Scoring Rules: An Alternative Parameterization♦

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Summary. This note presents an alternative parameterization of *any* scoring rule that satisfies the score-expansion property. This parameterization is based on the vector that specifies, for every number of alternatives k, $k \geq 3$, the minimal size of a coalition that can veto an alternative which is preferred by everybody outside the coalition. Our result sheds new light on the commonly used plurality and Borda rules, as well as the inverse plurality rule and any "vote for t alternatives rule".

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1. Introduction

Various successful attempts were made to characterize some specific scoring rules,[3] such as the plurality rule (Richelson (1978), Ching (1996)), the Borda rule (Young (1974) and Saari (1990)), and, more recently, the inverse plurality rule[4] and the "vote for *t* alternatives rule" (Baharad and Nitzan (2005)). All these scoring rules satisfy the score-expansion property. This note shows that such scoring rules are parameterized by a vector that specifies, for every number of alternatives k, $k \geq 3$, the minimal size of a coalition that can veto an alternative which is preferred by everybody outside the coalition. This parameterization is an alternative to the direct one which is based on the fixed vector of k scores assigned to the alternatives.

2. The Framework

Let A be a finite set of a alternatives, $a \geq 3$, and let $N=\{1,\ldots,n\}$ be a finite set of individuals. Suppose that the preference relation L_i of individual i, $i \in N$, is an irreflexive and transitive relation on A. The set of these relations is denoted by $\mathcal{L}$. A preference profile is an n-tuple $L=(L_1, L_2,\ldots, L_n)$ of such preference relations. The set of preference profiles is denoted $\mathcal{L}^n$. For any k-element subset B of A, $k=3,\ldots,a$, a social choice rule V maps $\mathcal{L}^n \times (2^A \setminus \{\phi\})$ to $2^A \setminus \{\phi\}$, such that $V(L,B) \subseteq B$ for all $L \in \mathcal{L}^n$ and all $B \in 2^A \setminus \{\phi\}$.

Let $\{S_1, S_2,\ldots, S_k\}$ be a monotone sequence of real numbers, $S_1 \leq S_2 \leq \ldots \leq S_k$. Suppose that the individuals rank k alternatives assigning S_1 scores to the one ranked last, S_2 to the one ranked next to the last, and so on until the best among the k alternatives is assigned the score S_k. Given any k-element subset B of A, $k \geq 3$, a **scoring rule** selects the alternatives in B with the maximal total score. The applicability of a scoring rule to any k- alternative set B, $B \subseteq A$, means that a particular scoring rule V is parameterized by the collection of (a-2) vectors of scores,

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[3] Young (1975) presented a characterization of the set containing all scoring rules, using four well known properties: Anonymity, Neutrality, Reinforcement and Continuity.

[4] Under this rule every voter is required to assign 1 point to every candidate but his least desirable one.

$S^V \equiv \{S^{V_k} = (S_1^{V_k},\ldots,S_k^{V_k}) \mid S_1^{V_k} \le S_2^{V_k} \le \ldots \le S_k^{V_k}, S_1^{V_k} < S_k^{V_k}, k = 3,\ldots a = |A|\}$. A scoring rule V satisfies the **score-expansion property (SE)** if S^V satisfies the following requirement: For every m and t, such that $3 \le m < t \le a, \{S_1^{V_m},\ldots,S_m^{V_m}\} \subset \{S_1^{V_t},\ldots,S_t^{V_t}\}$. Note that the most common scoring rules, e.g., Borda, plurality, inverse plurality, and, in fact, any "vote for *t* alternatives rule" satisfy SE.

3. The Result

To introduce the alternative parameterization, let us first define the weak requirement of having veto power.

A coalition *M*, $M \subseteq N$, has **veto power** under the social choice rule *V* at *B*, $B \subseteq A$, if and only if there exists some $x \in B$ and some profile *L*, such that $x \notin V(L,B)$ while $\forall i \in N \setminus M$, $xL_i y$ for every *y* in *B*.

That is, a coalition *M* has veto power under *V* at *B*, if all the members outside *M* cannot always guarantee the selection of their most favorable alternative *x*.

Let us denote by $\boldsymbol{p}^a = \{p_3,\ldots,p_a\}$ the (*a*-2)-component vector of the minimal sizes veto coalitions. That is, for any $B, B \subseteq A, |B| = k, k = 3,\ldots a,$ a coalition of voters *M*(*k*) has veto power under *V*, if and only if $|M(k)| \ge p_k$.

By Theorem 2 in Baharad and Nitzan (2002), for a scoring rule parameterized by S^V,

$$p_k = \left(1 - \frac{S_k - S_1}{2S_k - S_1 - \bar{S}(k)}\right)n.$$ [5]

where $\bar{S}(k) = \sum_{i=1}^{k-1} S_i / (k-1)$.

[5] For the sake of simplicity we omit the superscript V_k from S_k and S_1.

The vector $\boldsymbol{p}^a$ enables an alternative parameterization of a scoring rule that satisfies SE. In other words, any such scoring rule is uniquely parameterized by the minimal sizes of the veto coalitions corresponding to the different k's, viz. by $\boldsymbol{p}^a$.

Theorem: A scoring rule *V* that satisfies SE is parameterized by S^V, if and only if it is parameterized by $\boldsymbol{p}^a$.

Proof:

$\Rightarrow$ Suppose that the scoring rule *V* that satisfies SE is parameterized by S^V. Then, by the definition of p_k, the (*a*-2) vectors of scores S^{V_k}, *k*=3,…*a*, uniquely determine the vector $\boldsymbol{p}^a$. That is, *V* is parameterized by $\boldsymbol{p}^a$.

$\Leftarrow$ To prove sufficiency, suppose that two scoring rules V^b and V^c that satisfy SE are parameterized by $\boldsymbol{p}^a$ and denote by S^{V^b} and S^{V^c} the (*a*-2) vectors of scores that parameterize these rules. Suppose that *k*=3 and, with no loss of generality, let $S_1^{V_3^b}$=0, $S_1^{V_3^c}$=0, $S_3^{V_3^b}$=1 and $S_3^{V_3^c}=1$. In this case, since $p_3=\left(1-\frac{S_3-S_1}{2S_3-S_1-\bar{S}(k)}\right)n$, $\bar{S}(k)$ is equal for V^b and V^c . This means that $S_2^{V_3^b}=S_2^{V_3^c}$. In other words, $S^{V_3^b}=S^{V_3^c}$. Suppose now that *k*=4. Since, by assumption, V^b and V^c satisfy the score-expansion property, then:

(1) $\{S_1^{V_3^c},S_2^{V_3^c},S_3^{V_3^c}\}\subset\{S_1^{V_4^c},S_2^{V_4^c},S_3^{V_4^c},S_4^{V_4^c}\}$

and, by definition:

(2) $p_4=\left(1-\frac{S_4-S_1}{2S_4-S_1-\bar{S}(k)}\right)n$

From (1) and (2) we obtain that $S^{V_4^b}=S^{V_4^c}$. Increasing *k* successively, one obtains in a similar way that $S^{V_k^b}=S^{V_k^c}$, for *k*=5,…,*a*. That is, $S^{V^b}=S^{V^c}$. ***Q.E.D.***

Remark 1: To demonstrate that the score-expansion property is essential for the alternative parameterization of a scoring rule, suppose that the scoring rules V^b and

V^c are parameterized, respectively, by $S^{V^b} \equiv \{S^{V_3^b} = (0, 4, 11), S^{V_4^b} = (0, 4, 11, 15)\}$ and $S^{V^c} \equiv \{S^{V_3^c} = (0, 4, 11), S^{V_4^c} = (0, 5, 10, 15)\}$. Clearly, V^b satisfies SE, while V^c does not. The two scoring rules are parameterized by the same $\boldsymbol{p}^a$, the vector of the minimal sizes veto coalitions: $\boldsymbol{p}^a = (0.4n, 0.45n)$. Such a case is impossible when the scoring rules satisfy SE.

Remark 2: Our alternative parameterization by $\boldsymbol{p}^a$ of scoring rules that satisfy the score-expansion property is of different dimensionality to the direct parameterization by S^V. In the former case, (k-2) numbers parameterize the scoring rule. These numbers are the minimal sizes of the coalitions that can veto an alternative which is preferred by everybody outside the coalition. In the latter case, k numbers, the scores, parameterize the scoring rule.

We conclude with an illustration of the proposed parameterization.

Example: In Table 1 we introduce the values of $\boldsymbol{p}^a$, a=6 for five different scoring rules: The plurality rule, the Borda rule, the inverse plurality rule defined by the scores S=(0,1, 1, …,1), , the "vote for the top two candidates" rule defined by the scores S=(0,0, … , 0, 1, 1), and the "quadratic Borda" rule, which is defined by the scores $S=\{0,1,4,9,\ldots,(k-1)^2\}$. Note that p_k was originally defined as the number of members in a coalition. However, for the sake of simplicity, in Table 1 we present p_k as a fraction of n .

Table 1

k The rule	3	4	5	6
Plurality	0.5	0.5	0.5	0.5
Borda	0.43	0.4	0.38	0.38
Inverse Plurality	0.33	0.25	0.2	0.17
Vote for the top two	0.33	0.4	0.43	0.44
The "quadratic Borda"	0.47	0.45	0.44	0.43

Note that, as implied by the theorem, every rule is characterized by a different $\boldsymbol{p}^a$.

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