The Inverse Plurality Rule – An Axiomatization♦

By

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Abstract

This note characterizes the 'inverse plurality rule', where voters specify only their least preferred alternative. This rule is characterized by a new minimal veto condition (MV) and the four well known conditions that characterize scoring rules; namely, Anonymity (A), Neutrality (N), Reinforcement (RE) and Continuity (C). Our new characterization result is related to the characterizations of approval voting and of the widely used plurality rule.

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1. Introduction

The plurality rule requires voters to vote for a single alternative that is their most preferred one. Nevertheless, in many voting situations voters might prefer to specify the alternative which is their *least* desired one. We refer to this interesting and easy to implement voting mechanism as the "inverse plurality rule".[1] Although these two rules represent polar voting mechanisms, both are special cases of scoring rules that can be conceived as restricted versions of the flexible "approval voting" rule (Brams and Fishburn (1978)). This note characterizes the inverse plurality rule by using the four conditions that characterize scoring rules; namely, Anonymity (A), Neutrality (N), Reinforcement (RE) and Continuity (C) and a new weak 'Minimal Veto' property (MV).

We conclude this note by briefly discussing the relationship between the characterization of the inverse plurality rule and the axiomatic characterization of 'approval voting', the plurality rule and, in general, any "vote for t alternatives scoring rule".

2. The Framework

Let A be a finite set of k alternatives, $k \geq 3$, and let $N=\{1,\ldots,n\}$ be a finite set of individuals. Suppose that the preference relation L_i of individual i, $i \in N$, is a strict linear order (complete, transitive and asymmetric relation) over A. The set of these orders is denoted by $\mathcal{L}$. A preference profile is an n-tuple $L=(L_1, L_2,\ldots, L_n)$ of such linear orders. The set of preference profiles is denoted $\mathcal{L}^n$. A social choice rule V is a mapping from $\mathcal{L}^n$ to the set of non-empty subsets of A. This rule specifies the collective choice for any preference profile, V: $\mathcal{L}^n \rightarrow 2^A$. It is assumed that V is defined for any A and N.

Let $\{S_1, S_2,\ldots, S_k\}$ be a monotone sequence of real numbers, $S_1 \leq S_2 \leq \ldots \leq S_k$, such that $S_1 < S_k$. Suppose that the individuals sincerely rank the alternatives assigning S_1 scores to the one ranked last, S_2 to the one ranked next to the last, and so on until the best alternative which is assigned the score S_k . The difference between the scores

[1] In Huang and Chua (2000), Lepelley and Mbih (1994) and Saari (1995) this rule is referred to as the anti-plurality rule. In Myerson (2002) this rule is called negative voting.

assigned to two alternatives supposedly represents each voter's preference between the two alternatives. A **scoring rule** selects the alternatives with the maximal total score. For any number of alternatives k, the **inverse plurality rule** is a scoring rule defined by the following scores: $\{S_1,S_2,\ldots,S_k\}=\{0,1,\ldots,1\}$. The most common scoring rule is the **plurality rule;** a scoring rule defined by the scores: $\{S_1,S_2,\ldots,S_k\}=\{0,\ldots,0,1\}$. The informational requirements of the plurality rule and of the inverse plurality rule are very modest. In the former case an individual has to report just his most preferred alternative. In the latter case the application of the rule is possible when every individual reports just his worst alternative.

The following well known properties require that the voting rule is unbiased toward alternatives as well as toward voters and that it ensures some 'positive relationship' between individual preferences and the social choice:

Anonymity (A): If the names of the voters are permuted, then the outcome of the voting rule is not affected.

Neutrality (N): If the names of the alternatives are permuted in the preferences of the voters on A, then the alternative/s selected by the voting rule change accordingly.

Reinforcement (RE): Suppose that two disjoint groups of individuals N_1, N_2 face the set of alternatives A and, given their preference profiles, V selects, respectively, B_1 and B_2. Then $B_1 \cap B_2 \neq \emptyset$ implies that, given the union of the profiles of the two groups, V selects $B_1 \cap B_2$.

Continuity (CO): Suppose that two disjoint groups of individuals N_1, N_2 face the set of alternatives A and, given their preference profiles, V selects, respectively, alternative a and alternative b. Then V selects alternative a, given the profile of the expanded group $(mN_1) \cup N_2$, if N_1 is replicated sufficiently many times (the integer m is sufficiently large).

A coalition M has veto power under the social choice rule V, if and only if there exists some $x \in A$ and some profile L, such that $x \notin V(L)$ while $\forall i \in N \setminus M \text{ and } \forall y \in A, xL_i y$. A weak requirement of veto power is that there exists at least one profile L, such that a minority group M of $\lceil n/k \rceil$ members[2, 3] containing that individual has a veto power: the other n-$\lceil n/k \rceil$ individuals cannot guarantee the selection of their most favorable consensus alternative, even when coordinated strategic voting is allowed.[4] Notice that (i) since $k \geq 3$, for $n>2$, the $\lceil n/k \rceil$ group is a minority group and (ii) if $n \leq k$, then $\lceil n/k \rceil=1$; that is, in such a case every individual has the above minimal veto power. Our '**Minimal Veto**' condition requires that the social choice rule assigns a minimal veto power to every minority group M of $\lceil n/k \rceil$ members, irrespective of the size of A and N. That is,

Minimal Veto (MV): A social choice rule V satisfies the minimal veto condition if and only if, for any k and n, $n>2$, a coalition M of $\lceil n/k \rceil$ individuals has a veto power.[5]

3. The Result

By the main result in Young (1975),

Lemma 1: A social choice rule is a scoring rule, if and only if it satisfies A, N, RE and CO.

By Theorem 2 in Baharad and Nitzan (2002),

Lemma 2: Suppose that a majority group of size αn ($\frac{1}{2}<\alpha<1$ and α being a fraction with a denominator n) unanimously prefers a certain alternative a. Then, under a scoring rule defined by the sequence $\{S_1,S_2,\ldots,S_k\}$, the minority group of size $(1-\alpha)n$ can veto the selection of alternative a iff:

$$\alpha < \frac{S_k - S_1}{2S_k - S_1 - \bar{S}} \quad , \tag{1}$$

where $\bar{S} = \sum_{i=1}^{k-1} S_i /(k-1)$.

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[2] $\lceil t \rceil$ is the smallest integer that is not smaller than t.

[3] In the next section it is shown that $\lceil n/k \rceil$ is the smallest number of individuals that always have the minimal veto power.

[4] See Baharad and Nitzan (2002).

[5] This condition is satisfied by Moulin's (1986) 'voting by veto' method. It also satisfies Kolm's (1997) requirement of assigning every individual the equal right of 'not being ignored'.

The next lemma establishes that the inverse plurality rule is the only scoring rule that satisfies MV.

Lemma 3: A scoring rule satisfies MV, if and only if it is the inverse plurality rule.

Proof: By Lemma 2, a minority of size $(1-\alpha)n$ has a minimal veto power if α is smaller than $\overline{\alpha}$:

$$(2) \qquad \frac{S_k - S_1}{2S_k - S_1 - \overline{S}} = \overline{\alpha}$$

S_k and S_1 are constants. $1-\alpha$ is therefore minimal (α is maximal) when $\overline{S}$ is maximal. Recall that $\overline{S}=(S_{k-1}+S_{k-2}+\ldots S_2+S_1)/(k-1)$. Since, with no loss of generality, we can let $S_1=0$, $\overline{S}=(S_{k-1}+S_{k-2}+\ldots S_2)/(k-1)$ is maximal when $S_{k-1}\ldots S_2$ are maximal. However, by definition, all of these scores are bounded by S_k. $\overline{S}$ is therefore maximal when $S_k=S_{k-1}=\ldots=S_2=1$ and $S_1=0$. Notice that this system of scores defines the inverse plurality rule. We conclude the sufficiency part of the proof by showing that the inverse plurality rule assigns the minimal veto power to $\lceil n/k \rceil$ individuals. To prove this we verify that $\overline{\alpha}=\frac{k-1}{k}$. Since we already know that $\overline{\alpha}$ is obtained under the inverse plurality rule defined by $S_k=S_{k-1}=S_{k-2}=\ldots=S_2=1$,

$$(3) \qquad \overline{S}=\frac{k-2}{k-1}$$

By substituting $\overline{S}$ into (2) , we obtain that $\alpha=\frac{k-1}{k}$ and, therefore, $(1-\alpha)n=\frac{n}{k}$. That is, the inverse plurality rule assigns the minimal veto power to $\lceil n/k \rceil$ individuals.

To prove that the inverse plurality is necessary for MV, notice that this property is defined for any k and n. In those cases where k and n satisfy the equality: $\alpha n=m(k-1)$, for some integer m, inequality (1) becomes the necessary and sufficient condition for a minority group of size $(1-\alpha)n$ to have the required minimal veto power. In those cases, therefore, the inverse plurality is the only voting rule that satisfies MV. Notice that inequality (1) is a condition that always (independent of k

and n) results in the same scoring rule- the inverse plurality rule. Hence, the inverse plurality rule is the only rule that always guarantees MV . ■

Lemmas 1, 2 and 3 imply that

Theorem: A social choice rule is the inverse plurality rule, if and only if it satisfies A, N, RE, CO and MV.

4. Plurality, inverse plurality and approval voting

The most widely used scoring rule is the plurality rule. An axiomatic characterization of the plurality rule appears in Richelson (1978). Whereas MV requires that a coalition M of $\lceil n/k \rceil$ individuals has minimal veto power, the condition MV' is defined as the requirement that no minority [6] sub-group of individuals has our minimal veto power. It can be verified that the only scoring rule satisfying MV' is the plurality rule[7]. Using again Young's characterization of scoring rules we obtain:

Corollary 1: The plurality rule is characterized by A, N, RE, CO and MV'.

The plurality rule and the inverse plurality rule are the two restricted (non-flexible) extreme versions of 'approval voting'; "vote for one alternative" and "vote for (k-1) alternatives" scoring rules. Their characterization complements the characterization of 'approval voting'. More specifically, in his characterization of 'approval voting', Fishburn (1978) introduces the disjoint equality (DE) condition. In our context, this condition requires that, $\forall L_i$ *and* L_j *in* $\mathcal{L}$, $V(L_i) \cap V(L_j) = \varnothing \Rightarrow V(L_i, L_j) = V(L_i) \cup V(L_j)$. In terms of veto power, DE implies that regardless of individual i's preference intensity, the selection of $V(L_j)$ cannot be prevented as long as the preference profile of the two-member society of i and j contains L_j.

6 A coalition M is said to be a minority if and only if its cardinality is less than the cardinality of the complement coalition $N \setminus M$.

7 See Theorem 1 in Lepelley and Merlin (1998) or Theorem 4.1 in Sanver (2002).

By the main result in Fishburn (1978), 'approval voting' is neutral and satisfies RE and DE.[8] Notice that both the plurality and inverse plurality rules satisfy DE. The inverse plurality trivially satisfies DE because under it (assuming $k\geq 3$), $V(L_i)\cap V(L_j)=\emptyset$ is never satisfied. Finally note that, by Corollary 1, the axiomatization of the plurality rule requires that we replace DE with the stronger requirement that the social choice rule V is anonymous and satisfies CO and MV'. Similarly, the axiomatization of the inverse plurality rule requires that we substitute the innocuous DE with the stronger combination of anonymity and conditions CO and MV.

5. Concluding Remark

This study focused on the axiomatization of the inverse plurality rule using the veto power property. Note that in the same spirit it is possible to characterize any "vote for *t*-alternatives scoring rule". Such a rule is defined by :

$\{S_1,\ldots,S_{k-t},S_{k-t+1},\ldots,S_k\}=\{0,\ldots,0,1,\ldots,1\}$, where $S_i=0$ if $1\leq i\leq k-t$ and $S_i=1$ if $k-t< i\leq k$.

We shoe below that every such rule satisfies a different q- veto condition. The q-veto condition is defined as follows:

q-Veto (qV): A social choice rule V satisfies the q-veto condition if and only if a coalition M of voters has veto power under V if and only if $|M|\geq q$.

By (1), $q=(1-\alpha)n=\left(1-\dfrac{S_k-S_1}{2S_k-S_1-\overline{S}}\right)n$. Note that $S_k=1$, $S_0=0$ and $\overline{S}=\dfrac{t-1}{k-1}$. Hence,

a "vote for *t*-alternatives scoring rule" satisfies the *q*-veto condition where

$$q=\left(1-\frac{k-1}{2k-1-t}\right)n=\left(\frac{k-t}{2k-1-t}\right)n\ . \tag{4}$$[9]

This implies that a "vote for *t*-alternatives scoring rule" is characterized by A, N, RE, CO and $\left(\dfrac{k-t}{2k-1-t}\right)n$-Veto.

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[8] For an alternative axiomatization of 'approval voting' see Baigent and Xu (1991).

[9] It can be verified that under the special cases of the plurality rule (t=1) and the inverse plurality rule ($t=k-1$), q is equal, respectively, to $n/2$ and (n/k).

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