priors

If Ω is a countable state space, an *improper* prior for a type function t_i is a non-negative and non-zero function $p \colon \Omega \to \mathbb{R}$ such that for each $\pi \in \Pi_i$, $p(\pi) < \infty$ and $p(\pi) t_i(\omega)(\pi) = p(\omega)$ for all $\omega \in \pi$ Note that although for any $\pi \in \Pi_i$, $p(\pi) < \infty$, the possibility that $p(\Omega) = \infty$ is not ruled out, so that p may not be normalisable.

A prior for a type function t_i is a probability distribution $p \in \Delta(\Omega)$ such that for each $\pi \in \Pi_i$ and $\omega \in \pi$, the equation $p(\pi) t_i(\omega)(\pi) = p(\omega)$ is satisfied. Obviously, a prior is in particular an improper prior (normalising if necessary in the finite case), so that when we use the term ‘improper prior’ we will mean both concepts, but the term ‘prior’ alone will mean a normalisable prior.

Let J be an index set for the elements in Π_i, i.e., write $(\pi_i^j)_{j \in J}$ for the set of atoms in the partition Π_i. In addition, let $t_i^j := t_i(\omega)$ for $\omega \in \pi_i^j$, so that the set of distinct types is $(t_i^j)_{j \in J}$. Suppose that $(\alpha_j)_{j \in J}$ is a collection of real numbers that are either zero or positive and satisfy $\sum_{j \in J} \alpha_j = 1$. Then it is immediate that $\sum_{j \in J} \alpha_j t_i^j$ is a prior for player i. Conversely, if p is a prior, then by setting $\alpha_j := \sum_{\omega \in \pi_i^j} p(\omega)$, we can write $p = \sum_{j \in J} \alpha_j t_i^j$.

Denoting the set of all priors of player i by P_i, it follows from the above paragraph that P_i is the closed convex hull of i’s types. In particular, this makes P_i closed as a subset of $\Delta(\Omega)$.

An improper prior does not allow us to talk about the ‘probabilities’ that player i assigns to events at the ex ante stage, but it still allows us to discuss the relative likelihood that he ascribes to pairs of events; and the interim probability assessments of the player i.e., his types constitute a disintegration of the improper prior.

A *common improper* prior for a type space τ is a $p \in \Delta(\Omega)$ that is an improper prior for each player i.[1]

Note also that if p is a common improper prior, then for any constant $\gamma > 0$, γp is also a common improper prior. Hence the set of common improper priors forms a cone (which is why cones are of interest here). If p is a common improper prior and $p(\Omega) < \infty$ then $[p(\Omega)]^{-1} p$ is a common prior. Thus, for a finite space, a type space has a common prior if and only if it has a common improper prior. In light of this, we extend the definition of consistency to countable spaces: a type space τ is *consistent* when it has a common improper prior and *inconsistent* otherwise.

2.4 Common knowledge

An event $E \subseteq \Omega$ is *self-evident* [2] if for all $\omega \in E$ and each $i \in I$

$$\Pi_i(\omega) \subseteq E. \tag{3}$$

In particular, every element of the meet, $M \in \Pi$, is self-evident.

An event E is *common knowledge at* $\omega \in \Omega$ if and only if there exists a self-evident event $F \ni \omega$ such that for all $i \in I$

$$F \subseteq K_i(E). \tag{4}$$

In fact, the element of the meet containing ω is also known as the *common knowledge component* of ω, because it is the smallest self-evident set containing ω.

[1] Contrasting a prior for t_i with the types $t_i(\omega)(\cdot)$, the latter are referred to as the posterior probabilities of i.

[2] This definition is based on Monderer and Samet (1989), as is most of Sect. 2.4.

Working with a connected space is thus particularly convenient for theorems involving common knowledge, because if Π is connected, then an event E is common knowledge at ω if and only if $E = \Omega$.

2.5 Characterisation of the existence of common priors

We adopt the notation that for n-tuples $x_1, x_2 \in \mathbb{R}^{\Omega}$, $x_1 > x_2$ means that $x_1(\omega) > x_2(\omega)$ for all $\omega \in \Omega$, and $x_1 > 0$ means that $x_1(\omega) > 0$ for all ω.

Definition 1 An m-tuple of random variables $\{f_1, \ldots, f_m\}$ is a *bet* if

$$\sum_{i=1}^{m} f_i = 0.$$

Given an m-player type space τ, a bet f is an *agreeable bet* if $E_i f_i > 0$ for all i.

In the two-player case, the condition that $\sum_{i=1}^{m} f_i = 0$ is the same as $f_1 = -f_2$. Hence in this case we will sometimes say that a bet is agreeable if there is a random variable f such that $E_1 f > 0 > E_2 f$.

We can also slightly tweak the definition, replacing strict inequality with weak inequality, to obtain:

Definition 2 An m-tuple of random variables $\{f_1, \ldots, f_m\}$, for an m-player type space τ, is a *weakly agreeable bet* if $\sum_{i=1}^{m} f_i = 0$ and $E_i f_i \geq 0$ for all i, with $E_j f_j > 0$ for at least one $j \in I$.

We will say that an agreeable bet $\{f_1, \ldots, f_m\}$ is bounded if $|f_i|$ is bounded for all $i \in I$. In addition, given two sequences of r.v. $f = \{f_1, \ldots, f_m\}$ and $g = \{g_1, \ldots, g_m\}$, we define $f + g := \{f_1 + g_1, \ldots, g_1 + g_m\}$

The characterisation of the existence of common priors in finite spaces is accomplished by:

A finite type space τ has a common prior if and only if there is no agreeable bet.

The functions f_i, which sum to zero, can be interpreted as a bet between the players. The condition $E_i(f_i|\omega) > 0$, for each state ω, amounts to saying that the positivity of $E_i f_i$ is always common knowledge amongst the players. Thus, a necessary and sufficient condition for the existence of a common prior is that there is no bet for which it is always common knowledge that all players expect a positive gain. This establishes a fundamental, and remarkable, two-way connection between posteriors and priors.

The most accessible proof of this result is in Samet (1998). It was proved by Morris (1994) for finite type spaces and independently by Feinberg (2000) for compact type spaces. Bonanno and Nehring (1999) proved it for finite type spaces with two agents.

Anne	1	1/2	1/2	1/2	1/2	1/2	1/2	...
Ben	1/2	1/2	1/2	1/2	1/2	1/2	1/2	...

Fig. 1 The partition profile of Example 1

3 Main results

3.1 Agreeable betting

Theorem 1 *Let τ be a type space over $\{\Omega, \Pi\}$, where Ω is countable.*

(a) *If τ has a common prior, then there is no bounded weakly agreeable bet relative to τ.*
(b) *If τ has no common improper prior, then there exists a bounded agreeable bet relative to τ.*

3.2 Counterexamples

The converses to the statements in Theorem 1 do not obtain. Example 1 shows that the converse to Theorem 1b does not obtain, by exhibiting a type space with both a common improper prior and a bounded agreeable bet.

Example 1 The state space is $\Omega = \{\omega_0, \omega_1, \ldots\}$. There are two players, Anne and Ben. Anne's knowledge partition, Π_A, is given by

$$\{\{0\}, \{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots\}.$$

Ben's partition, Π_B, is given by

$$\{\{0, 1\}, \{2, 3\}, \{4, 5\}, \ldots\}.$$

Anne's type function, t_A, is given by $t_A(\omega_0, \omega_0) = 1$, and $t_A(\omega_n, \omega_n) = 0.5$ for all $n \geq 1$. Ben's type function, t_B, is given by $t_B(\omega_n, \omega_n) = 0.5$ for all $n \geq 0$. See Fig. 1.

Let $p(\omega) = 1$ for all $\omega \in \Omega$. Then p is a common improper prior.

Define a bounded random variable f as follows:

$$f(\omega_n) = \begin{cases} 1 & \text{if } \; n = 0 \\ 1 + \sum_{i=1}^{n} \frac{1}{2^i} & \text{if } \; n \text{ is even} \\ -(1 + \sum_{i=1}^{n} \frac{1}{2^i}) & \text{if } \; n \text{ is odd} \end{cases}$$

Then $E_A(f|\omega) > 0 > E_B(f|\omega)$ for all $\omega \in \Omega$, hence $\{f, -f\}$ is a bounded agreeable bet.

Example 1 exhibits a type space with a common improper prior and no common prior, but it cannot serve as a counterexample to Theorem 1a because of the existence of a bounded agreeable bet. Feinberg (2000), however exhibits a type space with neither a common prior nor a bounded agreeable bet. Example 1 and Feinberg's example together show that the no betting criterion is insufficiently subtle to be used as a tool for determining when a type space has a common improper prior but no common prior, as both examples satisfy that property, but one has a bounded agreeable bet and the other does not.

3.3 Unbounded bets

The restriction to bounded bets in the statement of Theorem 1a is necessary. Example 2 shows that the statement in Theorem 1a does not hold for unbounded bets.

Example 2 The state space is $\Omega = \{1, 2, \ldots\}$. There are two players, Anne and Ben. Anne's knowledge partition, Π_A, is given by

$$\{\{1\}, \{2, 3\}, \{4, 5\}, \{6, 7\}, \ldots\}.$$

Ben's partition, Π_B, is given by

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots\}.$$

Anne's type function, t_A, is given by

$$t_A(n, n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ \frac{2}{3} & \text{if} \quad n \text{ is even} \\ \frac{1}{3} & \text{if} \quad n \text{ is odd}, \quad n > 1 \end{cases}$$

Ben's type function, t_B, is given by

$$t_B(n, n) = \begin{cases} \frac{2}{3} & \text{if} \quad n \text{ is odd} \\ \frac{1}{3} & \text{if} \quad n \text{ is even} \end{cases}$$

See Fig. 2.

Let $p(n) = 2^{-n}$ for all n. Then p is a common prior.

Fix $\varepsilon > 0$. Define an unbounded random variable f as follows:

$$f(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ -(2f(n-1) + \varepsilon) & \text{if} \quad n \text{ is even} \\ 2f(n-1) + \varepsilon & \text{if} \quad n \text{ is odd}, \quad n > 1 \end{cases}$$

Then $E_A f > 0 > E_B f$, hence $\{f, -f\}$ is an unbounded agreeable bet.

Anne	1 |	2/3	1/3 |	2/3	1/3 |	2/3	1/3 ...
Ben	2/3	1/3 |	2/3	1/3 |	2/3	1/3 |	2/3 ...

Fig. 2 The partition profile of Example 2

3.4 Further comments

The familiar common prior characterisation from the finite state case can be recapitulated if the set of priors of at least one player [3] is compact; i.e. if this compactness condition is met than there is a common prior if and only if there is no bounded agreeable bet. An immediate corollary of this is that if the partition Π_j of at least one player is finite then the standard common prior characterisation in terms of bets holds. Proofs of these claims can be found in Hellman (2012). A full characterisation of the existence of common priors in countable state spaces has been obtained in Lehrer and Samet (2013).

In contrast, a complete understanding of when a countable knowledge space admits an unbounded bet is open as of this writing. Both Lehrer and Samet (2013) and Hellman (2012) present sufficient conditions for the existence of unbounded bets.

4 Acceptable bets and countable spaces

First note that for the proof of Theorem 1 it suffices to assume that the knowledge space Π is connected. For part (a), if there exists a common prior p over a non-connected Π, we can decompose Ω into disjoint connected components, $\Omega = \sum_j S_j$, and then p restricted to each S_j is a common prior; if there is no acceptable bet over S_j, then there can be no acceptable bet over Ω. For part (b), again decompose Ω disjointly as $\Omega = \sum_j S_j$. If there is no common improper prior over Ω, then there can be no *cip* over each S_j (otherwise that *cip* could serve as a *cip* for all of Ω). Next, suppose that we show the existence of a bounded agreeable bet f_{S_j} over each S_j. Since an agreeable bet may be arbitrarily scaled by a positive real number and remain agreeable, we may suppose w.l.o.g. that f_{S_j} is bounded above by 1 for each j. Then the function $f(\omega) := f_{S_j}(\omega)$ for $\omega \in S_j$ is a bounded agreeable bet over Ω.

We will therefore assume that for all type spaces τ over $\{\Omega, \Pi\}$ in this section, Π is connected.

4.1 Common prior $\Rightarrow$ no betting

The proof of Theorem 1a is a straightforward extension of the proof of the same statement in the finite state space case.

Proof of Theorem 1(a) Suppose by contradiction that there exists a bounded weakly agreeable bet f. First note that for any $j \in I$,

$$E_p(f_j) = \sum_{\omega' \in \Omega} f_j(\omega') p(\omega')$$

This quantity is well-defined given the assumptions that f is bounded, and that p is a (proper) common prior.

[3] Compare with a similar result in Ng and Wong (2005), where compactness of the prior sets of *all* players is required.

Next, as p is a prior for each $j \in I$, it follows that $E_p(f_j) = E_p(E_j f_j)$, where $E_j f_j$ is regarded as a function from Ω to $\mathbb{R}$.

As f is a bet, $\sum_{i \in I} f_i = 0$. Hence

$$0 = E_p\left(\sum_i f_i\right) = \sum_i E_p f_i = \sum_i E_p(E_i f_i). \tag{5}$$

But by the assumption that f is a weakly agreeable bet, there is at least one player i such that $E_i f_i > 0$, in which case $E_p E_i f_i > 0$, hence

$$\sum_i E_p(E_i f_i) > 0,$$

contradicting Eq. (5). □

4.2 No betting ⇒ common improper prior

Definition 3 Let τ be a type space over (Ω, Π), and let $X \subseteq \Omega$ be a subset of Ω. Define Π *restricted to* X, denoted Π^X, to be the partition profile over X given by $\Pi_i^X(\omega) := \Pi_i(\omega) \cap X$ for any state ω. Furthermore, let τ^X, a type function τ *restricted to* X, to be any type function over (X, Π^X) that satisfies the property that for any $\omega \in \Omega$, $t_i(\Pi_i^X) t_i^X(\omega) = t_i(\omega)$.

In the special case in which X is a positive subset of Ω, τ^X is explicitly given by:

$$t_i^X(\omega) := \frac{t_i(\omega)}{t_i(\Pi_i^X(\omega))}$$

for any $\omega \in X$ and any $i \in I$.

Intuitively, Π_i^X is the partition of X derived from the partition Π_i of Ω by 'ignoring all states outside of X'. It then follows intuitively that $t_i^X(\omega)$, for each state $\omega \in X$, is $t_i(\omega)$ scaled relative to the other states in $\Pi_i^X(\omega)$ in such a way that $\sum_{\omega \in X} t_i^X(\omega) = 1$.

For a random variable f, denote

$$E_i^X(f \mid \omega) := \sum_{\omega' \in \Pi_i^X(\omega)} t_i^X(\omega') f(\omega').$$

A set of random variables $f = \{f_1, \ldots, f_m\}$ is an agreeable bet relative to τ^X if for all $\omega \in X$, $\sum_i f_i(\omega) = 0$, and $E_i^X(f|\omega) > 0$ for all $i \in I$.

It is straightforward that if μ is a common improper prior for τ, $X \subseteq \Omega$ and $\mu(X) > 0$ then μ^X, meaning the restriction of μ to X, is a common improper prior for τ^X. Hence if X is finite and $\mu(X) > 0$ then $\mu^X/\mu(X)$ is a common prior for τ^X.

Lemma 1 *Let $(d_n)_{n=1}^{\infty}$ be an increasing sequence of integers. Let $(C_n)_{n=1}^{\infty}$ be a sequence satisfying the properties that each C_n is a non-trivial cone in $\mathbb{R}^{d_n}$ and $\varphi_n(C_{n+1}) \subseteq C_n$, where φ_n is the projection from $\mathbb{R}^{d_{n+1}}$ to $\mathbb{R}^{d_n}$. Assume in addition* *that for each $m > n$ the projection of C_m to C_n is a non-trivial cone. Then there exists a sequence of non-trivial cones $(C'_n)_{n=1}^{\infty}$ such that $C'_n \subseteq C_n$ and $\varphi_n(C'_{n+1}) = C'_n$ for all n.*

Proof For each n define a sequence $(C_n^k)_{n=1}^{\infty}$. To begin with, set $C_n^1 = C_n$ and then set $C_n^{k+1} = \varphi_n(C_{n+1}^k)$. The sets C_n^k are closed cones. Moreover, $C_n^{k+1} = \varphi_n \varphi_{n+1} \cdots \varphi_{n+k-1}(C_{n+k}^1)$, hence C_n^{k+1} is the projection of C_{n+k} and therefore by the assumption in the statement of the lemma it is a non-trivial cone. We show by induction that $C_n^{k+1} \subseteq C_n^k$ for all $k \geq 1$. For $k = 1$ this holds by the properties satisfied by the sequence $(C_n)_{n=1}^{\infty}$. If it holds for k then, since projection functions preserve inclusion, $C_n^{k+1} = \varphi_n(C_{n+1}^k) \subseteq \varphi_n(C_{n+1}^{k-1}) = C_n^k$.

Define $C'_n = \cap_{k \geq 1} C_n^k$. As an intersection of a decreasing sequence of non-trivial closed cones, C'_n is a non-trivial cone. Moreover, $\varphi_n(C'_{n+1}) = \varphi_n(\cap_{k \geq 1} C_{n+1}^k) = \cap_{k \geq 1} \varphi_n(C_{n+1}^k) = \cap_{k \geq 1} C_n^{k+1} = C'_n$, where the second equality follows because C_{n+1}^k is decreasing in k. □

Proposition 1 *Let τ be a type space over $\{\Omega, \Pi\}$. There exists a common improper prior for τ if and only if there exists an increasing sequence $(\Omega_n)_{n=1}^{\infty}$ of finite subsets of Ω such that $\cup_n \Omega_n = \Omega$ and for each n there is a common prior for τ^{Ω_n}.*

Proof Suppose that μ is a common improper prior for τ. Choose a finite subset Ω_1 of Ω such that $\mu(\Omega_1) > 0$ and then let $(\Omega_n)_{n=1}^{\infty}$ be any sequence of finite subsets of Ω satisfying $\Omega_{n-1} \subseteq \Omega_n$ such that $\cup_n \Omega_n = \Omega$. We have that $\mu(\Omega_n) > 0$ for all n (since $\Omega_1 \subseteq \Omega_n$).

Let μ^{Ω_n} be the restriction of μ to τ^{Ω_n}. By construction, for each i, each $\Pi_i^{\Omega_n}$ in the partition of i restricted to Ω_n such that $\mu(\Pi_i^{\Omega_n}) > 0$ and each $\omega \in \Omega_n$, we have

$$t_i^{\Omega_n}(\omega) = \frac{\mu(\omega)}{\mu(\Pi_i^{\Omega_n})} = \frac{\mu^{\Omega_n}(\omega)}{\mu^{\Omega_n}(\Pi_i^{\Omega_n})}. \tag{6}$$

Hence, μ^{Ω_n} satisfies the conditions of being an improper prior for i over Ω_n. Since we want a proper prior, we normalise to $\overline{\mu}^{\Omega_n}(\omega) := \mu^{\Omega_n}(\omega)/\mu(\Omega_n)$ instead. As this can be done for all i, we have that $\overline{\mu}^{\Omega_n}$ is a common prior for τ^{Ω_n}.

In the other direction, suppose there exists a sequence $(\Omega_n)_{n=1}^{\infty}$ satisfying the condition in the statement of the proposition. Let $d_n = |\Omega_n|$ and let C_n be the non-trivial cone of common improper priors for τ^{Ω_n} in $\mathbb{R}^{d_n}$.

For any $\mu \in C_{n+1}$, write as above μ^{Ω_n} to denote the restriction of μ to τ^{Ω_n}. By the same reasoning as that around Eq. (6), $t_i^{\Omega_n}(\omega) = \mu(\omega)/\mu(\Pi_i^{\Omega_n}) = \mu^{\Omega_n}(\omega)/\mu^{\Omega_n}(\Pi_i^{\Omega_n})$ for all i, all n and each $\omega \in \Omega_n$. It follows that $\varphi_n(C_{n+1}) \subseteq C_n$ for all n, where φ_n is the projection from $\mathbb{R}^{d_{n+1}}$ to $\mathbb{R}^{d_n}$

We show now that for $m > n$ the projection of C_m on C_n is a non-trivial cone. To see this, let $\mu \in C_m$. If $\mu(\Omega_n) > 0$ then the projection is straightforwardly non-trivial. If $\mu(\Omega_n) = 0$ and $\mu' \in C_n$ then $\mu + \mu' \in C_m$ and again we have that the projection is non-trivial. Consider the sequence of non-trivial cones $(C'_n)_{n=1}^{\infty}$ constructed in Lemma 1. Choose $\overline{\mu} \in C'_1$.

Next, consider a sequence of probability distributions $p_0 = \overline{\mu}$, $p_1 = \varphi_1^{-1}(\overline{\mu})$, $p_2 = \varphi_2^{-1}(\varphi_1^{-1}(\overline{\mu}))$, …. Since $\cup_n \Omega_n = \Omega$, for each $\omega \in \Omega$ there is a k such that $p_m(\omega)$ is determined for all $m \geq k$. It follows that the pointwise limit of $p_0, p_1, \ldots$, which is a function $p : \Omega \to \mathbb{R}$, exists and is well-defined.

It remains to show that p is a common improper prior. Let $\pi \in \Pi_i$ for any player i. Since p_n is a common prior for τ^{Ω_n}, it follows that $p_n(\pi^{\Omega_n})t_i^{\Omega_n}(\omega) = p_n(\omega)$. Hence, in the limit $p(\pi)t_i(\omega) = p(\omega)$, and therefore $p(\pi) < \infty$ as well. As this is true for each i, p is a *cip*. □

Proof of Theorem 1(b) Let τ be a type space over $\{\Omega, \Pi\}$. If Ω is finite, then Proposition 5 suffices. Suppose therefore that Ω is countably infinite. Applying Proposition 1, let $(\Omega_n)_{n=1}^{\infty}$ be an arbitrary increasing sequence of finite subsets such that $\cup_n \Omega_n = \Omega$. Since we are assuming that τ has no *cip*, there must be infinitely many n's such that there is no common prior for Ω_n. Thus we may assume without loss of generality that for each n there is no common prior for τ^{Ω_n}. By the standard characterisation of common priors in finite spaces, for each n there is an agreeable bet f^n on Ω_n bounded by 1. Let $\hat{f}^n$ be the extension of f^n to all of Ω, with $\hat{f}^n(\omega)$ defined to be 0 at every point $\omega \notin \Omega_n$. Then $f = \sum_n 2^{-n} \hat{f}^n$ is a bounded agreeable bet over Ω. □

Acknowledgements The results in this paper formed part of the author's PhD thesis, with the research supported in part by the European Research Council under the European Commission's Seventh Framework Programme (FP7/2007–2013)/ERC Grant Agreement No. 249159, and by ISF Grants 538/11 and 212/09. Special thanks are due to Ehud Lehrer and Dov Samet for many helpful conversations as well as crucial assistance in formulating and proving Lemma 1 and Proposition 1. The author also thanks the anonymous referees for enabling a much improved presentation, and Aviad Heifetz for helpful correspondence.

Appendix

Chains

Definition 4 A *chain* [4] of length $n \geq 0$, for a partition profile Π, from one state to another, is defined by induction on n. A state ω_0 is a chain of length 0 from ω_0 to ω_0. A chain of length $n+1$, from ω_0 to ω, is a sequence $c \xrightarrow{i} \omega$, where c is a chain of length n from ω_0 to ω', and $\omega \in \Pi_i(\omega')$. Thus, a chain of positive length n is a sequence $c = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} \omega_n$, such that $\omega_{s+1} \in \Pi_{i_s}(\omega_s)$ for $s = 0, \ldots, n-1$.

We write $\omega \to \omega'$ when there is a chain from ω to ω', in which case we say that ω and ω' are connected by a chain. The binary relation $\to$ is the transitive closure of the union of the relations $\xrightarrow{i}$, and it is an equivalence relation.

Definition 5 Given a chain $c = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} \omega_n$, its *reverse chain* c^{-1} is defined as

$$c^{-1} := \omega_n \xrightarrow{i_{n-1}} \omega_{n-1} \xrightarrow{i_{n-2}} \cdots \xrightarrow{i_0} \omega_0.$$

[4] The definition is taken from Hellman and Samet (2012).

A chain c is *alternating* if no two consecutive states ω_s and ω_{s+1} in c are the same, and no two consecutive agents i_s and i_{s+1} in c are the same.

Hellman and Samet (2012) prove that a partition profile Π is connected if and only if every two states are connected by at least one chain.

Positive, zero, and singular states

Definition 6 Given a type space τ, a state $\omega \in \Omega$ is:

- *positive* if $t_i(\omega) > 0$ for all $i \in I$;
- *zero* if $t_i(\omega) = 0$ for all $i \in I$;
- *singular* if it is neither positive nor zero.

Based on the categorisation of states in Definition 6, define the following:

Definition 7 Given a type space τ,

- A subset $S \subseteq \Omega$ is *i-positive* if $t_i(\omega) > 0$ for all $\omega \in S$.
- A subset $S \subseteq \Omega$ is *positive* if it is i-positive for all i (equivalently, if every $\omega \in S$ is a positive state). A chain c satisfying the condition that every element $\omega \in c$ is a positive state is a *positive chain*.
- A subset $S \subseteq \Omega$ is *i-non-singular* if $t_i(\omega) = 0$ for every singular $\omega \in S$.
- A subset $S \subseteq \Omega$ is *non-singularly positive* if it is positive, and every maximal chain c entirely contained in S satisfies the property that for every $\omega \in c$ and every $i \in I$, $\pi_i(\omega)$ is i-non-singular.

A subset S of Ω is thus non-singularly positive if it is positive, and for every $\omega \in S$, and every $i \in I$, every $\omega' \in \Pi_i(\omega)$ satisfies the condition that either

- $\omega' \in S$, or
- ω' is a zero state, or
- ω' is a singular state such that $t_i(\omega') = 0$.

Note that it is immediate by definition that if Ω is a positive state space, then trivially the entire space Ω is non-singularly positive.

Lemma 2 *If S is a non-singularly positive subset of Ω, then for any $\omega \in S$, every $\omega' \in \Omega$ that is connected to ω via a positive chain is also an element of S. It follows that every non-singularly positive subset S can be decomposed as $S = \cup T_j$, where each T_j is a non-singularly positive subset such that all of the members of T_j are connected to each other by positive chains.*

Proof Suppose that $S \subseteq \Omega$ is non-singularly positive, and let $\omega_0 \in S$ be chosen arbitrarily. Suppose that ω_0 is connected to ω_n by a positive chain $c = \omega_0 \overset{i_0}{\to} \omega_1 \overset{i_1}{\to} \cdots \overset{i_{n-1}}{\to} \omega_n$. Since ω_0 is in S and ω_1 is in the same partition element of player i_0 as ω_0, by Definition 7, ω_1 must also be in S, and continuing the same argument by induction, we conclude that all the elements in c are members of S. □

Type ratios

Definition 8 Let τ be a type space and (ω_1, ω_2) an ordered pair of positive states in $\pi \in \Pi_i$. The *type ratio* of (ω_1, ω_2) relative to i is [5] $\text{tr}^i_\tau(\omega_1, \omega_2) = t_i(\pi, \omega_1)/t_i(\pi, \omega_2)$. If a chain $c = \omega_0 \overset{i_0}{\to} \omega_1 \overset{i_1}{\to} \cdots \overset{i_{n-1}}{\to} \omega_n$ of length $n > 0$ is a positive chain, the *type ratio* of c is $\text{tr}_\tau(c) = \times_{k=0}^{n-1} \text{tr}^{i_k}_\tau(\omega_k, \omega_{k+1})$. For a positive chain c of length 0, $\text{tr}_\tau(c) = 1$. Thus, if $c = c' \overset{i}{\to} \omega$ where c' is a positive chain from ω_0 to ω' and ω' is a positive state, $\text{tr}_\tau(c) = \text{tr}_\tau(c')\text{tr}^i_\tau(\omega', \omega)$. [6]

We note here for later use two equalities involving type ratios that follow immediately from the definitions:

- For any chain c,

$$\text{tr}(c^{-1}) = [\text{tr}(c)]^{-1}, \tag{7}$$

- If $c = \omega_1 \overset{i}{\to} \omega_2 \overset{i}{\to} \omega_3$ (i.e., $\omega_2, \omega_3 \in \Pi_i(\omega_1)$), then

$$\text{tr}(c) = \text{tr}^i(\omega_1, \omega_3). \tag{8}$$

The following proposition extends the results of Proposition 4 in Hellman and Samet (2012), from positive type spaces to general type spaces.

Proposition 2 *Let τ be a type space with a connected knowledge space. Then there exists a common improper prior for τ if and only if Ω has a non-singularly positive subspace S with respect to τ, and for each ω_0 and ω in S, and chains c and c' entirely contained in S from ω_0 to ω*, $\text{tr}_\tau(c) = \text{tr}_\tau(c')$.

Proof Suppose that there exists a common improper prior p for τ. Let $S = \{\omega \in \Omega \mid p(\omega) > 0\}$. S is guaranteed to be positive, because $p \neq 0$. We next show that S is non-singularly positive: Suppose that for arbitrary $i \in I$ and $\omega \in S, \omega' \in \Pi_i(\omega)$. Furthermore, suppose that $\omega' \notin S$. Then $p(\omega') = 0$, while $p(\omega) > 0$. Hence $p(\Pi_i(\omega)) > 0$, and by the definition of an improper prior,

$$t_i(\omega') = \frac{p(\omega')}{p(\Pi_i(\omega))} = 0.$$

It follows from Definition 7 that S is non-singularly positive.

To complete this part of the proof, note that for any pair of states $\omega_1, \omega_2 \in S$ such that ω_1 and ω_2 are in the same element of Π_i for some $i \in I$, $\text{tr}^i_t(\omega_1, \omega_2) = p(\omega_1)/p(\omega_2)$. It then easily follows from the definition of the type ratio of a chain that for any chain c entirely contained in S and connecting ω_0 and ω, one has $\text{tr}_\tau(c) = p(\omega_0)/p(\omega)$.

[5] The type ratio defined in Hellman and Samet (2012) is the inverse of the one defined here, i.e., there $\text{tr}^i_t(\omega_1, \omega_2) = t_i(\pi, \omega_2)/t_i(\pi, \omega_1)$. Which definition is used is immaterial, as long as one keeps to it consistently in an exposition. The definition chosen here is more convenient for the equations developed in this paper.

[6] When we discuss only one type space we omit the subscript τ in tr_τ.

Conversely, suppose that Ω has a non-singularly positive subspace S with respect to τ, and that for each ω_0 and ω in S, any pair of chains c and c' entirely contained in S connecting ω_0 to ω satisfy $\text{tr}_\tau(c) = \text{tr}_\tau(c')$. Using Lemma 2, we may assume that S is connected (replacing S by a connected subset of itself if necessary).

We will construct a *cip* p. For $\omega \notin S$, set $p(\omega) = 0$. Otherwise, fix $\omega_0 \in S$ and for each $\omega \in S$, let $p(\omega) = \text{tr}(c)$ for some chain c from ω_0 to ω contained in S.

To see that p is a *cip* consider $\pi \in \Pi_i$. Suppose first that $\pi \cap S = \emptyset$. Then for all $\omega \in \pi$, $p(\omega) = 0$, hence $p(\pi) = 0$, and $p(\pi)t_i(\omega) = p(\omega)$ is satisfied.

Suppose instead that $\pi \cap S \neq \emptyset$, and that $\omega \in \pi \cap S$. Let c be a chain from ω_0 to ω entirely contained in S. For $\omega' \in \pi \cap S$, consider the chain $c' = c \overset{i}{\to} \omega'$. Then, by the definitions of tr and p, $p(\omega') = \text{tr}(c') = \text{tr}(c)\text{tr}^i(\omega, \omega') = p(\omega)t_i(\pi)(\omega')/t_i(\pi)(\omega)$. Thus,

$$p(\pi) = \sum_{\omega' \in \pi \cap S} p(\omega') = [p(\omega)/t_i(\pi)(\omega)] \sum_{\omega' \in \pi \cap S} t_i(\pi)(\omega') = p(\omega)/t_i(\pi)(\omega) < \infty,$$

and $p(\omega) = p(\pi)t_i(\pi)(\omega)$.

Finally, suppose that $\pi \cap S \neq \emptyset$, and that $\omega \in \pi$ is such that $\omega \notin S$. By construction, $p(\omega) = 0$, and by the assumption that S is non-singularly positive, it must be the case that $t_i(\omega) = 0$. We have already shown that $p(\pi) < \infty$, hence $p(\pi)t_i(\omega) = p(\omega)$ is satisfied. □

Definition 9 A chain $c = \omega_0 \overset{i_0}{\to} \omega_1 \overset{i_1}{\to} \cdots \overset{i_{n-1}}{\to} \omega_n$, where $\omega_{s+1} \in \Pi_{i_s}(\omega_s)$ for $s = 0, \ldots, n-1$, is a *cycle* [7] if $\omega_n = \omega_0$. If with respect to a cycle c of length n there is a pair $s, s' \in \{0, \ldots, n-1\}$ such that $s' > s+1$ and $\omega_{s'} \in \Pi_{i_s}(\omega_s)$, then we say that c has a *self-crossing point* at $\omega_{s'}$. A cycle c is a *non-crossing* cycle if it is alternating, and has no self-crossing points, i.e., for every pair $s, s' \in \{0, \ldots, n-1\}$ such that $s' > s+1$, $\omega_{s'} \notin \Pi_{i_s}(\omega_s)$.

Definition 9 leads to an immediate corollary of Proposition 2:

Corollary 1 *Let τ be a type space with a connected knowledge space. Then there exists a common improper prior for τ if and only if Ω has a non-singularly positive subspace S with respect to τ, and for each ω in S, every cycle $\overline{c} = \omega \to \omega$ that is entirely contained in S satisfies* $\text{tr}(\overline{c}) = 1$.

Proof It suffices to note the following: suppose that c_1 and c_2 are two distinct chains entirely contained in S connecting a pair of states ω and ω'. Then $\overline{c} := c_1 c_2^{-1}$ is a cycle connecting ω to itself. By Eq. (7), $\text{tr}(c_1) = \text{tr}(c_2)$ if and only if $\text{tr}(\overline{c}) = 1$. □

In fact, we can do even better, and show that it suffices to check the type ratios only of non-crossing cycles, instead of all cycles:

Proposition 3 *Let τ be a type space with a connected knowledge space. Then there exists a common improper prior for τ if and only if Ω has a non-singularly positive subspace S with respect to τ, and every non-crossing cycle $\overline{c}$ that is entirely contained in S satisfies* $\text{tr}(\overline{c}) = 1$.

[7] Cf. a similar definition in Rodrigues-Neto (2009).

Proof If there exists a *cip*, then by Corollary 1, there is a non-singularly positive $S \subseteq \Omega$ such that every cycle contained in S has type ratio equal to 1, hence in particular every non-crossing cycle satisfies the same property.

In the other direction, if there does not exist a *cip*, then for every non-singularly positive $S \subseteq \Omega$, there is at least one cycle $\overline{c}$ entirely contained in S such that $\text{tr}(\overline{c}) \neq 1$. Suppose that $\overline{c}$ is not non-crossing.

If $\overline{c}$ fails to be non-crossing because it is not alternating, this 'flaw' can easily be corrected: if two consecutive states ω_s and ω_{s+1} in $\overline{c}$ are identical, since that implies that $\text{tr}^i(\omega_s, \omega_{s+1}) = 1$, the state ω_{s+1} is redundant and can be removed from $\overline{c}$ without changing the type ratio. Similarly, if $\omega_s, \omega_{s+1}, \omega_{s+2}$ and ω_{s+3} are consecutive states that are all members of the same partition element of player i, then $\text{tr}^i(\omega_s, \omega_{s+1})\text{tr}^i(\omega_{s+2}, \omega_{s+3}) = \text{tr}^i(\omega_s, \omega_{s+3})$, hence we may remove ω_{s+1} and ω_{s+2} from $\overline{c}$ without changing the type ratio.

We will therefore assume that $\overline{c}$ is alternating but not non-crossing, and that we can then write $\overline{c} = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} \omega_n = \omega_0$, where $\omega_{s+1} \in \Pi_{i_s}(\omega_s)$ for $s = 0, \ldots, n-1$, where there exists at least one pair r, k such that $k > r + 1$, and $\omega_k \in \Pi_{i_r}(i_r)$.

We can 'shorten' $\overline{c}$ into another cycle:

$$\widehat{c} = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \ldots \omega_r \xrightarrow{i_r} \omega_k \xrightarrow{i_k} \cdots \xrightarrow{i_{n-1}} \omega_n = \omega_0.$$

If $\text{tr}(\widehat{c}) \neq 1$, then we have a cycle of type ratio not equal to 1, with a number of self-crossing points that is strictly less than the number of self-crossing points in $\overline{c}$, and we can continue by induction to apply the same process to $\text{tr}(\widehat{c})$.

Suppose, therefore, that $\text{tr}(\widehat{c}) = 1$. Denote:

$$c_0 = \omega_0 \xrightarrow{i_0} \omega_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{r-1}} \omega_r,$$
$$c_k = \omega_k \xrightarrow{i_k} \omega_{k+1} \xrightarrow{i_{k+1}} \cdots \xrightarrow{i_{n-1}} \omega_n,$$

and

$$c_l = \omega_r \xrightarrow{i_r} \omega_{r+1} \xrightarrow{i_{r+1}} \cdots \xrightarrow{i_{k-1}} \omega_k.$$

Then $\overline{c} = c_0 c_l c_k$, and $\widehat{c} = c_0(\omega_r, \omega_k)c_k$. By assumption, $1 = \text{tr}(\widehat{c}) = \text{tr}(c_0)\text{tr}^{i_r}(\omega_r, \omega_k)\text{tr}(c_k)$. It follows that $[\text{tr}^{i_r}(\omega_r, \omega_k)]^{-1} = \text{tr}^{i_r}(\omega_k, \omega_r) = \text{tr}(c_0)\text{tr}(c_k)$. We also assumed that $\text{tr}(\overline{c}) \neq 1$, so $1 \neq \text{tr}(c_0 c_l c_k) = \text{tr}(c_0)\text{tr}(c_k)\text{tr}(c_l) = \text{tr}^{i_r}(\omega_k, \omega_r)\text{tr}(c_l)$.

Writing out the last inequality in full yields

$$\text{tr}(\omega_k \xrightarrow{i_r} \omega_r \xrightarrow{i_r} \omega_{r+1} \xrightarrow{i_{r+1}} \cdots \xrightarrow{i_{k-1}} \omega_k) \neq 1.$$

But by Eq. (8), $\text{tr}(\omega_k \xrightarrow{i_r} \omega_r \xrightarrow{i_r} \omega_{r+1}) = \text{tr}(\omega_k \xrightarrow{i_r} \omega_{r+1})$, hence

$$\text{tr}(\omega_k \xrightarrow{i_r} \omega_{r+1} \xrightarrow{i_{r+1}} \cdots \xrightarrow{i_{k-1}} \omega_k) \neq 1.$$

We deduce then that the cycle $\widetilde{c} = \omega_k \overset{i_r}{\rightarrow} \omega_{r+1} \overset{i_{r+1}}{\rightarrow} \cdots \overset{i_{k-1}}{\rightarrow} \omega_k$ satisfies both that $\mathrm{tr}(\widetilde{c}) \neq 1$, and that it has a number of self-crossing points that is strictly less than the number of self-crossing points in $\overline{c}$. We can continue by induction to apply the same process to $\mathrm{tr}(\widetilde{c})$.

After applying this reasoning as often as necessary, we arrive at the existence of a cycle entirely contained in S with no self-crossing points, i.e., a non-crossing cycle, whose type ratio is not equal to 1, which is what we needed to show. □

Note that it follows from the definitions that if (ω_1, ω_2) is an ordered pair of positive states in $\pi \in \Pi_i^X$ then

$$\mathrm{tr}^i_{\tau^X}(\omega_1, \omega_2) = \frac{t_i^X(\omega_1)}{t_i^X(\omega_2)} = \frac{t_i(\omega_1)}{\tau_i(\pi)} \frac{\tau_i(\pi)}{t_i(\omega_2)} = \frac{t_i(\omega_1)}{t_i(\omega_2)} = \mathrm{tr}^i_\tau(\omega_1, \omega_2), \tag{9}$$

from which it further immediately follows that for any chain c of τ whose elements are entirely contained in X,

$$\mathrm{tr}_{\tau^X}(c) = \mathrm{tr}_\tau(c). \tag{10}$$

We need one more definition.

Definition 10 Let τ be a type space over (Ω, Π), let $X \subseteq \Omega$ be a positive subset of Ω, and let f be a random variable. A state $\omega \in X$ is a *surplus state for player* i relative to f and τ^X if $E_i^X(f|\omega) > 0$. In the context of a sequence $f = \{f_1, \ldots, f_m\}$ of r.v., we will say that ω is an i*-surplus state* if f_i is a surplus state for player i.

Constructing disagreements

Proposition 4 *Let τ be a type space over (Ω, Π), let S be a finite connected subset of positive states in Ω, and let $X \subseteq S$. Suppose that there exists an agreeable bet relative to τ^X. Then there exists an agreeable bet relative to τ^S.*

Proof Let f be an agreeable bet relative to τ^X. If $X = S$, there is nothing to prove. If $X \subset S$, then by the assumption of the connectedness of S, we can find at least one player i and a point $\omega' \notin X$ such that $\Pi_i(\omega') \cap X \neq \emptyset$. By the assumption of positivity, $t_i(\omega') > 0$, and by the assumption that f is an agreeable bet, every state $\omega \in \Pi_i(\omega')$ is an i-surplus state relative to f.

Denote $Y := X \cup \omega'$, and let ε be the (by the i-surplus state assumption) positive value

$$\varepsilon := \sum_{\omega'' \in \Pi_i(\omega') \cap X} f_i(\omega'') t_i^X(\omega''). \tag{11}$$

Next, let $\overline{f}_i(\omega')$ be a negative real number satisfying

$$0 > \overline{f}_i(\omega') > \frac{-(1 - t_i^Y(\omega'))}{t_i^Y(\omega')} \varepsilon\,, \tag{12}$$

and for $j \neq i$, set $\overline{f}_j(\omega') := -\overline{f}_i(\omega')/(m-1) > 0$, where $m = |I|$. Clearly, by construction, $\sum_{j \in I} \overline{f}_j(\omega') = 0$. Complete the definition of $\overline{f}$ by letting $\overline{f}(\omega'') := f(\omega'')$ for all $\omega'' \in X$.

It is straightforward to check that $\overline{f}$ is an agreeable bet relative to τ^Y. Now simply repeat this procedure as often as necessary to extend the agreeable bet to every state in the finite set S. □

Lemma 3 *Let τ be a type space over (Ω, Π), and let X be a non-crossing cycle such that* $\mathrm{tr}(X) \neq 1$. *Then there exists a random variable f that is an agreeable bet relative to τ^X.*

Proof Write the non-crossing cycle as $X = \omega_1 \overset{i_1}{\to} \omega_2 \overset{i_2}{\to} \ldots \omega_n \overset{i_n}{\to} \omega_{n+1} = \omega_0$, where $\omega_{s+1} \in \Pi_{i_s}(\omega_s)$ for $s = 1, \ldots, n$. Assume without loss of generality that $\mathrm{tr}(X) < 1$ (otherwise simply reverse the ordering of states in the cycle). To cut down on notational clutter, write $r_s := \mathrm{tr}^{i_s}(\omega_s, \omega_{s+1})$, hence $\mathrm{tr}(X) = r_1 r_2 \cdots r_n$. Furthermore, denote by P the set such that $i \in P$ if and only if i is one of $i_1, i_2, \ldots i_n$ used in the presentation above of the cycle X.

We will several times make use of the following simple technical observation: suppose that i is a player, π is a partition element of Π_1, and $\omega', \omega'' \in \pi$. Furthermore, suppose that g is a random variable satisfying the property that $g(\omega) = 0$ for all $\omega \in \pi$ such that $\omega \neq \omega', \omega''$. Then:

$$\left\{ \begin{array}{lll} E_i(g|\omega') = 0 & \text{if} & g(\omega'') = -\mathrm{tr}^i(\omega', \omega'')g(\omega'), \\ E_i(g|\omega') > 0 & \text{if} & g(\omega'') > -\mathrm{tr}^i(\omega', \omega'')g(\omega'), \\ E_i(g|\omega') < 0 & \text{if} & g(\omega'') < -\mathrm{tr}^i(\omega', \omega'')g(\omega'). \end{array} \right\} \tag{13}$$

We now proceed to the construction of an agreeable bet, in stages.

First note that by the assumption that $r_1 r_2 \cdots r_n < 1$, we may choose a $\delta_n > 1$ such that $r_1 r_2 \cdots r_n \delta_n < 1$. We may then further define δ_{n-1} such that $\delta_n > \delta_{n-1} > 1$ and so on, to yield a sequence $\delta_n > \delta_{n-1} > \ldots > \delta_2 > 1$. Use this to define $\overline{f}$ by:

$$\begin{array}{ll} \overline{f}_{i_1}(\omega_1) = -1 & \\ \overline{f}_{i_n}(\omega_1 = \omega_{n+1}) = 1 & \\ \overline{f}_{i_{s-1}}(\omega_s) = \delta_s r_1 r_2 \cdots r_{s-1} & \text{for } \; 2 \leq s \leq n \\ \overline{f}_{i_s}(\omega_s) = -\delta_s r_1 r_2 \cdots r_{s-1} & \text{for } \; 2 \leq s \leq n \\ \overline{f}_j(\omega) = 0 & \text{for all other } \omega \text{ and } j. \end{array}$$

Using Eq. (13) repeatedly (here is also where we use the assumption that X is non-crossing, which ensures that in each partition element of every player i there are at most two states at which $\overline{f}_i$ takes on non-zero values) we deduce that $E_{i_s}(\overline{f}_{i_s}|\omega_s) > 0$ for $1 \leq s \leq n+1$.

We still need to ensure that the players who are not in P have positive expectations at the states participating in X. To do so, note the following: since $r_1 r_2 \cdots r_n \delta_n < 1$, we can choose an ε_{n+1} such that $1 - \varepsilon_{n+1} > r_1 r_2 \cdots r_n \delta_n$. Similarly, since for any $2 \leq s \leq n$, $\delta_s r_1 r_2 \ldots r_{s-1} > \delta_{s-1} r_1 r_2 \ldots r_{s-1}$, we can choose ε_s such that

$$\delta_s r_1 r_2 \ldots r_{s-1} - \varepsilon_s > \delta_{s-1} r_1 r_2 \ldots r_{s-1}.$$

At each state in $\omega_s \in X$, therefore, we intuitively can take away a positive part of the positive value of $\overline{f}_{i_{s-1}}(\omega_s)$ and 'redistribute' it among the other players. This enables the following construction:

$$\begin{array}{ll} f_{i_s}(\omega_s) = \overline{f}_{i_s}(\omega_s) & \text{for} \quad 1 \le s \le n \\ f_{i_{s-1}}(\omega_s) = \overline{f}_{i_{s-1}}(\omega_s) - \varepsilon_s & \text{for} \quad 1 \le s \le n \\ f_j(\omega_s) = \varepsilon_s/(m-2) & \text{for } 1 \le s \le n, \text{ for } \quad j \in I,\ j \neq i_s, i_{s-1} \\ f_j(\omega) = 0 & \text{for all other } \omega \text{ and } j. \end{array}$$

By construction, $f = \{f_1, \ldots, f_m\}$ is an agreeable bet relative to τ^X, which is what we needed to show. □

The following proposition, one half of the no betting characterisation for finite type spaces, has several proofs in the literature, all of which ultimately rely on convex separation theorem, or equivalents thereof. The proof presented here (building on the previous lemmas) is, in contrast, constructive and entirely combinatorial.

Proposition 5 *Let τ be a type space over (Ω, Π), where Ω is a finite state space. If τ has no common prior then there exists an agreeable bet relative to τ.*

Proof Recalling that in a finite state space every improper prior can be normalised, and is therefore a proper prior, we may refer to all the previous lemmas and apply them restricting attention to the special case of common priors, rather than the more general common improper prior.

Let $S \subseteq \Omega$ be the set of non-singularly positive states in Ω, and (using Lemma 2) decompose S disjointly as $S = \cup_{j=1}^{k} T_j$, where each T_j is non-singularly positive and connected. By Proposition 3, for each $1 \le j \le k$, there is a non-crossing cycle c_{T_j} contained in T_j such that $\text{tr}(c_{T_j}) \neq 1$. Lemma 3 together with Proposition 4 imply that there exists $f^{T_j} = \{f_1^{T_j}, \ldots, f_k^{T_j}\}$ such that $f_i^{T_j}(\omega) = 0$ for all $\omega \notin T_j$ and $i \in I$, and f^{T_j} is an agreeable bet relative to τ^{T_j}.

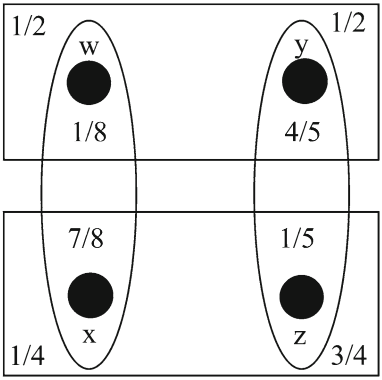
Let Z be the set of zero states, and denote by Q the complement of $S \cup Z$. Define f^Q in stages as follows.

In stage 0, let W_0 be the set of singular states in Q. For every $\omega \in W_0$ decompose I as $I = J \cup K$, where $\tau_j(\omega) > 0$ for all $j \in J$ and $\tau_i(\omega) = 0$ for $i \in K$, and choose an arbitrary $i' \in K$. Then define $g_j^0(\omega) = 1/|J|$ for every $j \in J$, with $g_{i'}^0(\omega) = -1$, and $g_i^0(\omega) = 0$ for all other $i \in K, i \neq i'$. Set $g_k^0(\omega) = 0$ for all states ω not in W_0 and all players $k \in I$.

Note, for the rest of this proof, that by definition every state in $Q \setminus W_0$ is a positive state.

In stage 1, let W_1 be the set of all states $\omega \in Q \setminus W_0$ satisfying the property that there is at least one player i, and a state $\omega' \in W_0$, such that $\omega \in \pi_i(\omega')$. For each $\omega \in W_1$, choose ω' as just described. If $t_i(\omega') > 0$, then by construction ω' is an i-surplus state relative to g^0. Hence we can apply a process similar to that described in the paragraph preceding and following Eq. (12) to extend g^0 to a function g^1 such that $g_i^1(\omega) < 0$, but for all $j \neq i$, $g_j^1(\omega) > 0$, $\sum_{j \in I} g_j^1(\omega) = 0$, and is a surplus state for all players.

Fig. 3 The knowledge space in Example 3



If $t_i(\omega') = 0$, then note that since ω is positive but not contained in any non-singularly positive set, there must be a chain $c = \omega \stackrel{i_0}{\rightarrow} \omega_1 \stackrel{i_1}{\rightarrow} \cdots \stackrel{i_{n-1}}{\rightarrow} \omega_n$ entirely contained in $Q \setminus W_0$ such that there is a player i and a state $\omega' \in X_0$ satisfying $\omega_n \in \pi_i(\omega')$ and $t_i(\omega') > 0$. But then we can apply the same argument as in the previous paragraph to extend g^0 to g^1 by induction over all the states $\omega_n, \omega_{n-1}, \ldots, \omega$, yielding a function such that $\sum_{j \in I} g^1_j(\omega) = 0$, and is a surplus state for all players.

In all stages $l > 1$, in stage $l - 1$, a function g^{l-1} has been defined such that each state $\omega \in W_{l-1}$ is an i-surplus state for every player i relative to g^{l-1}. Denote by W_l the set containing every state $\omega \in Q \setminus W_{l-1}$ satisfying the property that there is at least one $i \in I$ and $\omega \in W_{l-1}$ such that $\omega' \in \pi_i(\omega)$. Since ω is an i-surplus state relative to g^{l-1}, we can again apply the same technique as in the previous paragraphs to extend g^{l-1} to g^l.

By the finiteness of Ω, this iterative process ends after a finite number of stages r. Finally, set $f^Q := g^r$, and define

$$f := f^{T_1} + f^{T_2} + \ldots f^{T_k} + f^Q.$$

It is straightforward to check that f, by construction, is an agreeable bet. □

As mentioned in the introduction, putting together the elements of the proofs in this section yields an algorithm that can be applied in finite type spaces. The algorithm determines whether the space has a common prior, by listing the connected non-singularly positive subspaces and checking whether there is such a subspace such that every non-crossing cycle contained in it satisfies $\mathrm{tr}(c) = 1$ (see Proposition 3). If it does have a common prior, the algorithm then constructs the common prior, by the method in the proof of Proposition 2. If the space does not have a common prior, the algorithm constructs an agreeable bet by the method in the proof of Proposition 5, thus finding a sequence of random variables about whose expected values the players ‘agree to disagree’.

Example 3 This simple example illustrates how to compute an acceptable bet given a knowledge space without a common prior. The space of states is$\Omega = \{w, x, y, z\}$.

Player 1's partition of Ω is $\{\{w, x\}, \{y, z\}\}$, with corresponding posteriors $\{\{1/8, 7/8\}, \{4/5, 1/5\}\}$ and Player 2's partition is $\{\{w, y\}, \{x, z\}\}$, with corresponding posteriors $\{\{1/2, 1/2\}, \{1/4, 3/4\}\}$. See Fig. 3.

The sequence w, x, y, z forms a cycle, with the corresponding type ratios $\frac{1}{7}, \frac{1}{3}, \frac{1}{4}$ and 1. Multiplying them together gives $\frac{1}{84}$, hence the type ratio of the cycle is not equal to one, and therefore this type space has no common prior, hence there must exist an agreeable bet.

To compute an agreeable bet, choose real numbers $1 < \delta_2 < \delta_3 < \delta_4 < 84$. For example, let $\delta_2 = 2, \delta_3 = 6$ and $\delta_4 = 8$. Use these to define a function $f(w) := -1$, $f(x) := \delta_2 \frac{1}{7} = \frac{2}{7}$, $f(y) := -\delta_3 \frac{1}{7} \frac{1}{3} = \frac{6}{21}$ and $f(z) := \delta_4 \frac{1}{7} \frac{1}{3} \frac{1}{4} = \frac{8}{84} = \frac{2}{21}$. $\{f, -f\}$ is an agreeable bet.

References

Aumann RJ (1976) Agreeing to disagree. Ann Stat 4(6):1236–1239

Aumann RJ (1998) Common priors: a reply to Gul. Econometrica 66:929–938

Bonanno G, Nehring K (1999) How to make sense of the common prior assumption under incomplete information. Int J Game Theory 28:409–434

Feinberg Y (2000) Characterizing common priors in the form of posteriors. J Econ Theory 91:127–179

Gul F (1998) A comment on Aumann's Bayesian view. Econometrica 66:923–927

Harsányi JC (1967–1968) Games with incomplete information played by Bayesian players. Manag Sci 14: 159–182, 320–334, 486–502

Heifetz A (2006) The positive foundation of the common prior assumption. Games Econ Behav 56:105–120

Hellman Z (2012) Common priors and uncommon priors. PhD Thesis, Hebrew University of Jerusalem

Hellman Z, Samet D (2012) How common are common priors. Games Econ Behav 74:517–525

Lehrer E, Samet D (2011) Agreeing to agree. Theor Econ 6:269287

Lehrer E, Samet D (2013) Belief consistency and trade consistency. Working paper, Tel Aviv University

Monderer D, Samet D (1989) Approximating common knowledge with common beliefs. Games Econ Behav 1:170–190

Morris S (1994) Trade with heterogenous prior beliefs and asymmetric information. Econometrica 62: 1327–1347

Ng MC, Wong NC (2005) The no trade principle in general environments, Working Paper

Rodrigues-Neto JA (2009) From posteriors to priors via cycles. J Econ Theory 144:876–883

Samet D (1998) Common priors and separation of convex sets. Games Econ Behav 24:172–174

Simon RS (2000) The common prior assumption in belief spaces: an example. Discussion Paper 228, The Centre for the Study of Rationality. Hebrew University