On The Selection of the Same Winner by All Scoring Rules

By

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Abstract

Different scoring rules can result in the selection of any of the k competing candidates, given the same preference profile, Saari (2001). It is also possible that a candidate, and even a Condorcet winning candidate, cannot be selected by any scoring rule, Saari (2000). These findings are balanced by Saari's result (1992) that specifies the necessary and sufficient condition for the selection of the same candidate by all scoring rules. This condition is, however, indirect. We provide a sufficient condition that is stated directly in terms of the preference profile and therefore its testability does not require the verdict of any voting rule.

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1. Introduction

Even with the same preference profile, different voting methods can generate conflicting election outcomes. This assertion was proved by Fishburn (1981), Nurmi (1987), Saari (1984), (1989), (1992), (2000), (2001), allowing the voting methods to be scoring rules[1] and the election outcomes to be the possible ranking of the alternatives (candidates) or the selected candidate. In fact, it has been shown that there is no limit to the difference between the outcomes of the election. It is also possible that a candidate, even if he is a Condorcet winner, cannot be selected by any scoring rule, Fishburn (1974), (1977). Saari (1992) has balanced the above findings, namely, that sometimes "anything can happen" and sometimes "something and, in particular, the selection of a desirable outcome like a Condorcet winner, cannot happen", by providing the necessary and sufficient condition for the selection of the same candidate by all scoring rules. This condition subjects the preference profile to (k-1) tests (k being the number of candidates): namely, the same candidate must be elected by (k-1) scoring rules: the "vote for one candidate rule", the "vote for two candidate rule",…, the "vote for (k-1) candidates rule". Our objective is to derive a condition that does not involve a large number of tests and does not depend on k and on the verdict of any voting rule. This condition is a sufficient condition. The following section presents our framework. Section 3 contains the new result and an illustration of its usefulness. Section 4 contains a brief summary.

2. The framework

Let A={$\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$…} be a finite set of k alternatives, $k \geq 2$, and let N={1,…,n} be a finite set of individuals, n≥2. Suppose that the preference relation L_i of individual i, $i \in N$, is a strict linear order (complete, transitive and asymmetric relation) over A. The set of these orders is denoted $\mathcal{L}$. A preference profile is an n-tuple L=$(L_1, L_2, ..., L_n)$ of such linear orders. The set of preference profiles is denoted $\mathcal{L}^n$. A

[1] The most well known such rules are the plurality rule and the Borda rule.

social choice rule V is a mapping from $\mathcal{L}^n$ to Λ, the set of non-empty subsets of A. In this study we focus on social choice rules that are usually referred to as scoring rules.

Let $S=\{S_1, S_2,\dots, S_k\}$ be a monotone sequence of real numbers, $S_1 \le S_2 \le \dots \le S_k$ and $S_1<S_k$. Each of the n voters ranks the candidates assigning S_{k+1-i} points to the candidate ranked in the i^{th} position. That is, each voter assigns S_1 points to the one ranked last, S_2 points to the one ranked next to the last, and so on until S_k , the number of points assigned to the candidate ranked first. A **scoring rule** V_S selects the candidates that receive the maximal total score. The set of sequences defining scoring rules is denoted Ω.

3. The robustness conditions

Saari (1992) has established that the necessary and sufficient condition for the invariance of the election outcome to the applied scoring rule is that each of the (k-1) "vote for t candidates rules", $t=\{1,\dots,k-1\}$ selects the same candidate. Our objective is to state a condition directly in terms of the preference profile that does not involve the verdict of any voting rule.

An α-majority is a group of size αn, where α is a fraction with a denominator n and $\frac{1}{2}<\alpha\le 1$. An α-majority *supports* a candidate ***a*** whenever this candidate is its unanimous most preferred candidate. In such a case the candidate is called a **clear Condorcet winner**. The corresponding (1-α)-minority *supports* candidate ***a*** in some degree q, $q=(k+1-t)$, $2 \le t \le k$, if for all members of the minority group the ranking of candidate ***a*** is smaller than or equal to t. Notice that $q=1,\dots,k-1$. The maximal minority support means that $q=k-1$, in which case candidate ***a*** is ranked second, $t=2$, by every member of the (1-α)-minority. By definition, support in degree q implies support in degree s, $1\le s<q$.

An α-majority *opposes* g candidates, $g=0, \dots,k-2$, if its members are unanimous regarding the set of the g least preferred candidates. That is, there exists a set A^g consisting of g candidates, $A^g \subset A$, such that the ranking of every candidate in this set according to any member of the α-majority group is larger than (k-g).

When an α-majority supports candidate ***a*** and opposes g candidates, *balancedness* within the α-majority group regarding the ranking of the remaining

(k-1-g) candidates requires equalization of the average ranking of those (k-1-g) candidates, denoted r_α^{k-1-g}. If $\alpha n=h(k-1-g)$, for some integer h, then $r_\alpha^{k-1-g}=(2+k-g)/2$. This average ranking is achieved when the reported preferences of the α-majority are cyclical over the (k-1-g) candidates that are ranked as second, third, and so on up to the $(k\text{-}g)^{\text{th}}$ position.[2] Let $\boldsymbol{L}^{\boldsymbol{CR}}=\{$L=($L_1, \ldots, L_n$) $\in \mathcal{L}^n$: $\exists\, \boldsymbol{a} \in A$, such that an α-majority, α=1, supports $\boldsymbol{a}$, *or*

[(i) an α-majority supports $\boldsymbol{a}$, $\alpha<1$, αn = $h(k\text{-}q)$, where h is an integer, $h\geq 1$ and $q=(k+1-t)$ is the degree of support of candidate $\boldsymbol{a}$ by the (1- α)-minority.

(ii) the α-majority opposes (q-1) candidates.

(iii) $r_\alpha^{k-q}=\frac{(2+k-(q-1))}{2}=\frac{(3+k-q)}{2}$

(iv) $\alpha \geq \frac{(k-q)}{(k-q+1)}=\frac{(t-1)}{t}$] $\}$

Condition (i) requires uniformity within a majority group regarding the most preferred candidate, namely, the existence of a clear Condorcet winner; Condition (ii) requires uniformity within the majority regarding the (q-1) least preferred candidates, given the degree of support q the majority consensus candidate receives from the minority members; Condition (iii) means balancedness within the majority group regarding the ranking of the (k-q) remaining candidates. The requirement that αn = $h(k\text{-}q)$, where h is an integer, is needed in order to enable the existence of such complete balancedness. Finally, condition (iv) requires that the existing majority is equal to or larger than $(t-1)/t$. Clearly, the set $\boldsymbol{L}^{\boldsymbol{CR}}$ is not empty and it can be partitioned into k subsets of preference profiles, $\boldsymbol{L}^{\boldsymbol{CR}}(\boldsymbol{a})$, $\boldsymbol{a}\in A$, where majority support is given to candidate $\boldsymbol{a}$. The overall support from the majority and the minority ensuring the selection of a clear Condorcet winner under any scoring rule is specified in the following result.

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[2] In this case each of these (k-1-g) candidates is ranked h times in the (k-1-g) possible positions 2,...,k-g. Hence $r_\alpha^{k-1-g}=\frac{h(2+3+\ldots+k-g)}{h(k-1-g)}=\frac{h(k-1-g)(2+k-g)}{h(k-1-g)2}=\frac{(2+k-g)}{2}$.

Theorem 1: $\forall \boldsymbol{a} \in A$, [$L \in \boldsymbol{L}^{CR}(\boldsymbol{a}) \Rightarrow \forall S \in \Omega$, $\boldsymbol{a} \in V_S(L)$].

Proof: Consider a profile L in $\boldsymbol{L}^{CR}(\boldsymbol{a})$. If candidate $\boldsymbol{a}$ is supported unanimously, $\alpha = 1$, then, clearly, $\forall S \in \Omega$, $\boldsymbol{a} \in V_S(L)$**.** If candidate $\boldsymbol{a}$ is supported by an α-majority, $\alpha<1$, then, by (iv), we can let $\alpha = \frac{(k-q)}{(k-q+1)}$. By assumption, the $(1-\alpha)-$minority supports candidate $\boldsymbol{a}$ in degree $q=(k+1-t)$. By (i), the total score received by candidate $\boldsymbol{a}$, $S(\boldsymbol{a})$, satisfies:

(1) $$S(\boldsymbol{a}) \geq \frac{(k-q)}{(k-q+1)} n\ S_k + \frac{1}{(k-q+1)} n\ S_q = \text{A}$$

By (ii) and (iii), $\forall S \in \Omega$, the average score $\bar{S}$ assigned by the α-majority to any candidate other than $\boldsymbol{a}$ is smaller than or equal to

(2) $$\bar{S} = \sum_{i=q}^{k-1} S_i /(k-q)$$

Hence, the total score $S(\boldsymbol{b})$ received by any candidate $\boldsymbol{b}$, $\boldsymbol{b} \neq \boldsymbol{a}$, satisfies:

(3) $$S(\boldsymbol{b}) \leq \frac{(k-q)}{(k-q+1)} n\ \bar{S} + \frac{1}{(k-q+1)} n\ S_k = \text{B}$$

We establish that $S(\boldsymbol{a}) \geq S(\boldsymbol{b})$ by proving that A≥B (see (1) and (3)), namely, that:

(4) $$(k-q)S_k - S_k \geq (k-q)\ \bar{S} - S_q$$

By (2), inequality (4) becomes:

(5) $$(k-q)S_k - S_k \geq (k-q) \sum_{i=q}^{k-1} S_i /(k-q) - S_q = \sum_{i=q+1}^{k-1} S_i$$

or,

(6) $$(k-q-1)S_k \geq \sum_{i=q+1}^{k-1} S_i$$

Since $S_k \geq S_i$, (6) is satisfied. The same inequality certainly holds when $\alpha > \frac{k-q}{k-q+1}$, which completes the proof. ■

The following example illustrates the conditions in Theorem 1. Let $A=\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{e},\boldsymbol{f}\}$ (that is k=6) and n=30. First, let α=0.8. This requires that 24 voters support the same candidate, say ***a*** (condition (i)). Now let q=3. This requires that all the 0.8 majority members oppose the same 2 = q-1 candidates, say candidates ***e*** and ***f*** (condition (ii)). Notice that α=0.8>0.75 = $\frac{6-3}{6-3+1}$ (condition (iv)). To satisfy condition (iii), $r_{\alpha}^{k-q} = r_{0.8}^{6-3} = r_{0.8}^{3} = \frac{(3+k-q)}{2} = 3$, the 0.8 majority members must rank the remaining candidates ***b***,***c***,***d*** in a balanced way. Finally, the second requirement in condition (i) is that $q=(k+1-t)=3$ is the degree of support of candidate ***a*** by the (1- α)-minority. It can be easily verified that the following two profiles (profile 1 and profile 2) simultaneously satisfy these requirements:

Profile 1:

8 voters: ***a>b>c>d>e>f*** , 8 voters: ***a>c>d>b>f>e*** , 8 voters: ***a>d>b>c>e>f***

3 voters: ***c>d>a>b>f>e*** , 3 voters: ***d>b>a>c>f>e***

Profile 2:

4 voters: ***a>b>c>d>e>f*** , 4 voters: ***a>b>d>c>e>f***

4 voters: ***a>c>d>b>f>e*** , 4 voters: ***a>c>b>d>f>e***

4 voters: ***a>d>b>c>e>f*** , 4 voters: ***a>d>c>b>e>f***

3 voters: ***c>d>a>b>e>f*** , 3 voters: ***d>b>a>c>f>e***

Remark: When a clear Condorcet winner exists, invariance of its selection with respect to all scoring rules allows tradeoff between the degree of support by the minority group, q, and the extent of support by the majority group, α. This tradeoff is directly obtained from (iv). Specifically, $\frac{d\alpha}{dq} = \frac{-1}{(k-q+1)^2} = -\frac{1}{t^2}$.

4. Summary

In this study we derived a new result that specifies sufficient conditions on a preference profile for the robustness of a voting outcome ***a*** with respect to the applied scoring rule. These conditions are stated in terms of the majority support in candidate ***a***, the majority opposition to some other candidates and the majority preferences with respect to the remaining candidates. Specifically, this condition consists of three requirements: uniformity within the majority regarding the $(q-1)$ least preferred candidates, given the degree of support q the majority consensus candidate receives from the minority members; balancedness within the majority group regarding the ranking of the $(k-q)$ candidates that are neither uniformly supported nor uniformly opposed; and a sufficiently large majority support that must be equal to or larger than $(k-q)/(k-q+1)$. Our three conditions are sufficient and they are stated in terms of the preference profile. They do not involve the verdict of $(k-1)$ scoring rules as required by Saari's (1992) necessary and sufficient conditions for the robustness of the election outcome.

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