Theory Dec. (2011) 71:151–162
DOI 10.1007/s11238-009-9157-z

Condorcet vs. Borda in light of a dual majoritarian approach

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Published online: 1 July 2009

Abstract Many voting rules and, in particular, the plurality rule and Condorcet-consistent voting rules satisfy the simple-majority decisiveness property. The problem implied by such decisiveness, namely, the universal disregard of the preferences of the minority, can be ameliorated by applying unbiased scoring rules such as the classical Borda rule, but such amelioration has a price; it implies erosion in the implementation of the widely accepted "majority principle". Furthermore, the problems of majority decisiveness and of the erosion in the majority principle are not necessarily severe when one takes into account the likelihood of their occurrence. This paper focuses on the evaluation of the severity of the two problems, comparing simple-majoritarian voting rules that allow the decisiveness of the smallest majority larger than 1/2 and the classical Borda method of counts. Our analysis culminates in the derivation of the conditions that determine, in terms of the number of alternatives k, the number of voters n, and the relative (subjective) weight assigned to the severity of the two problems, which of these rules is superior in light of the dual majoritarian approach.

Keywords Majority decisiveness · Condorcet criterion · Erosion of majority principle · The Borda method of counts

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1 Introduction

There are two main classes of studies dealing with the celebrated Condorcet principle that suggests the selection of an alternative that is preferred to any other alternative by a majority of the voters. The first focuses on the merit of this simple, easy to implement, appealing, and hardly disputable principle that assigns a majority, any majority, the power to decide what the chosen alternative is. The second class of studies focuses on the drawbacks of this notable principle. This latter literature is especially concerned about the non-existence of a Condorcet winner in a relatively large proportion of preference profiles, and, in particular, in those cases that are referred to as cases of "the voting paradox". It is well known that this proportion approaches one for a sufficiently large number of voters and alternatives (see Fishburn (1973)). In light of this finding, "Condorcet-consistent rules" have been proposed as a remedy to the problem of implementing the Condorcet criterion. Under such rules, a Condorcet winner is chosen whenever it exists. A weaker, more compelling demand is that the voting rules should pick the simple-majority consensus, the candidate ranked first by a majority, when one exists. In other words, a minimal requirement for good majoritarian rules is that they satisfy the simple-majority decisiveness property. In this article, this requirement is referred to as "the majority principle". Yet another requirement inspired by the Condorcet principle is that a voting rule should not select a candidate who is a Condorcet loser, a candidate who is defeated by all other candidates in pair-wise comparisons based on a simple majority. The most well-known example of a voting rule, that does not satisfy this latter property as well as the Condorcet-consistency property but does allow the decisiveness of a simple majority, is the widely used plurality rule. The most well-known example of a strictly monotone positional rule that does not satisfy the Condorcet-consistency property[1] as well as the weaker property of allowing simple-majority decisiveness, yet satisfies the third requirement of avoiding the selection of a Condorcet loser is the Borda rule, Brams and Fishburn (2002); Mueller (2003); Nurmi (2002); Saari (2001).

The purpose of this study is to contribute to the debate between the supporters of majoritarian and positionalist election rules by providing a more balanced assessment of voting rules that allow simple-majority decisiveness, such as the plurality rule and Condorcet-consistent rules that abide by the majority principle. Our analysis focuses on one of the drawbacks of these rules, namely, their disregard of the minority preferences. Baharad and Nitzan (2002) have studied this drawback clarifying how it can be ameliorated by applying unbiased scoring rules. The proposed resolution of the problem associated with majority decisiveness is, nevertheless, incomplete as argued in Baharad and Nitzan (2007). First, it does not take into account the negative aspect that accompanies the amelioration of majority decisiveness, namely, the erosion in the majority principle. Such erosion occurs whenever the alternative that is most preferred by the majority cannot be enforced (selected), regardless of the minority reported preferences. Second, no attempt was made to quantitatively estimate the

[1] In Baharad and Nitzan (2003) we have shown that the Borda rule satisfies the generalized Condorcet consistency property, if a Condorcet winner is defined as an alternative that defeats every other alternative in pair-wise special-majority comparisons and the required special majority is equal to $(k-1)/k$.

severity of the majority decisiveness problem, namely, to measure the vulnerability of some common voting rules to this problem,[2] on the one hand, and to the erosion of the majority principle, on the other hand. A plausible measure of the severity of the first problem is the proportion of preference profiles under which there exists an alternative that is unanimously preferred by (at least) a decisive majority of the voters. The rationale behind this measure is that such profiles enable the realization of majority decisiveness. The proposed measure of the second problem is the proportion of all preference profiles in which a majority of the voters shares a common view regarding the most preferred alternative, which differs from the minority's consensus, and the majority is indecisive. That is, the indecisive majority cannot enforce the selection of its most favorable alternative, independent of the preferences reported by the minority. In this study, these measures that are proposed in Baharad and Nitzan (2007), are used to compare the Borda method of counts and majoritarian rules that satisfy the simple-majority decisiveness requirement under alternative decision-making settings represented by different combinations of the number of voters n and the number of alternatives k.

In the first part of this article, we present the severity measures in the context of majoritarian rules that allow simple-majority decisiveness (henceforth, simple-majoritarian rules) and the Borda rule. As already noted, mitigating majority decisiveness results in some vulnerability to erosion in the majority status, a phenomenon that can be considered as inseparable from the amelioration of majority decisiveness. In order to meaningfully compare a majoritarian rule that allows decisiveness of a simple majority to some positionalist election rule, the severity of the two problems (decisiveness and erosion) must be taken into account. Clearly, different weights or valuations can be assigned to these problems. Our analysis culminates in the derivation of the condition that determines, in terms of the number of alternatives k, the number of voters n, and the relative (subjective) weight assigned to the severity of the two problems, whether a majoritarian rule that allows simple-majority decisiveness is superior, equivalent, or inferior to the Borda method of counts according to the applied criterion.

In the next section, we measure the effectivity of majority decisiveness for any given combination of n and k under simple-majoritarian rules, and under the Borda method of counts. The erosion in the majority principle for the Borda rule is examined in Sect. 3. We then present in Sect. 4, the comparison between any simple-majoritarian rule and the Borda method of counts. The outcome of this comparison depends on k, n, and the subjective weight β assigned to the severity of the two problems. Section 5 contains brief concluding remarks.

2 Severity of simple-majority decisiveness

Let $N = \{1, \ldots, n\}, n \geq 3$, denote a finite set of voters; A, a finite set consisting of k distinct alternatives, $k \geq 3$; and $L(A)$, the set of linear orderings (complete,

[2] Notice that a Condorcet-consistent voting rule does not allow the classical decisiveness of a particular permanent majority group because, by definition, it does not allow factionalism. That is, under such a rule the incidence of the majority power is not restricted to a specific group.

transitive, and asymmetric relations) over A. Voter i's preferences, $i \in N$, are denoted by $u^i \in L(A)$. A preference profile is an n-tuple $u = \{u_i\}_{i \in N} \in L(A)^n$, the set of linear profiles on A. A voting rule, V, is a mapping from $L(A)^n$ to the set of non-empty subsets of A. $V(u)$ specifies the alternatives selected by the voters at the preference profile u.

The notion of α-majority decisiveness focuses on situations where more than αn, $(1/2 < \alpha < 1)$, of the voters can enforce the selection of their most preferred alternative. α-majority decisiveness implies that in **all** such situations, the $(1-\alpha)$-minority preferences have no effect on the outcome of the voting.

For the sake of simplicity, throughout this study, we adopt the commonly made impartial culture assumption, namely, all preference profiles are assumed to be equally likely.[3] The applied measure for the severity of α-majority decisiveness is the likelihood that majority decisiveness is realized.[4] Under the impartial culture assumption, it is equal to the proportion of preference profiles under which there exists an alternative that is unanimously preferred by more than αn of the voters. The severity of simple-majority decisiveness characterizes the decisiveness of voting rules that satisfy the simple-majority decisiveness property. These rules, the simple-majoritarian rules, include all Condorcet-consistent voting rules, such as the Copeland method, and the widely used plurality rule which is not Condorcet-consistent.

In a preference profile that allows the realization of simple-majority decisiveness, the majority consensus is a Condorcet winner. The unanimous consent among the majority members regarding the most-preferred alternative clearly implies the existence of a Condorcet winner. However, the reverse implication does not hold. That is, an alternative can be a Condorcet winner without being the most-preferred alternative of the majority. Hence, the proportion of preference profiles with a Condorcet winner is clearly larger than the proportion of profiles that allow the realization of simple-majority decisiveness.[5]

2.1 Severity measure of majority decisiveness (SMD)

Given a voting rule, V, a coalition of voters, T, is **decisive**, if at every preference profile, $u^{a(T)}$, where alternative a is the T-majority consensus, T has a message in $L(A)^t$, $t = |T|$, that ensures the selection of a.

Let us first consider the severity of simple majority decisiveness, where the decisive majority is any majority the size of which is larger than 1/2. The probability that a specific alternative, say alternative "a," is preferred by *exactly* $\left(\frac{n+1}{2}\right)$ of the voters (the other $\left(\frac{n-1}{2}\right)$ of the voters choose another alternative from the remaining $k-1$ alternatives, but not a) is:

$$\left(\frac{1}{k}\right)^{\frac{n+1}{2}} \cdot \left(\frac{k-1}{k}\right)^{\frac{n-1}{2}} \tag{1}$$

[3] See, e.g., Gehrlein (1983); Gehrlein and Fishburn (1976); Gehrlein and Lepelley (1998); Lepelley and Merlin (2001); Merlin et al. (2002); Merlin (1984); Nurmi and Uusi-Heikkila (1986).

[4] See Baharad and Nitzan (2007).

[5] Fishburn (1973) presents the larger values of the probability of having a Condorcet winner.

However, there are many combinations of $\left(\frac{n+1}{2}\right)$ voters that can choose alternative a. The number of such combinations is $\binom{n}{(n+1)/2}$.

Thus, the probability that alternative a is preferred by $\left(\frac{n+1}{2}\right)$ of the voters is equal to

$$\left(\frac{1}{k}\right)^{\frac{n+1}{2}} \cdot \left(\frac{k-1}{k}\right)^{\frac{n-1}{2}} \cdot \binom{n}{(n+1)/2} \tag{2}$$

The probability that alternative a is preferred by $\left(\frac{n+3}{2}\right)$ of the voters is:

$$\left(\frac{1}{k}\right)^{\frac{n+3}{2}} \cdot \left(\frac{k-1}{k}\right)^{\frac{n-3}{2}} \cdot \binom{n}{(n+3)/2} \tag{3}$$

In a similar manner, we can take into account any other majority group the size of which is larger than $n/2$, that is, the size of which is equal to $\left(\frac{n+5}{2}\right)$, $\left(\frac{n+7}{2}\right)$, ..., $\left(\frac{n+n}{2}\right)$. Hence, the probability that alternative a is preferred by more than half of the voters is given by

$$\sum_{i=1,3,\ldots,n} \left(\frac{1}{k}\right)^{\frac{n+i}{2}} \cdot \left(\frac{k-1}{k}\right)^{\frac{n-i}{2}} \cdot \binom{n}{(n+i)/2} \tag{4}$$

In order to obtain the general expression for the probability of having a decisive majority (recall that under simple majority rules, this majority is any majority that exceeds 1/2) that unanimously chooses the same alternative, we should take into account unanimity with respect to all the alternatives, and not just for a. This means that (4) has to be multiplied by k. To sum up, under the impartial culture assumption, the probability of a profile that enables the realization of majority decisiveness, i.e., the probability of having a majority that unanimously chooses the same alternative is given by SMD:

$$\text{SMD} = k \cdot \sum_{i=1,3,\ldots,n} \left(\frac{1}{k}\right)^{\frac{n+i}{2}} \cdot \left(\frac{k-1}{k}\right)^{\frac{n-i}{2}} \cdot \binom{n}{(n+i)/2} \tag{5}$$

2.2 Severity of majority decisiveness under the Borda rule

In an unbiased setting, where the voting rule is anonymous and neutral,[6] majority decisiveness can be eliminated by applying a scoring rule and, in particular, by using

6 Unbiasedness toward voters (anonymity) requires invariance of the voting rule with respect to permutations of voters' preferences; if the preference relations of the voters are permuted, then the outcome of the voting rule is not affected. Unbiasedness toward alternatives requires appropriate variance of the voting rule with respect to permutations of the alternatives in A; if the alternatives are permuted in the preferences of the voters on A, then the alternative/s selected by the voting rule change accordingly.

the Borda rule. This rule occupies a special place among all the positional scoring rules since it is less susceptible than all other rules to many unsettling possibilities and anomalies, Brams and Fishburn (2002); Nurmi (1999), Saari (1995) and Saari (2001). Under the Borda rule, a simple majority of 1/2 is not decisive; however, as shown by Baharad and Nitzan (2002), a special majority is decisive.[7] We measure the severity of the special-majority decisiveness under the Borda rule, $\mathrm{SMD^B}$, in a similar manner to SMD, (see Eq. 6):

Under the Borda rule, the size of the decisive majority is equal to $DM^B = \left(\frac{2k-2}{3k-2}\right) \cdot n$. In such a case, we have to sum up the probabilities of having a majority the size of which is equal to or larger than DM^B. In this summation, we ignore non-integers, so DM^B is rounded-up. In an analogous derivation to the derivation of (5), we get that the severity of majority decisiveness under the Borda rule is equal to:[8]

$$\mathrm{SMD^B} = k \cdot \sum_{i=0}^{n-DM^B} \left(\frac{1}{k}\right)^{DM^B+i} \cdot \left(\frac{k-1}{k}\right)^{n-(DM^B+i)} \cdot \binom{n}{DM^B+i} \qquad (6)$$

The difference between the two measures is that in the case of SMD, we take into account every majority group the size of which is equal to $\frac{n+i}{2}$, $i = 1, 3, \ldots, n$, whereas in the case of $\mathrm{SMD^B}$, we only consider majority groups the size of which exceeds the size of the decisive majority under the Borda rule.

3 Erosion in the majority principle and its severity

In any preference profile that gives rise to erosion in the majority principle, the majority members share a common view regarding the most-preferred alternative, which differs from the minority's consensus, and the majority is indecisive. That is, it cannot enforce the selection of its most-favorable alternative, independent of the preferences reported by the minority. Clearly, the erosion problem does not exist under a majoritarian rule that allows decisiveness of a simple majority, since under such a rule the outcome of the voting always coincides with the majority (any majority) consensus. For such a rule, there is just one problem the severity of which is represented by SMD. In contrast, for the Borda method of counts, we consider both $\mathrm{SMD^B}$ and the severity of the erosion problem, $\mathrm{EMP^B}$, presented below. The superiority of the Borda rule in terms of immunity to majority decisiveness is clouded by its vulnerability to the problem of erosion in the majority principle. Such erosion means that the majority is unable to enforce the selection of its most favorable alternative, irrespective of the preferences reported by the minority. The problem of erosion in the majority principle can be considered as a problem that accompanies any amelioration of the simple-majority decisiveness problem. However, as in the case of simple-majority decisiveness, the existence of erosion in the majority principle does not imply that the problem is

7 This special majority is equal to $(2k-2)/(3k-2)$. Hence, for a sufficiently large k, this majority is a 2/3 majority.

8 SMD and $\mathrm{SMD^B}$ can be directly obtained by applying Eq. 1 in Baharad and Nitzan (2007).

severe. Furthermore, while the severity of one of the problems can be significant, that of the other one can be negligible. The extent of the problem of simple-majority decisiveness has been evaluated in the previous section. In order to compare the severity of the decisiveness problem under rules that allow simple-majority decisiveness and under the Borda rule, and then to make the more meaningful comparison where the severity of the two problems is taken into account, we have computed the proportion of preference profiles that give rise to erosion in the majority principle under the Borda rule. The formal derivation is as follows:

Letting i denote the size of non-decisive majorities, then under the Borda rule, i ranges from $\lfloor 0.5n + 1 \rfloor$ to $DM^B - 1$, where $\lfloor t \rfloor$ is the largest integer that is equal to or smaller than t; $\left(\frac{1}{k}\right)^i$ and $\left(\frac{1}{k}\right)^{n-i}$ are, respectively, the probabilities that a majority of size i and a minority of size $n - i$, choose, unanimously, two different alternatives. The number of possible partitions of the voters to groups of size i and $n - i$ is $\binom{n}{i}$. The number of pairs of different alternatives unanimously chosen by the majority and minority groups is $\left(\frac{k!}{(k-2)!}\right)$. Hence, the probability of erosion in the majority principle is:[9]

$$EMP^B = \sum_{i=\lfloor 0.5n+1 \rfloor}^{DM^B-1} \left(\frac{1}{k}\right)^i \cdot \left(\frac{1}{k}\right)^{n-i} \cdot \binom{n}{i} \cdot \left(\frac{k!}{(k-2)!}\right) \tag{7}$$

Notice that when DM^B is equal to $\lfloor 0.5n + 1 \rfloor$, the cost associated with erosion in the majority status is 0. In such a case, no terms are summed up in (7).

4 Borda vs. Condorcet: a comparison

4.1 The critical weight β

An analytical comparison between a simple-majoritarian rule and the Borda method of counts hinges on the weights assigned to the two problems. Different subjective weights may alter the comparison results. For given n and k, the following theorem identifies the critical weight β^* assigned to the problem of erosion in the majority principle, such that $SMD^B + \beta \cdot EMP^B = SMD$, namely, such that a simple-majoritarian rule and the Borda rule are equivalent when their vulnerability to the two problems is taken into account. A weight larger (smaller) than β^* clearly results in the superiority (inferiority) of any simple-majoritarian rule over (relative to) the Borda rule.

[9] EMP^B can be directly obtained by applying Eq. 2 in Baharad and Nitzan (2007).

Theorem $SMD^B + \beta \cdot EMP^B = SMD$ *when* $\beta = \frac{\sum_{\lfloor i=0.5n+1 \rfloor}^{DM^B-1} (k-1)^{n-i} \binom{n}{i}}{(k-1) \sum_{\lfloor i=0.5n+1 \rfloor}^{DM^B-1} \binom{n}{i}} = \beta^*$

Proof Recall that

$$\text{SMD} = k \cdot \sum_{i=1,3,\ldots,n} \left(\frac{1}{k}\right)^{\frac{n+i}{2}} \cdot \left(\frac{k-1}{k}\right)^{\frac{n-i}{2}} \cdot \binom{n}{(n+i)/2},$$

$$\text{SMD}^B = k \cdot \sum_{i=0}^{n-DM^B} \left(\frac{1}{k}\right)^{DM^B+i} \cdot \left(\frac{k-1}{k}\right)^{n-(DM^B+i)} \cdot \binom{n}{DM^B+i} \text{ and}$$

$$\text{EMP}^\text{B} = \sum_{i=\lfloor 0.5n+1 \rfloor}^{DM^B-1} \left(\frac{1}{k}\right)^{i} \cdot \left(\frac{1}{k}\right)^{n-i} \cdot \binom{n}{i} \cdot \left(\frac{k!}{(k-2)!}\right).$$

Notice that SMD can be decomposed as follows:

$$\text{SMD} = k \cdot \sum_{\lfloor i=0.5n+1 \rfloor} \left(\frac{1}{k}\right)^{i} \cdot \left(\frac{k-1}{k}\right)^{n-i} \cdot \binom{n}{i} = \text{SMD1} + \text{SMD2}, \text{ where}$$

$$\text{SMD1} = k \cdot \sum_{\lfloor i=0.5n+1 \rfloor}^{DM^B-1} \left(\frac{1}{k}\right)^{i} \cdot \left(\frac{k-1}{k}\right)^{n-i} \cdot \binom{n}{i} \text{ and}$$

$$\text{SMD2} = k \cdot \sum_{i=DM^B}^{n} \left(\frac{1}{k}\right)^{i} \cdot \left(\frac{k-1}{k}\right)^{n-i} \cdot \binom{n}{i}.$$

Since $\text{SMD}^\text{B} = \text{SMD2}$, the equality $SMD^B + \beta \cdot EMP^B = SMD$ requires that β satisfies $\text{SMD1} = \beta \cdot \text{EMP}^\text{B}$, which implies that

$$k \cdot \sum_{\lfloor i=0.5n+1 \rfloor}^{DM^B-1} \left(\frac{1}{k}\right)^{i} \cdot \left(\frac{k-1}{k}\right)^{n-i} \cdot \binom{n}{i}$$
$$= \beta \cdot k \cdot \sum_{\lfloor i=0.5n+1 \rfloor}^{DM^B-1} \left(\frac{1}{k}\right)^{i} \cdot \left(\frac{1}{k}\right)^{n-i} \cdot \binom{n}{i} \cdot \left(\frac{k-1}{k-2}\right).$$

From the above equality, we obtain that

$$\sum_{\lfloor i=0.5n+1 \rfloor}^{DM^B-1} \frac{(k-1)^{n-i}}{k^n} \cdot \binom{n}{i} = \beta \cdot \sum_{\lfloor i=0.5n+1 \rfloor}^{DM^B-1} \left(\frac{1}{k^n}\right) \cdot \binom{n}{i} \cdot \left(\frac{k-1}{k-2}\right).$$

Table 1 The break-even values β^* that equate SMD and $(\mathrm{SMD}^{\mathrm{B}} + \beta \cdot \mathrm{EMP}^{\mathrm{B}})$

k	n				
	3	5	7	9	11
3	=	=	=	8	16
4	=	=	9	27	81
5	=	4	16	64	256
6	=	5	25	125	625
7	=	6	36	216	>1000

= means that the simple-majoritarian rule and the Borda method of counts are equivalent

Hence,

$$\beta = \frac{\sum_{\lfloor i=0.5n+1 \rfloor}^{DM^B-1} (k-1)^{n-i} \binom{n}{i}}{(k-1) \sum_{\lfloor i=0.5n+1 \rfloor}^{DM^B-1} \binom{n}{i}} = \beta^*.$$

□

Table 1 presents the critical weight, β^*, that has to be assigned to $\mathrm{EMP}^{\mathrm{B}}$, such that the severity of the problem of simple-majority decisiveness is equal to the weighted sum $\mathrm{SMD}^{\mathrm{B}} + \beta \cdot \mathrm{EMP}^{\mathrm{B}}$, the weighted sum of the measures of severity of the special majority decisiveness problem, and the problem of erosion in the majority principle corresponding to the Borda rule.

The critical values of the weight β enable a straightforward comparison between the Borda rule and a majoritarian rule that allows simple-majority decisiveness. For example, when $k = 5$ and $n = 5$, a subjective weight $\beta = 4$, the weight assigned to $\mathrm{EMP}^{\mathrm{B}}$ is four times larger than the weight assigned to SMD, yields an equality between SMD and $\mathrm{SMD}^{\mathrm{B}} + \beta \cdot \mathrm{EMP}^{\mathrm{B}}$. A larger β results in the superiority of the majoritarian rule relative to the Borda rule (that is, under such a weight, $\mathrm{SMD} < \mathrm{SMD}^{\mathrm{B}} + \beta \cdot \mathrm{EMP}^{\mathrm{B}}$), whereas a smaller β, ($\beta < 4$), yields the opposite result.

By Table 1, when $n = 3$, independent of k, any simple- majoritarian rule is equivalent to the Borda rule. This is due to the fact that in such a case the Borda rule is also majoritarian in the sense that it allows a simple-majority (two voters) decisiveness. For the same reason, the two rules are equivalent in terms of the proposed criterion when $k \leq 4$, $n = 5$ and when $k = 3$, $n = 7$. For other values of relatively small k and n, the critical values of the weight assigned to $\mathrm{EMP}^{\mathrm{B}}$ monotonically increase with k and with n. Note, however, that for $k \geq 6$ and $n \geq 9$, the values of the critical weight β^* exceed 100. The actual weight in these cases is likely to be smaller than β^*, which implies that the Borda rule is superior to any majoritarian rule in such voting situations. In other words, in typical voting settings, even the assignment of the slightest weight to the problem of simple-majority decisiveness, namely, to the disregard of the minority inability to effectively express its preferences, justifies the use of the classical Borda method of counts.

Table 2 SMD vs. $SMD^B + EMP^B$

n:	3		5		7		9		11	
K	$SMD^B + EMP^B$	SMD	$SMD^B + EMP^B$	SMD	$SMD^B + EMP^B$	SMD	$SMD^B + EMP^B$	SMD	$SMD^B + EMP^B$	SMD
3	0.778	0.778	0.630	0.630	0.520	0.520	0.166	0.435	0.132	0.366
4	0.625	0.625	0.414	0.414	0.077	0.282	0.046	0.196	0.032	0.137
5	0.520	0.520	0.098	0.290	0.032	0.167	0.017	0.098	0.010	0.058
6	0.444	0.444	0.059	0.213	0.016	0.106	0.007	0.054	0.004	0.028
7	0.388	0.388	0.038	0.163	0.009	0.071	0.004	0.032	0.002	0.014

4.2 The symmetric case ($\beta = 1$)

It can be easily verified that for any n and k, $\sum_{\lfloor i=0.5n+1 \rfloor}^{DM^B-1} (k-1)^{n-i} \binom{n}{i} \geq (k-1) \sum_{\lfloor i=0.5n+1 \rfloor}^{DM^B-1} \binom{n}{i}$. This implies that in the symmetric case where the two types of problems are equally weighted, β is smaller than or equal to β^*. Therefore, by the theorem, in the symmetric case any simple-majoritarian rule is inferior or equivalent to the Borda rule.

Assuming that equal weights are assigned to EMP and SMD ($\beta = 1$), Table 2 presents the comparison between SMD and the sum of SMD^{B} and EMP^{B} for different pairs of the parameters, n and k.

The table reveals the much stronger severity of the simple-majority decisiveness problem relative to the severity of the problem of erosion in the majority principle. Consequently, even though the erosion problem is taken into account, as expected, the sum of SMD^{B} and EMP^{B} is still smaller than or equal to SMD, for any k and n.

5 Concluding remarks

Majoritarian voting rules are vulnerable to the problem of simple-majority decisiveness. That is, under such rules, more than 50% of the voters can enforce the selection of their most preferred alternative, regardless of the reported preferences of the minority. Positionalist voting rules are not subject to the problem of simple-majority decisiveness, however, they are susceptible to the lesser problem of a special-majority decisiveness and, consequently, to some erosion in the implementation of the majority principle. The celebrated Borda method of counts is vulnerable to the decisiveness of a $\frac{2k-2}{3k-2}$ majority. The existence of the two problems does not imply that they are severe. In other words, the likelihood of occurrence of realization of majority decisiveness or of the erosion in the majority principle can be very small. In this study, we apply a measure of severity of the two problems proposed in Baharad and Nitzan (2007), namely, the proportion of preference profiles under which the problems occur. The weights assigned to each of the problems need not be equal. Our main result provides the condition that determines whether a simple-majoritarian rule is superior, equivalent, or inferior to the Borda method of counts. It implies that in a symmetric case where the two types of problems are equally weighted, any simple-majoritarian rule is inferior or equivalent to the Borda rule. Our analysis also conveys the following clear message: In typical voting settings where n is sufficiently large, even the slightest awareness to the problem implied by the decisiveness of a simple majority justifies the use of the Borda rule relative to any simple-majoritarian rule.[10]

[10] For over two hundred years, the debate between the proponents of majoritarian and positionalist election rules centered on the comparison between the plurality and the Borda rules, the two most widely used scoring rules, Brams and Fishburn (2002); Nurmi (1999, 2002); Saari (1990, 2001). Our analysis that sheds new light on this debate naturally raises the question which scoring rule is optimal according to the applied criterion. The study of this question is undertaken in Baharad and Nitzan (2007). It reveals that, in "small size" voting bodies, the optimal scoring rule can be the plurality rule or the Borda rule. However, in

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Footnote 10 continued
voting contexts where the number of voters is typically considerably larger than the number of candidates, neither of these voting rules need be optimal.