Vector Measure Games Based on Measures with Values in an Infinite Dimensional Vector Space*

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Abstract

The definition of vector measure game is generalized in this paper to include all cooperative games of the form $f \circ \mu$, where μ is a nonatomic vector measure of bounded variation that takes values in a Banach space. It is shown that if f is weakly continuously differentiable on the closed convex hull of the range of μ then the vector measure game $f \circ \mu$ is in pNA_∞ and its value is given by the diagonal formula. Moreover, every game in pNA_∞ has a representation that satisfies this condition. These results yield a characterization of pNA_∞ as the set of all differentiable games whose derivative satisfies a certain continuity condition. *Journal of Economic Literature* Classification Numbers: C71, D46, D51.

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Introduction

A list of axioms, adapted from those which uniquely characterize the Shapley value for finite-player cooperative games, determines a unique value on certain classes of nonatomic cooperative games—games involving an infinite number of players, each of which is individually insignificant (Aumann and Shapley, 1974). Concrete criteria for identifying a given nonatomic game as belonging to such a class of games, and a formula for computing the value, are known for certain kinds of vector measure games (Aumann and Shapley, 1974). In a vector measure game, the worth of a coalition S depends only on the value that a particular vector measure on the space of players takes in S. In this context, the term "vector measure" usually refers to an $\mathbf{R}^n$-valued measure, that is, to a vector of n scalar measures. However, certain nonatomic games are more naturally described in terms of measures that take values in an infinite dimensional Banach space.

Consider, for example, a (transferable utility) market game v, where the worth of a coalition S is

$$v(S) = \max \left\{ \int_S u(x(i), i)\, dm(i) \,\middle|\, \int_S x(i)\, dm(i) = \int_S a(i)\, dm(i) \right\}, \qquad (1)$$

the maximum aggregate utility that S can guarantee to itself by an allocation x of its aggregate endowment $\int_S a\, dm$ among its members (Aumann, 1964. In this formula, $u(\xi, i)$ is the utility that player i gets from the bundle ξ and m, the population measure, is a nonatomic probability measure.) Since $v(S)$ depends on S only through certain integrals over S, if $m(S \backslash T) = m(T \backslash S) = 0$ then $v(T) = v(S)$. Thus, $v(S)$ can be expressed as a function of the characteristic function χ_S of the coalition S, seen as an element of $L_1(m)$. The market game under consideration can therefore be viewed as a vector measure game based on the $L_1(m)$-valued vector measure defined by $S \mapsto \chi_S$.

Only one previous work known to me deals with vector measure games based on vector measures with values in an infinite dimensional vector space. Sroka (1993) studied games based on vector measures of bounded variation with values in a relatively compact subset of a Banach space with a shrinking Schauder basis. The vector measure considered above in connection with the market game is not of this kind: its range is not a relatively compact subset of $L_1(m)$, and $L_1(m)$ itself does not have a shrinking

Schauder basis. (Only a separable Banach space with a separable dual space can have such a basis. Considering the range of the measure in question to be a subset of L_2, say, rather then L_1, would not help, for the measure would not then be of bounded variation. Note that if the range space were taken to be L_∞ then the above set function would not even be a measure; specifically, it would not be countably additive.) In the first part of this paper, these limitations on the range of the vector measure and on the space in which it lies are dispensed with. Thus, the results of Aumann and Shapley are generalized to a much larger class of vector measure games.

As the above market game example demonstrates, the present interpretation of "vector measure games" is broad enough to include all games in which the worth of a coalition is not affected by the addition or subtraction of a set of players of measure zero—the measure in question being a fixed nonatomic scalar measure on the space of players. All the games that belong to one of the spaces of games on which Aumann and Shapley have proved the existence of a unique value have this property, and can therefore be represented as vector measure games. This representation is, however, not unique. It is therefore desirable to reformulate the conditions for a vector measure game to belong to one of these spaces in a language that does not make an explicit reference to vector measures. Such an alternative formulation is presented in the second part of the paper, where the above conditions are stated as differentiability and continuity conditions on a suitable extension of the game, an ideal game, that assigns a worth to every ideal, of "fuzzy", coalition, in which some players are only partial members.

The last part of the paper contains an example that shows how these general results can be applied to market games. Another application, involving cooperative games derived from a particular class of nonatomic noncooperative congestion games, is given in a separate paper (Milchtaich, 2004). Two rather technical lemmas, which are of some independent interest, are given in the Appendix.

Preliminaries

The *player space* is a measurable space $(I, \mathcal{C})$. A member of the σ-field $\mathcal{C}$ is called a *coalition*. A *set function* is a function from $\mathcal{C}$ into a real Banach space

X. The *variation* of a set function v is the extended real-valued function $|v|$ defined by

$$|v|(S) = \sup \sum_{i=1}^{n} \|v(S_i) - v(S_{i-1})\| \qquad (S \in \mathcal{C}),$$

where the supremum is taken over all finite nondecreasing sequences of coalitions of the form $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S$. A set function v is of *bounded variation* if $|v|(I) < \infty$. A *game* is a real-valued set function v such that $v(\emptyset) = 0$. A game v is *monotonic* if $T \subseteq S$ implies $v(T) \leq v(S)$. BV denotes the normed linear space of all games of bounded variation endowed with the operations of pointwise addition and multiplication by a (real) scalar and with the *variation norm* (called the "variation" in Aumann and Shapley, 1974) $\|v\|_{BV} = |v|\,(I)$. The monotonic games span this space. The subspace of BV that consists of all finitely additive real-valued set functions of bounded variation is denoted FA.

A *vector measure* is a countably additive set function. A (finite, signed) *measure* is a real-valued vector measure. A vector measure μ is *nonatomic* if for every $S \in \mathcal{C}$ such that $\mu(S) \neq 0$ there is a subset $T \subseteq S$ such that $\mu(T), \mu(S \backslash T) \neq 0$. The variation $|\mu|$ (also called the total variation measure) of a vector measure of bounded variation μ is a measure (Diestel and Uhl, 1977, p. 3). $|\mu|$ is nonatomic if and only if μ is nonatomic. The subspace of BV that consists of all nonatomic (finite, signed real-valued) measures is denoted NA.

An *ideal coalition* is a measurable function $h : I \to [0,1]$. For a vector measure μ and an ideal coalition (or a linear combination of ideal coalitions) h, $\mu(h)$ denotes the integral $\int h \, d\mu$. See Dunford and Schwartz (1958, Section IV.10) for a definition of integration with respect to a vector measure. The space $\mathcal{I}$ of ideal coalitions is topologized by the *NA-topology*, defined as the smallest topology on $\mathcal{I}$ with respect to which all functions of the form $\mu(\cdot)$, $\mu \in NA$, are continuous. As a base for the neighborhood system of an ideal coalition h one can take the collection of all open sets of the form $\{g \in \mathcal{I} | \max_{1 \leq i \leq k} |m_i(g-h)| < \epsilon\}$, with $\epsilon > 0$ and $m_1, m_2, \ldots, m_k \in NA$.

The *range* of a vector measure μ is the set $\mu(\mathcal{C})$. If the range of μ (or, more precisely, the subspace it spans) is finite dimensional then μ is automatically of bounded variation. If in addition μ is nonatomic then its range is compact and convex (Lyapunoff theorem). The range of a general nonatomic vector measure need not be compact nor convex. However, for every vector measure μ the set $\mu(\mathcal{I}) = \{\mu(h) | h \in \mathcal{I}\}$ is convex and weakly compact (Diestel and Uhl, 1977, p. 263). This set coincides with the closed convex hull of $\mu(\mathcal{C})$, and if μ is nonatomic then it is also the weak closure of $\mu(\mathcal{C})$ (Diestel and Uhl, 1977, p. 264). We will call $\mu(\mathcal{I})$ the *extended range* of μ.

EXAMPLE. Let m be a probability measure on $(I, \mathcal{C})$, and define $\mu : \mathcal{C} \to L_1(m)$ by $\mu(S) = \chi_S$. Then μ is a vector measure of bounded variation whose variation is m. Therefore, μ is nonatomic if and only if m is nonatomic. The range of μ consists of all (equivalence classes of) characteristic functions of measurable subsets of I. This is a closed, but not convex, subset of $L_1(m)$, and if m is nonatomic then it is also not compact. The extended range of μ is the set of all (equivalence classes of) measurable functions from I into the unit interval. Indeed, for every $h \in \mathcal{I}$, $\mu(h) = h$. Note that in this example the weak compactness of the extended range of μ follows immediately from Alaoglu's theorem and from the fact that the relative weak topology on this set coincides with the relative weak* topology on it when seen as a subset of $L_\infty(m)$.

We will say that a real-valued function f defined on a convex subset C of a Banach space X is *differentiable* at $x \in C$ if there exists a continuous linear functional $Df(x) \in Y^*$, where Y is the subspace of X spanned by $C - C = \{y - z | y, z \in C\}$ and Y^* is its dual space, such that for every $y \in C$

$$f(x + \theta(y - x)) = f(x) + \theta \langle y - x, Df(x) \rangle + o(\theta)$$

as $\theta \to 0_+$. (The angled brackets $\langle \cdot, \cdot \rangle$ denote the operation of applying an element of Y^* to an element of Y.) This continuous linear functional, which is necessarily unique, will be called the *derivative* of f at x. The function f is (*weakly*) *continuously differentiable* at x if it is differentiable in a (respectively, weak) neighborhood of x in C and Df is (respectively, weakly) continuous at x. The function f is (*weakly*) *continuously differentiable* if it is (respectively, weakly) continuously differentiable in the whole of its domain. The restriction of a (weakly) continuously differentiable function to a convex subset of its domain is (respectively, weakly) continuously differentiable. Continuous differentiability and weak continuous differentiability are equivalent for functions with compact domain. This follows from the fact that the relativization of the weak topology to a compact subset of a Banach space coincides with the relative norm topology (because every set that is closed, and hence compact, with respect to the relative norm topology is compact, and hence closed, also with respect to the relative weak topology). A real-valued function f defined on a bounded convex subset C of a Banach space X is (weakly) continuous at every point x at which it is (respectively, weakly) continuously differentiable.[1] If X is a Euclidean space and C is compact then f is continuously differentiable if and only if it can be extended to a continuous function on X with continuous first-order partial derivatives.

The closed linear subspace of BV that is generated by all powers (with respect to pointwise multiplication) of nonatomic probability measures is denoted pNA. There exists a unique continuous linear operator $\varphi : pNA \to FA$ that satisfies $\varphi(\mu^k) = \mu$ for every nonatomic probability measure μ and positive integer k, called the (Aumann-Shapley) *value* on pNA. See Aumann and Shapley (1974) for an axiomatic characterization of the value.

For a game v, define $\|v\|_\infty = \inf\{m(I) | m \in NA$, and $|v(S) - v(T)| \leq m(S \backslash T)$ for every $S, T \in \mathcal{C}$ with $T \subseteq S\}$ ($\inf \emptyset = \infty$). The collection of all games v such that $\|v\|_\infty < \infty$ is a linear subspace of BV, denoted AC_∞, and $\|\cdot\|_\infty$ is a norm on this space. The $\|\cdot\|_\infty$-closed linear subspace of AC_∞ that is generated by all powers of nonatomic probability measures is denoted pNA_∞. This space is a proper subset of pNA.

Vector Measure Games

A composed set function of the form $f \circ \mu$, where μ is a nonatomic vector measure of bounded variation and f is a real-valued function defined on $\mu(\mathcal{I})$ such that $f(0) = 0$, will be called a *vector measure game*. Aumann and Shapley (1974) proved that if the range of μ is finite dimensional then a sufficient condition for a vector measure game $f \circ \mu$ to be in pNA (actually in pNA_∞) is that f be continuously differentiable. The value of such a vector measure game is given by the so-called diagonal formula. Sroka

[1]Proof: If U is a convex neighborhood of x in C in which f is differentiable then it follows from the mean value theorem that $|f(y) - f(x)| \leq \sup_{0<\theta<1} |\langle y - x, Df(x + \theta(y - x))\rangle| \leq |\langle y - x, Df(x)\rangle| + \|y - x\| \sup_{z \in U} \|Df(z) - Df(x)\|$ for every $y \in U$. By a suitable choice of U the last two terms can be made arbitrarily small.

(1993) generalized this result to the case where the range of μ is a relatively compact subset of a Banach space with a shrinking Schauder basis. These results are generalized further in the following theorem.

THEOREM 1. *Let μ be a nonatomic vector measure of bounded variation with values in a Banach space X. If f is a weakly continuously differentiable real-valued function defined on the extended range of μ such that $f(0) = 0$, then $f \circ \mu$ is in pNA_∞ and its value is given by the (diagonal) formula*

$$\varphi(f \circ \mu)(S) = \int_0^1 \langle \mu(S), Df(t\mu(I)) \rangle \, dt \qquad (S \in \mathcal{C}). \tag{2}$$

If X is finite dimensional then the converse is also true: a vector measure game $f \circ \mu$ is in pNA_∞ only if f is continuously differentiable on the range of μ.

The restriction that X is finite dimensional can not be removed. For example, if m is Lebesgue measure on the unit interval and μ is as in the Example in the previous section then the function $f : \mu(\mathcal{I}) \ (\subseteq L_1(m)) \to \mathbf{R}$ defined by $f(h) = \int_0^1 t^{-1/2} h(t) \, dt$ is not differentiable according to the present definition. Nevertheless, $f \circ \mu \in NA$. The question of what conditions on f, if any, are both necessary and sufficient for a general vector measure game $f \circ \mu$ to be in pNA_∞, or in pNA, remains open (cf. Kohlberg, 1973; Aumann and Shapley, 1974, Theorem C; Tauman, 1982). Note that if the range of μ is relatively compact (this is automatically the case if X is a reflexive space or a separable dual space; see Diestel and Uhl, 1977, p. 266) then by Mazur theorem (Dunford and Schwartz, 1958, p. 416) the extended range of μ is compact. Therefore, in such a case f is weakly continuously differentiable if and only if it is continuously differentiable.

If a vector measure game $f \circ \mu$ is monotonic, then for it to be in pNA it suffices that f be continuous, rather than differentiable, at 0 and $\mu(I)$.

PROPOSITION 1. *Let μ be a nonatomic vector measure of bounded variation, and let $f : \mu(\mathcal{I}) \to \mathbf{R}$ be weakly continuously differentiable in $\mu(\{h \in \mathcal{I} \mid 0 < |\mu|(h) < |\mu|(I)\})$ and continuous at 0 and at $\mu(I)$. If $f \circ \mu$ is a monotonic game then it is in pNA and its value is given by* (2).

The following lemma, which is of some independent interest, is used in the proofs of Theorem 1 and Proposition 1.

LEMMA 1. *Let $\mu : \mathcal{C} \to X$ be a nonatomic vector measure of bounded variation, and let f be a real-valued function defined on $\mu(\mathcal{I})$. Define $\hat{\mu} : \mathcal{C} \to L_1(|\mu|)$ by $\hat{\mu}(S) = \chi_S$. (Note that $|\hat{\mu}| = |\mu|$.) Then there exists a unique real-valued function $\hat{f}$, defined on $\hat{\mu}(\mathcal{I})$, such that*

$$\hat{f}(\hat{\mu}(h)) = f(\mu(h)) \qquad (h \in \mathcal{I}). \tag{3}$$

For every $h \in \mathcal{I}$, if f is (weakly) continuous at $\mu(h)$ then $\hat{f}$ is (respectively, weakly) continuous at $\hat{\mu}(h)$, and if f is weakly continuously differentiable at $\mu(h)$ then $\hat{f}$ is weakly continuously differentiable at $\hat{\mu}(h)$ and $D\hat{f}(\hat{\mu}(h))$ satisfies

$$\langle \hat{\mu}(g), D\hat{f}(\hat{\mu}(h)) \rangle = \langle \mu(g), Df(\mu(h)) \rangle \qquad (g \in \mathcal{I}). \tag{4}$$

It follows from Lemma 1 that, conceptually, there is only one kind of vector measures that needs to be considered in the present context, namely, vector measures that map coalitions into their characteristic functions. One may thus wonder whether vector measures need to be considered at all. An alternative approach might be to express the above conditions for a game to be in pNA_∞ or in pNA directly in terms of a particular "extension" of the game into a function on $\mathcal{I}$. We will see in the next section that these results can indeed be reformulated in such a manner.

Differentiable Ideal Games

An *ideal game* is a real-valued function on $\mathcal{I}$ that vanishes at (the constant function) 0. We will say that an ideal game v^* is *monotonic* if $h \leq g$ implies $v^*(h) \leq v^*(g)$, and that v^* is *differentiable* at $h \in \mathcal{I}$ if there exists a (necessarily unique) nonatomic measure $Dv^*(h)$, called the *derivative* of v^* at h, such that for every $g \in \mathcal{I}$

$$v^*(h + \theta(g - h)) = v^*(h) + \theta\, Dv^*(h)(g - h) + o(\theta)$$

as $\theta \to 0_+$. An ideal game is *differentiable* if it is differentiable at every point in $\mathcal{I}$. An ideal game v^* is a *continuous extension* of a game v if $v^*(\chi_S) = v(S)$ for every $S \in \mathcal{C}$ and v^* is continuous (with respect to the NA-topology). Aumann and Shapley (1974, Proposition 22.16) showed that a continuous extension is always unique, and that a sufficient condition for a game to have such an extension is that there exists a sequence in pNA that converges to the game in the *supremum norm* $\|v\|' = \sup_{S \in \mathcal{C}} |v(S)|$. The set of all games that satisfy this condition is closed under pointwise addition and multiplication by a real scalar, and is denoted pNA'.

THEOREM 2. *An ideal game v^* is the continuous extension of some game v in pNA_∞ if and only if there is a nonatomic probability measure m such that, for every $h \in \mathcal{I}$, the derivative $Dv^*(h)$ exists and is absolutely continuous with respect to m, $d(Dv^*(h))/dm$ is essentially bounded, and $d(Dv^*(\cdot))/dm$ is continuous at h as a function into $L_\infty(m)$. The value of v is then given by*

$$(\varphi v)(S) = \int_0^1 Dv^*(t)(S)\, dt \qquad (S \in \mathcal{C}). \tag{5}$$

The necessary and sufficient condition for an ideal game v^* to be the continuous extension of a game in pNA_∞ that is given in Theorem 2 is apparently stronger then the sufficient condition obtained by Hart and Monderer (1997). In fact, Hart and Monderer's condition is equivalent to the requirement that $d(Dv^*(\cdot))/dm$ be continuous as a function into $L_1(m)$. The above condition is equivalent to the requirement that there is a representation of the game as a vector measure game that satisfies the conditions of Theorem 1. Thus, we have the following result.

LEMMA 2. *A game v can be represented as a vector measure game $f \circ \mu$, with f weakly continuously differentiable on the extended range of μ, if and only if $v \in pNA_\infty$. A game v can be represented as a vector measure game $f \circ \mu$, with f weakly continuous on the extended range of μ, if and only if $v \in pNA'$.*

The following sufficient condition for a monotonic ideal game to be the continuous extension of a game in pNA is derived from Proposition 1.

PROPOSITION 2. *If v^* is a monotonic ideal game such that $\lim_{t\to 0_+} v^*(t) = 0$ and $\lim_{t\to 1_-} v^*(t) = v^*(1)$, and there exists a nonatomic probability measure m such that, for every $h \in \mathcal{I}$ with $0 < m(h) < 1$, the derivative $Dv^*(h)$ exists and is absolutely continuous with respect to m, $d(Dv^*(h))/dm$ is essentially bounded, and $d(Dv^*(\cdot))/dm$ is continuous at h as a function into $L_\infty(m)$, then the game v defined by $v(S) = v^*(\chi_S)$ is in pNA and its value is given by* (5).

Proofs

Proof of Lemma 1. For every $h \in \mathcal{I} - \mathcal{I}$ and every continuous linear functional $x^* \in X^*$, $x^*(\mu(h)) = \int h \, d(x^* \circ \mu) = \int h \, d(x^* \circ \mu)/d|\mu| \; d|\mu|$ by Theorem IV.10.8 of Dunford and Schwartz (1958). Taking the maximum over the unit sphere in X^*, we get $\|\mu(h)\|_X \leq \int |h| \; d|\mu| = \|\hat{\mu}(h)\|_{L_1(|\mu|)}$, since $\|d(x^* \circ \mu)/d|\mu|\|_{L_\infty(|\mu|)} \leq \|x^*\|_{X^*}$ for every x^*. Therefore, $\hat{\mu}(h) \mapsto \mu(h)$ is a well-defined continuous function from $\hat{\mu}(\mathcal{I})$ onto $\mu(\mathcal{I})$ that is continuous also with respect to the relative weak topologies on these spaces. It follows that $\hat{f}$ is well defined by (3) and that it is (weakly) continuous at $\hat{\mu}(h)$ if f is (respectively, weakly) continuous at $\mu(h)$. Also, if f is weakly continuously differentiable at $\mu(h)$ then Eq. (4) well defines a continuous linear functional $D\hat{f}(\hat{\mu}(h)) \in L_1(|\mu|)^*$ (which is in fact equal to, or rather identifiable with, $d(Df(\mu(h)) \circ \mu)/d|\mu| \in L_\infty(|\mu|)$). $||D\hat{f}(\hat{\mu}(g)) - D\hat{f}(\hat{\mu}(h))||_{L_1(|\mu|)^*} \leq ||Df(\mu(g)) - Df(\mu(h))||_{\mathrm{Sp}(\mu(\mathcal{I}))^*}$ holds for every $\hat{\mu}(g)$ in a weak neighborhood of $\hat{\mu}(h)$, and therefore $D\hat{f}(\cdot)$ is weakly continuous at $\hat{\mu}(h)$. By (3) and (4), and the definition of $Df(\mu(h))$, for every $\hat{\mu}(g) \in \hat{\mu}(\mathcal{I})$

$$\hat{f}(\hat{\mu}(h) + \theta(\hat{\mu}(g) - \hat{\mu}(h))) = \hat{f}(\hat{\mu}(h)) + \theta \left\langle \hat{\mu}(g) - \hat{\mu}(h), D\hat{f}(\hat{\mu}(h)) \right\rangle + o(\theta)$$

as $\theta \to 0_+$. This proves that $\hat{f}$ is differentiable at $\hat{\mu}(h)$ and that the continuous linear functional $D\hat{f}(\hat{\mu}(h))$ is indeed its derivative there. $\square$

Proof of Theorem 1. Suppose that f satisfies the condition of the theorem. In light Lemma 1, it can be assumed without loss of generality that $X = L_1(m)$, where m is a nonatomic probability measure, and that $\mu(S) = \chi_S$ ($S \in \mathcal{C}$). For a sub-σ-field $\mathcal{F}$ of $\mathcal{C}$, define an $L_1(m)$-valued nonatomic vector measure $\mu_\mathcal{F}$ by $\mu_\mathcal{F}(S) = E(\chi_S|\mathcal{F})$, where $E(\cdot|\mathcal{F})$ denotes conditional expectation. If $\mathcal{F}$ is finite then the range of $\mu_\mathcal{F}$, which is a subset of the convex hull of the range of μ, clearly spans a finite dimensional subspace of $L_1(m)$. Therefore, by an immediate extension of Proposition 7.1 of Aumann and Shapley (1974), $f \circ \mu_\mathcal{F} \in pNA_\infty$. To prove that $f \circ \mu$ is in pNA_∞ it suffices to show that

$$\lim_{\mathcal{F}} \|f \circ \mu_\mathcal{F} - f \circ \mu\|_\infty = 0 \tag{6}$$

as $\mathcal{F}$ varies over the finite subfields of $\mathcal{C}$, directed by inclusion. (That is, for every $\epsilon > 0$ there exists a finite measurable partition of I such that, if $\mathcal{F}$ is the field generated by some finer finite measurable partition, $\|f \circ \mu_{\mathcal{F}} - f \circ \mu\|_\infty < \epsilon$.)

Let $\mathcal{F}$ be a sub-σ-field of $\mathcal{C}$, and let $S, T \in \mathcal{C}$ be such that $T \subseteq S$. For $0 \leq t \leq 1$, define $h_t = \chi_T + t\chi_{S\setminus T}$ ($\in \mathcal{I}$). By the fundamental theorem of calculus, applied to the function $t \mapsto f(\mu_{\mathcal{F}}(h_t))$ ($= f(\mu_{\mathcal{F}}(T) + t\mu_{\mathcal{F}}(S\setminus T))$),

$$\begin{aligned}(f \circ \mu_{\mathcal{F}})(S) - (f \circ \mu_{\mathcal{F}})(T) &= \int_0^1 \langle \mu_{\mathcal{F}}(S\setminus T), Df(\mu_{\mathcal{F}}(h_t))\rangle \, dt \\ &= \int_0^1 \int_I \mu_{\mathcal{F}}(S\setminus T) Df(\mu_{\mathcal{F}}(h_t)) \, dm \, dt \qquad (7) \\ &= \int_0^1 \int_{S\setminus T} E(Df(\mu_{\mathcal{F}}(h_t))|\mathcal{F}) \, dm \, dt,\end{aligned}$$

where the last equality follows from the identity

$$\int_I E(h|\mathcal{F}) g \, dm = \int_I h E(g|\mathcal{F}) \, dm \qquad (h \in L_1(m), g \in L_\infty(m)),$$

applied to the functions $h = \chi_{S\setminus T}$ and $g = Df(\mu_{\mathcal{F}}(h_t))$. Note that the same notation is used in (7) for the derivative of f at a point and for the representation of that derivative as an element of $L_\infty(m)$. In the special case $\mathcal{F} = \mathcal{C}$ we get

$$(f \circ \mu)(S) - (f \circ \mu)(T) = \int_0^1 \int_{S\setminus T} Df(\mu(h_t)) \, dm \, dt. \qquad (8)$$

It follows from (7) and (8) that

$$\|f \circ \mu_{\mathcal{F}} - f \circ \mu\|_\infty \leq \sup_{h\in\mathcal{I}} \|E(Df(E(h|\mathcal{F}))|\mathcal{F}) - Df(\mu(h))\|_{L_\infty(m)}.$$

Hence, in order to complete the proof of (6) it suffices to show that

$$\lim_{\mathcal{F}} \|E(Df(E(h|\mathcal{F}))|\mathcal{F}) - Df(E(h|\mathcal{F}))\|_{L_\infty(m)} = 0 \qquad (9)$$

and

$$\lim_{\mathcal{F}} \|Df(E(h|\mathcal{F})) - Df(\mu(h))\|_{L_\infty(m)} = 0 \qquad (10)$$

uniformly in $h \in \mathcal{I}$ as $\mathcal{F}$ varies over the finite subfields of $\mathcal{C}$, directed by inclusion. Df is a weakly continuous function defined on a weakly compact subset of $L_1(m)$. Therefore, it is uniformly weakly continuous and its range is compact. Theorem IV.8.18 of Dunford and Schwartz (1958) asserts that, for every compact subset K of $L_\infty(m)$, $\lim_{\mathcal{F}} \|E(g|\mathcal{F}) - g\|_{L_\infty(m)} = 0$ uniformly in $g \in K$. This proves (9). The same theorem, together with the above identity, implies that, for every $g \in L_\infty(m)$, $\lim_{\mathcal{F}} \int E(h|\mathcal{F})g\, dm = \lim_{\mathcal{F}} \int hE(g|\mathcal{F})\, dm = \int hg\, dm$ uniformly in $h \in \mathcal{I}$. Thus, with respect to the weak topology on $L_1(m)$, $\lim_{\mathcal{F}} E(h|\mathcal{F}) = \mu(h)$ uniformly in $h \in \mathcal{I}$. This, and the uniform weak continuity of Df, together imply (10).

Since f is weakly continuous, it follows from Lemma 5 in the Appendix that the ideal game defined by $h \mapsto f(\mu(h))$ is a continuous extension of $f \circ \mu$. Therefore, by Theorem H of Aumann and Shapely (1974), the value of $f \circ \mu$ is given by

$$\varphi(f \circ \mu)(S) = \int_0^1 \left.\frac{d}{d\theta}\right|_{\theta=0} f(t\mu(I) + \theta\mu(S))\, dt \qquad (S \in \mathcal{C})$$

(and the derivative on the right hand side of this equation exists for almost every $0 < t < 1$). By definition of Df, this gives (2).

The proof of the second part of the theorem will be given after the proof of Theorem 2. $\square$

Proof of Proposition 1. We prove Proposition 1 by making the following two modifications to the proof of the first part of Theorem 1.

First, since f is continuous at 0, and since $\mu(h) \to 0$ is equivalent to $m(h) \to 0$, f is actually weakly continuous at 0. Similarly, f is weakly continuous at $\mu(I)$. Since f is weakly continuously differentiable, and hence weakly continuous, in $\mu(\{h \in I \,|\, 0 < m(h) < 1\}) = \mu(\mathcal{I}) \backslash \{0, \mu(I)\}$, the ideal game $h \mapsto f(\mu(h))$ is continuous. Aumann and Shapley (1974, p. 150) showed that a continuous extension of a monotonic game is a monotonic ideal game. It follows that, for every finite subfield $\mathcal{F}$ of $\mathcal{C}$, the restriction of f to $\mu_{\mathcal{F}}(\mathcal{I})$ can be viewed as a nondecreasing continuous function on the unit cube in $\mathbf{R}^n$, where n is the dimension of the subspace of $L_1(m)$ that is spanned by $\mu_{\mathcal{F}}(\mathcal{I})$ (which is equal to the number of atoms of nonzero m-measure of the field $\mathcal{F}$). This function is continuously differentiable outside of the origin and $(1, 1, \ldots, 1)$. Therefore, by an extension of Proposition 10.17 of Aumann and Shapely (1974, proposition and extension in p. 92), $f \circ \mu_{\mathcal{F}} \in pNA$.

Second, for fixed $\epsilon > 0$, let $0 < \delta < 1/2$ be such that $f(\mu(h)) < \epsilon$ and $f(\mu(1-h)) > f(\mu(I)) - \epsilon$ for every $h \in \mathcal{I}$ that satisfies $m(h) \leq \delta$.

If $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_{i_0} \subseteq \cdots \subseteq S_{i_1} \subseteq \cdots \subseteq I$ is a finite nondecreasing sequence of coalitions such that $m(S_{i_0}) = \delta$ and $m(S_{i_1}) = 1 - \delta$, and if $\mathcal{F}$ is a finite subfield of $\mathcal{C}$, then for every $i_0 < i \leq i_1$ Eqs. (7) and (8) hold for $S = S_i$ and $T = S_{i-1}$. The monotonicity of $f \circ \mu$ and of $f \circ \mu_{\mathcal{F}}$ implies that $\sum_{i=1}^{i_0} |(f \circ \mu_{\mathcal{F}})(S_i) - (f \circ \mu)(S_i) - (f \circ \mu_{\mathcal{F}})(S_{i-1}) + (f \circ \mu)(S_{i-1})| < 2\epsilon$, and a similar inequality holds for the sum over $i_1 + 1, i_1 + 2, \ldots$. It follows that $\|f \circ \mu_{\mathcal{F}} - f \circ \mu\|_{BV} < \sup_{\delta \leq m(h) \leq 1-\delta} \|E(Df(E(h|\mathcal{F}))|\mathcal{F}) - Df(\mu(h))\|_{L_\infty(m)}$ $+4\epsilon$. Since the set $\{\mu(h) \in \mu(\mathcal{I}) \mid \delta \leq m(h) \leq 1 - \delta\}$ is weakly compact, an argument similar to that given in the proof of Theorem 1 shows that the limit of the last supremum as $\mathcal{F}$ varies over the finite subfields of $\mathcal{C}$, directed by inclusion, is zero. Since ϵ is arbitrary, this proves that $\lim_{\mathcal{F}} \|f \circ \mu_{\mathcal{F}} - f \circ \mu\|_{BV} = 0$, and therefore $f \circ \mu \in pNA$. $\square$

Proof of Theorem 2. If v^* satisfies the condition of the theorem then $v^*(g) = v^*(h)$ for every g and h in $\mathcal{I}$ that are equal m-almost everywhere. Indeed, by the mean value theorem, there exists some $0 < \theta < 1$ such that

$$\begin{aligned} v^*(g) - v^*(h) &= Dv^*(h + \theta\,(g-h))(g-h) \\ &= \int (g-h)\, \frac{d\,(Dv^*(h + \theta\,(g-h)))}{dm}\, dm = 0. \end{aligned}$$

Therefore, there exists a unique function $f : \mu(\mathcal{I}) \to \mathbf{R}$, where μ is the $L_1(m)$-valued nonatomic vector measure of bounded variation defined by $\mu(S) = \chi_S$, such that $v^*(h) = f(\mu(h))$ $(h \in \mathcal{I})$. This function is differentiable. Indeed, its derivative at $\mu(h)$ is $Df(\mu(h)) = d(Dv^*(h))/dm \in L_\infty(m)$ $(= L_1(m)^*)$, and it follows that $Df(\mu(\cdot))$ is continuous on $\mathcal{I}$. Hence, Lemma 5 in the Appendix implies that Df is weakly continuous. Therefore, by Theorem 1, $f \circ \mu \in pNA_\infty$. The formula for the value of v $(= f \circ \mu)$ now follows from (2) and from the above expression for Df.

Conversely, suppose that v^* is the continuous extension of a game v in pNA_∞. There exists a sequence $\{v_n\}_{n \geq 1}$ of games in pNA_∞ whose continuous extensions satisfy the condition of the theorem such that $\|v_n - v\|_\infty < 4^{-n}$ for every n. Indeed, we can take these games to be polynomials in NA measures. For every $u \in pNA_\infty$ and for every $\epsilon > 0$ there is a nonatomic probability measure m such that $|u^*(h + \theta g) - u^*(h)| \leq |\theta|\,(\|u\|_\infty + \epsilon)\, m(g)$ for every $g, h \in \mathcal{I}$ and $\theta \in \mathbf{R}$ such that $h + \theta g \in \mathcal{I}$ (see that Appendix). It follows that if u^* is differentiable at h then

$$|Du^*(h)(g)| \leq (\|u\|_\infty + \epsilon)\, m(g) \leq \|u\|_\infty + \epsilon \qquad (g \in \mathcal{I}).$$

In particular, $\left|Dv_n^*(h)(g) - Dv_{n'}^*(h)(g)\right| \le \|v_n - v_{n'}\|_\infty$ for every n, n', and $g, h \in \mathcal{I}$, and therefore $Dv_n^*(\cdot)(\cdot)$ converges uniformly to a real-valued function $\gamma(\cdot,\cdot)$ on $\mathcal{I} \times \mathcal{I}$. For every h, $\gamma(h,\cdot)$ is (the continuous extension of) a nonatomic measure and, for every g, $\gamma(\cdot, g)$ is continuous on $\mathcal{I}$. Since, for every $g, h \in \mathcal{I}$ and $\theta \in \mathbf{R}$ such that $h + \theta g \in \mathcal{I}$ and for every n, $v_n^*(h+\theta g) - v_n^*(h) = \int_0^\theta Dv_n^*(h+tg)(g)\, dt$, in the limit we get $v^*(h+\theta g) - v^*(h) = \int_0^\theta \gamma(h+tg, g)\, dt$. This equation implies that v^* is differentiable at h (and $Dv^*(h) = \gamma(h,\cdot)$).

For every n, let m_n be a nonatomic probability measure such that $|Dv_n^*(h)(g) - Dv^*(h)(g)| \le 4^{-n} m_n(g)$ $(g, h \in \mathcal{I})$. The nonatomic probability measure $m = \sum_{k\ge 1} 2^{-k} m_k$ satisfies $4^{-n} m_n(g) \le 2^{-n} m(g)$. Therefore, $Dv_n^*(h) - Dv^*(h)$ is absolutely continuous with respect to m and

$$\left\| \frac{d(Dv_n^*(h) - Dv^*(h))}{dm} \right\|_{L_\infty(m)} \le 2^{-n}.$$

Since v_n satisfies the condition of the theorem, we may assume without loss of generality that, for every h, $Dv_n^*(h)$ is absolutely continuous with respect to m_n, its Radon-Nikodym derivative with respect to m_n is essentially bounded, and $d(Dv_n^*(\cdot))/dm_n$ is continuous at h as a function into $L_\infty(m_n)$. This remains true when m_n is replaced by m. It follows that, for every h, $Dv^*(h)$, too, is absolutely continuous with respect to m and its Radon-Nikodym derivative with respect to m is essentially bounded. And since the function $d(Dv^*(\cdot))/dm : \mathcal{C} \to L_\infty(m)$ is the uniform limit of the continuous functions $\{d(Dv_n^*(\cdot))/dm_n\}_{n\ge 1}$, this function is continuous, too. $\square$

Proof of Theorem 1 (continued). Suppose that $\dim X < \infty$. If a vector measure game $f \circ \mu$ is in pNA_∞ then by Theorem 2 its continuous extension v^* is differentiable. It is shown in the Appendix that $v^*(h) = f(\mu(h))$ $(h \in \mathcal{I})$. Hence, $Dv^*(h)(g-h) = d/d\theta|_{\theta=0_+} f(\mu(h+\theta(g-h)))$ for every g and h. The value of this derivative clearly depends only on $\mu(h)$ and on $\mu(g-h)$, and can therefore be written as $\hat{\gamma}(\mu(h))(\mu(g-h))$, where $\hat{\gamma}$ is a function from $\mu(\mathcal{I})$ to X^*. By Theorem 2, $Dv^*(\cdot)(g)$ is continuous for every $g \in \mathcal{I}$. Therefore, by Lemma 5 in the Appendix, $\hat{\gamma}(\cdot)(\mu(g))$ is continuous for every g. This proves that f is continuously differentiable on $\mu(\mathcal{I})$. $\square$

Proof of Lemma 2. If f is weakly continuously differentiable on the extended range of a nonatomic vector measure of bounded variation μ then $f \circ \mu \in pNA_\infty$ by theorem 1. Conversely, if v is in pNA_∞ then it is shown in the proof of Theorem 2 that v can be represented as a vector measure game $f \circ \mu$, with f weakly continuously differentiable.

If $f \circ \mu$ is a vector measure game such that f is weakly continuous on the extended range of μ, which is a subset of some Banach space X, then by Stone-Weierstrass theorem f can be uniformly approximated by polynomials in elements of the dual space X^*. Specifically, since $\mu(\mathcal{I})$ is weakly compact, and since the continuous linear functionals on X separate points in this set, for every $\epsilon > 0$ there are a finite sequence of functionals $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ and a polynomial p in n variables such that $|f(\mu(h)) - p(x_1^*(\mu(h)), x_2^*(\mu(h)), \ldots, x_n^*(\mu(h)))| < \epsilon$ for every $h \in \mathcal{I}$. Since $x_i^* \circ \mu \in NA$ for every i, it follows from Lemma 7.2 of Aumann and Shapley (1974) that $f \circ \mu \in pNA'$.

Conversely, it is shown in the Appendix that if v^* is the continuous extension of a game v in pNA' then there is a nonatomic probability measure m such that, for every $g, h \in \mathcal{I}$ that are equal m-almost everywhere, $v^*(g) = v^*(h)$. It follows that there exists a real-valued function f, defined on the extended range of the nonatomic vector measure of bounded variation μ defined as in the Example, such that $v^*(h) = f(\mu(h))$ ($h \in \mathcal{I}$). Since v^* is continuous, by Lemma 5 in the Appendix f is weakly continuous. $\square$

Proof of Proposition 2. It suffices to show that if v^* satisfies the conditions of the proposition then there exist a vector measure μ and a function f, which satisfy the conditions of Proposition 1, such that $v^*(h) = f(\mu(h))$ ($h \in \mathcal{I}$). Once we establish that $v^*(h) = 0$ for every h that is equal m-almost everywhere to 0 and that $v^*(g) = v^*(1)$ for every g that is equal m-almost everywhere to 1 we can proceed almost exactly as in the proof of Theorem 2: define μ and f as in that proof, and use the same arguments to show that $Df(\mu(\cdot))$ exists and is continuous in $\{h \in \mathcal{I} | 0 < m(h) < 1\} = \{h \in \mathcal{I} | \mu(h) \neq 0, \mu(I)\}$, and that Df is therefore weakly continuous in $\mu(\mathcal{I}) \backslash \{0, \mu(I)\}$. The proof for Eq. (5) is also similar. Hence, it only remains to show that $v^*(h) \to 0$ when $\mu(h) \to 0$ (or, equivalently, when $m(h) \to 0$) and that $v^*(g) \to v^*(1)$ when $\mu(g) \to \mu(I)$ (or, equivalently, when $m(g) \to 1$).

For every $g, h \in \mathcal{I}$ such that $h \leq g$ and $m(h) < m(g)$, $v^*(h + t(g - h))$

is nondecreasing as a function of t in the interval $[0,1]$ and has a nonnegative continuous derivative in the interior of that interval. For $h = 0$ (identically) and $g = 1$ (identically) this function is continuous also at the end points. Therefore, for every sequence $\{h_n\} \subseteq \mathcal{I}$ such that $\mu(h_n) \to 0$ and for every sequence $\{g_n\} \subseteq \mathcal{I}$ such that $h_n \leq g_n$ and $\mu(g_n) \to \mu(I)$,

$$\begin{aligned}\liminf_n \left[v^*(g_n) - v*(h_n)\right] &\geq \liminf_n \int_0^1 Dv^*(h_n + t\,(g_n - h_n))(g_n - h_n)\,dt \\ &= \liminf_n \int_0^1 \langle \mu(g_n - h_n), Df(\mu(tg_n + (1-t)h_n))\rangle\,dt \\ &\geq \int_0^1 \langle \mu(I), Df(t\mu(I))\rangle\,dt = v^*(1)\end{aligned}$$

by Fatou's lemma and the continuity of Df in $\mu(\mathcal{I})\backslash\{0,\mu(I)\}$. Taking $g_n = 1$ for every n shows that $v^*(h_n) \to 0$. Taking $h_n = 0$ for every n shows that $v^*(g_n) \to v^*(1)$. $\square$

Application: Market Games

We give a new proof to the following result, due to Aumann and Shapley (1974, Chapter VI).

THEOREM 3. *Suppose that $(\mathcal{I},\mathcal{C}) = ([0,1]$, the Borel sets$)$. Let m (the population measure) be a nonatomic probability measure, k (the number of different goods) a positive integer, a (the endowment) an m-integrable function from I into the interior of the k-dimensional nonnegative orthant $\mathbf{R}^k_+$, and u (the utility function) a real-valued function that is defined on $\mathbf{R}^k_+ \times \mathcal{I}$ and satisfies the following assumptions:*

$$\text{for every } \xi \in \mathbf{R}^k_+,\ u(\xi,\cdot) \text{ is a measurable function on } I; \tag{11}$$

$$\text{for every } i \in I,\ u(\cdot,i) \text{ is a continuous function on } \mathbf{R}^k_+; \tag{12}$$

$$\begin{gathered}\text{for every } i \in I,\ u(\cdot,i) \text{ is strictly increasing (in each} \\ \text{component separately), and } u(0,i) = 0;\end{gathered} \tag{13}$$

$$\begin{gathered}\text{for every } i \in I \text{ and } j,\ \partial u(\xi,i)/\partial\xi_j \text{ exists and is continuous} \\ \text{at each } \xi \in \mathbf{R}^k_+ \text{ for which } \xi_j > 0; \text{ and}\end{gathered} \tag{14}$$

$$u(\xi, i) = o(\sum_j \xi_j) \text{ as } \sum_j \xi_j \to \infty, \text{ integrably in } i, \tag{15}$$

that is, for every $\epsilon > 0$ there is an m-integrable function $\gamma : I \to \mathbf{R}$ such that, for every $\xi \in \mathbf{R}^k_+$ and $i \in I$, $\sum_j \xi_j \geq \gamma(i)$ implies $u(\xi, i) \leq \epsilon \sum_j \xi_j$. Then the maximum in (1) *is attained for every coalition S, and the market game v defined by this equation is in pNA. The value of this game coincides with the unique competitive payoff distribution of the (transferable utility) market.*

Proof. It follows from (11) and (12) that u is Borel measurable (Klein and Thompson, 1984, Lemma 13.2.3). Define an ideal game $v^* : \mathcal{I} \to \mathbf{R}$ by

$$v^*(h) = \max \int u(x) h \, dm, \tag{16}$$

where the maximum is taken over the set $\{x : I \to \mathbf{R}^k_+ \mid x \text{ is measurable and } \int xh \, dm = \int ah \, dm\}$ of all (feasible) *allocations* and $u(x)$ denotes the function on I whose value at i is $u(x(i), i)$. This maximum is attained (which means, in particular, that it is finite; Aumann and Shapley, 1974, Proposition 36.1). Furthermore, by Proposition 36.4 and the discussion in pp. 189–190 of Aumann and Shapley (1974), for every h such that $\int h \, dm > 0$ there is a unique vector $p(h)$ in the interior of $\mathbf{R}^k_+$, called the vector of *competitive prices* corresponding to h, such that, for some allocation x,

$$u(x(i), i) - p(h) \cdot x(i) = \max_{\xi \in \mathbf{R}^k_+} [u(\xi, i) - p(h) \cdot \xi] \tag{17}$$

for m-almost every i for which $h(i) \neq 0$ (the dot stands for scalar product). It is not difficult to see that such an allocation x maximizes the integral in (16). If x is an allocation that satisfies (17) for *every i* then the pair $(x, p(h))$ is called a *transferable utility competitive equilibrium* corresponding to h. Such an allocation always exists: Since u is Borel measurable on $\mathbf{R}^k_+ \times I$ and is continuous in the first argument, there exists a measurable function $x : I \to \mathbf{R}^k_+$ that satisfies (17) for every i for which the maximum on the right-hand side of that equation is attained (Wagner, 1977, Theorem 9.2). But, as we show next, this maximum is in fact attained for every i. It follows that, given an allocation x that satisfies (17) for m-almost every i such that $h(i) \neq 0$, we can change the values that x takes at those points where (17) does not hold in such a way that the new function be an allocation that satisfies (17) everywhere.

For every positive integer s there exists, by (15), an m-integrable function $\gamma_s : I \to (0, \infty)$ such that, for every i and $\xi \in \mathbf{R}^k_+$,

$$e \cdot \xi \geq \gamma_s(i) \text{ implies } u(\xi, i) < \frac{1}{s} e \cdot \xi, \tag{18}$$

where e is the vector with all components 1. If $s > 1/\min_j p_j(h)$ then $(1/s)\, e{\cdot}\xi \leq p{\cdot}\xi$. It then follows from (18) that, for every i, the maximum on the right-hand side of (17) is attained in, and only in, the set $\{\xi \in \mathbf{R}^k_+ \,|\, e{\cdot}\xi < \gamma_s(i)\}$. Consequentially, if (17) holds, $x(i) \neq 0$, and $\min_j p_j(h) > 1/s$, then $\gamma_s(i)/e{\cdot}x(i) > 1$, and hence

$$\begin{aligned} 0 &\leq u(x(i), i) - p(h) \cdot x(i) < u(\frac{\gamma_s(i)}{e \cdot x(i)} x(i), i) - p(h) \cdot x(i) \\ &< \gamma_s(i) - p(h) \cdot x(i) \end{aligned} \tag{19}$$

by (13) and (18). It follows that if x is an allocation that satisfies (17) for every i then $\int u(x)\, dm < \int \gamma_s dm < \infty$.

LEMMA 3. *The function $p(\cdot)$ that sends each element of the set $\mathcal{I}_m = \{h \in \mathcal{I} \,|\, m(h) > 0\}$ to the corresponding vector of competitive prices is continuous (with respect to the NA-topology).*

Proof. It suffices to show that if $\{h_n\}_{n \geq 0} \subseteq \mathcal{I}_m$ is such that $\int g h_n\, dm \to \int g h_0\, dm$ for every $g \in L_1(m)$ then $p(h_n) \to p(h_0)$. The idea (borrowed from Aumann and Shapley, 1974, p. 188) is to identify the competitive prices corresponding to $h \in \mathcal{I}_m$ with equilibrium prices of a suitable exchange economy $\mathcal{E}_h$ (see Hildenbrand, 1974, Chapter 2). There are $k+1$ kinds of goods in $\mathcal{E}_h$: the k original ones plus "money" (the 0-th good). The consumption set of each player i is $\mathbf{R}_+ \times \mathbf{R}^k_+ = \mathbf{R}^{k+1}_+$ (thus, no player is allowed to hold a negative amount of money), his utility function is given by $(\xi_0, \xi) \mapsto u(\xi, i) + \xi_0$, and his endowment is $(\gamma_s(i), a(i))$, where s is some positive integer, that does not depend on i, such that $\min_j p_j(h) > 1/s$. The population measure is $h\, dm$. As is readily verified (see (19)), an allocation x satisfies (17) if and only if the bundle $(\gamma_s(i) - p(h){\cdot}x(i) + p(h){\cdot}a(i), x(i))$ maximizes player i's utility in the set $\{(\xi_0, \xi) \in \mathbf{R}_+ \times \mathbf{R}^k_+ \mid \xi_0 + p(h){\cdot}\xi = \gamma_s(i) + p(h){\cdot}a(i)\}$. It follows that $(1, p(h))$ are equilibrium prices for $\mathcal{E}_h$. Moreover, by the uniqueness of the competitive prices corresponding to h, these are the only equilibrium prices for $\mathcal{E}_h$ that are of the form $(1, p)$, with $\min_j p_j > 1/s$.

If there is some positive integer s such that $\inf_{j,n} p_j(h_n) > 1/s$ then, for every i and n, we can choose the monetary endowment of player i in $\mathcal{E}_{h_n}$ to be equal to $\gamma_s(i)$. The exchange economies then differ only in their population measures. The assumption concerning $\{h_n\}$ implies, in this case, that the (preference–endowment) distribution of $\mathcal{E}_{h_n}$ tends to that of $\mathcal{E}_{h_0}$, and similarly for the aggregate endowments. It follows, by Proposition 4 in Section 2.2 of Hildenbrand (1974), that every cluster point of the corresponding sequence of normalized equilibrium prices $\{1/(1 + \sum_j p_j(h_n))\ (1, p(h_n))\}$ is a $(k+1)$-tuple of normalized equilibrium prices for $\mathcal{E}_{h_0}$. The first component of such a cluster point cannot be zero: the equilibrium price of money must be positive. Therefore, the sequence $\{p(h_n)\}$ must be bounded. If p is a cluster point of this sequence then $(1, p)$ are equilibrium prices for $\mathcal{E}_{h_0}$. And since $\min_j p_j > 1/s$, $p = p(h_0)$. This proves that $\{p(h_n)\}$ converges to $p(h_0)$.

It remains to show that $\inf_{j,n} p_j(h_n) = 0$ is impossible. We will prove this by assuming that $p_j(h_n) \to 0$ for some j and showing that this leads to a contradiction. If, for every n, $(x_n, p(h_n))$ is a transferable utility competitive equilibrium corresponding to h_n then in particular $u(x_n) - p(h_n)\cdot x_n \geq u(x_n + e^j) - p(h_n)\cdot(x_n + e^j)$, where e^j is the jth unit vector in $\mathbf{R}^k$, and therefore $0 < u(x_n + e^j) - u(x_n) \leq p_j(h_n) \to 0$. This is consistent with (12) and (13) only if $x_n(i) \to \infty$ for every i, and hence only if $\int \max\{e\cdot(2a - x_n), 0\}h_n\, dm \to 0$. But this contradicts the fact that $\int \max\{e\cdot(2a - x_n), 0\}h_n\, dm \geq e\cdot \int (2a - x_n)h_n\, dm = e\cdot \int ah_n\, dm \to e\cdot \int ah_0\, dm > 0$. $\square$

Proof of Theorem 3 (continued). Let g and h be two elements of $\mathcal{I}_m$, and let $(y, p(g))$ and $(x, p(h))$ be two corresponding transferable utility competitive equilibria. since (17) holds for every i,

$$\begin{aligned} v^*(g) - v^*(h) &= \int u(y) g\, dm - \int u(x) h\, dm \\ &= \int u(x)\,(g-h)\,dm - \int [u(x) - u(y)]\,g\,dm \\ &\leq \int u(x)\,(g-h)\,dm - \int p(h)\cdot(x-y)\,g\,dm \qquad (20) \\ &= \int u(x)\,(g-h)\,dm - p(h)\cdot \int (x-a)\,g\,dm \\ &= \int [u(x) - p(h)\cdot(x-a)]\,(g-h)\,dm. \end{aligned}$$

Similarly,

$$v^*(h) - v^*(g) \leq \int [u(y) - p(g) \cdot (y-a)] \, (h-g) \, dm. \tag{21}$$

If $h \leq g$ then the right-hand side of (21) is nonpositive. Hence, v^* is monotonic.

Let $0 < c_0 < c_1 < \cdots$ be such that the series $\gamma = (1/c_0)\, e{\cdot}a + \sum_{s \geq 1} (1/c_s)\, \gamma_s$ converges m-almost everywhere and $\int \gamma \, dm = 1$. Let m_γ be the nonatomic probability measure defined by $dm_\gamma = \gamma \, dm$. It follows from (21) and (22) that, for some $0 \leq \theta \leq 1$,

$$v^*(g) - v^*(h) = \int \frac{1}{\gamma} \left[u(x) - p(h) \cdot (x-a) + \theta \epsilon_{g,h} \right] (g-h) \, dm_\gamma,$$

where $\epsilon_{g,h} : I \to \mathbf{R}$ is defined by $\epsilon_{g,h}(i) = \max_{\xi \in \mathbf{R}^k_+} [u(\xi, i) - p(g){\cdot}(\xi - a(i))] - \max_{\xi \in \mathbf{R}^k_+} [u(\xi, i) - p(h){\cdot}(\xi - a(i))]$. If s is a positive integer such that $p_j(g), p_j(h) > 1/s$ for all j then, as shown above, both maxima are attained in the set $\{\xi \in \mathbf{R}^k_+ | e{\cdot}\xi < \gamma_s(i)\}$. Therefore, m-almost everywhere,

$$\begin{aligned} \frac{1}{\gamma} \left| \epsilon_{g,h} \right| &\leq \frac{1}{\gamma} \max_{e \cdot \xi \leq \gamma_s} \left| (p(g) - p(h)) \cdot (\xi - a) \right| \\ &\leq \frac{1}{\gamma} (\gamma_s + e \cdot a) \max_j \left| p_j(g) - p_j(h) \right| \leq c_s \max_j \left| p_j(g) - p_j(h) \right|. \end{aligned}$$

Since, by Lemma 3, $p(\cdot)$ is continuous on $\mathcal{I}_m$, $\max_j \left| p_j(g) - p_j(h) \right| \to 0$ when $g \to h$. This proves that v^* is differentiable at h, that its derivative there is absolutely continuous with respect to m_γ, and that

$$\frac{d\,(Dv^*(h))}{dm} = \frac{1}{\gamma} \left[u(x) - p(h) \cdot (x-a) \right] \in L_\infty(m_\gamma).$$

(The essential boundedness of $(1/\gamma)\,[u(x) - p(h) \cdot (x-a)]$ follows from (19) and from the definition of γ.) Since, for every g and h as above,

$$\left\| \frac{d\,(Dv^*(g))}{dm_\gamma} - \frac{d\,(Dv^*(h))}{dm_\gamma} \right\|_{L_\infty(m_\gamma)} = \left\| \frac{1}{\gamma} \epsilon_{g,h} \right\|_{L_\infty(m_\gamma)} \leq c_s \max_j \left| p_j(g) - p_j(h) \right|,$$

the function $d\,(Dv^*(\cdot))\,/dm_\gamma$ is continuous at h.

A transferable utility competitive equilibrium $(x, p(h))$ corresponding to an ideal coalition $h \in \mathcal{I}_m$ is easily seen to correspond also to th, for every $0 < t < 1$. Therefore, $v^*(th) = tv^*(h)$ for every $0 \le t \le 1$. (Incidentally, this shows that v^* is differentiable at 0 if and only if $v \in NA$.) In particular, $\lim_{t\to 0_+} v^*(t) = 0$ and $\lim_{t\to 1_-} v^*(t) = v^*(1)$, and if $(x, p(1))$ is a transferable utility competitive equilibrium corresponding to the ideal coalition $h = 1$ then $d\left(Dv^*(t)\right)/dm = (1/\gamma)\left[u(x) - p(1)\cdot(x-a)\right]$ for every $0 < t \le 1$. It follows, by Proposition 2, that the market game v is in pNA and its value is given by

$$(\varphi v)(S) = \int_S \left[u(x) - p(1)\cdot(x-a)\right] dm \qquad (S \in \mathcal{C}).$$

Thus, the value of v is equal to the competitive payoff distribution (Aumann and Shapley, 1974, p. 184) of the market. $\square$

Appendix

Approximation lemma

The method of approximation employed in the proofs of Theorem 1 and Proposition 1 can be used more generally for approximating games in pNA_∞, pNA, or pNA'. All three spaces of games are generated by powers of nonatomic probability measures, but the norm is different in each case. The norm on pNA' is the supremum norm $\|\cdot\|'$, the norm on pNA is the variation norm $\|\cdot\|_{BV}$, and pNA_∞ is endowed with the norm $\|\cdot\|_\infty$. These norms satisfy $\|\cdot\|' \le \|\cdot\|_{BV} \le \|\cdot\|_\infty$, and the spaces themselves satisfy $pNA_\infty \subseteq pNA \subseteq pNA'$. Each of the three norms can be "extended" in a natural way to a norm on the linear space of extensions of games in the respective space. Specifically, we define $\|v^*\|' = \sup_{h\in\mathcal{I}} |v^*(h)|$, $\|v^*\|_{BV} = \sup_{0\le h_0\le\cdots\le h_n\le 1} \sum_{i=1}^{n} |v^*(h_i) - v^*(h_{i-1})|$, or $\|v^*\|_\infty = \inf\{m(I) | m \in NA$, and $|v^*(g) - v^*(h)| \le m(g-h)$ for every $g, h \in \mathcal{I}$ with $h \le g\}$ when the ideal game v^* is the continuous extension of a game v in pNA', in pNA, or in pNA_∞, respectively. The extension operator $v \mapsto v^*$ is linear and norm-preserving on each of the three spaces (Aumann and Shapley, 1974, p. 151; Hart and Monderer, 1997). If $f \circ \mu$ is a vector measure game in pNA' such that the range of μ is finite dimensional, then the continuous extension v^* of $f \circ \mu$ is given by $v^*(h) = f(\mu(h))$. This can be shown as follows. First, since the range of a finite dimensional nonatomic vector measure coincides with its extended range, for every $h \in \mathcal{I}$ we can find a coalition S such that $\mu(h) = \mu(S)$. Second, for the same reason, for every given neighborhood of h in $\mathcal{I}$ we can choose S in such a way that χ_S is in that neighborhood. Therefore, by the continuity of v^*, for every given $\epsilon > 0$ we may assume that S satisfies $|v^*(\chi_S) - v^*(h)| < \epsilon$. But $v^*(\chi_S) = f(\mu(S)) = f(\mu(h))$.

For every game v in pNA' there exists a nonatomic probability measure m such that v is in $pNA'(m)$, the closed linear subspace of pNA' that is generated by powers of nonatomic probability measures that are absolutely continuous with respect to m. (If $\{\mu^{(k)}\}_{k\geq 1}$ is a sequence of vectors of nonatomic probability measures, $\mu^{(k)} = (\mu_1^{(k)}, \mu_2^{(k)}, \ldots, \mu_{n^{(k)}}^{(k)})$, and $\{p^{(k)}\}_{k\geq 1}$ is a sequence of polynomials such that $\left\| v - p^{(k)} \circ \mu^{(k)} \right\|' \to 0$, then m can be chosen as $\sum_{k\geq 1} 2^{-k} \left(1/n^{(k)}\right) (\mu_1^{(k)} + \mu_2^{(k)} + \cdots + \mu_{n^{(k)}}^{(k)})$.) For every $g, h \in \mathcal{I}$ such that $g = h$ m-almost everywhere, $v^*(g) = v^*(h)$. Similarly, if v is in pNA or in pNA_∞ then there exists a nonatomic probability measure m such that v is in $pNA(m)$ or in $pNA_\infty(m)$, respectively. These subspaces are defined in a similar way to $pNA'(m)$. For every finite subfield $\mathcal{F}$ of $\mathcal{C}$ and for every $g \in L_1(m)$, the conditional expectation $E(g|\mathcal{F})$ is defined as that function on I which is constant on each atom S of the field $\mathcal{F}$ and is equal there to $(1/m(S)) \int_S g\, dm$ ($= 0$, by convention, if $m(S) = 0$). The ideal game $v_{\mathcal{F}}^*$ defined by $v_{\mathcal{F}}^*(h) = v^*(E(h|\mathcal{F}))$ is easily seen to be continuous. Its "restriction" to $\mathcal{C}$ is the game $v_{\mathcal{F}}$ defined by $v_{\mathcal{F}}(S) = v_{\mathcal{F}}^*(\chi_S)$.

LEMMA 4. *Let X be one of the three spaces, $pNA(m)$, $pNA_\infty(m)$, or $pNA'(m)$, and let $\|\cdot\|$ be the norm on that space. Then a game v is in X if and only if $v_{\mathcal{F}}$ is in X for every finite field $\mathcal{F} \subseteq \mathcal{C}$ and* $\lim_{\mathcal{F}} \|v_F - v\| = 0$ *as $\mathcal{F}$ varies over the finite subfields of C, directed by inclusion.*

Proof. One direction in trivial: if v is the limit of a net in X then $v \in X$. Conversely, if v is in X and $\mathcal{F}$ is a finite subfield of $\mathcal{C}$ then it is not too difficult to see that $\|v_{\mathcal{F}}\| \leq \|v^*\|$ ($= \|v\|$). Hence, for every game u that is a linear combination of powers of nonatomic probability measures that are absolutely continuous with respect to m, $\|v_{\mathcal{F}} - u_{\mathcal{F}}\| \leq \|v - u\|$. Therefore, it suffices to prove that $u_{\mathcal{F}} \in X$ and $\lim_{\mathcal{F}} \|u_{\mathcal{F}} - u\| = 0$ for every such u. Consider, then, a game of the form η^k, where η is a nonatomic probability measure that is absolutely continuous with respect to m and k is a positive integer.

For every $\mathcal{F}$, $\eta_{\mathcal{F}}$ is a nonatomic probability measure that is absolutely continuous with respect to m. In fact, $d\eta_{\mathcal{F}}/dm = E(d\eta/dm|\mathcal{F})$. Therefore, $(\eta^k)_{\mathcal{F}} = (\eta_{\mathcal{F}})^k \in X$. Since $\|uw\|_\infty \leq 4\,\|u\|_\infty\,\|w\|_\infty$ for every $u, w \in pNA_\infty$ and $\|\lambda\|_\infty \leq 2\,\|\lambda\|_{BV}$ for every $\lambda \in NA$ (Monderer and Neyman, 1988), $\left\|\eta_{\mathcal{F}}^k - \eta^k\right\| \leq \left\|\eta_{\mathcal{F}}^k - \eta^k\right\|_\infty = \left\|\sum_{l=0}^{k-1} \eta_{\mathcal{F}}^{k-l-1}\eta^l\,(\eta_{\mathcal{F}} - \eta)\right\|_\infty \leq \sum_{l=0}^{k-1} 4^k\,\|\eta_{\mathcal{F}}\|_\infty^{k-l-1}\,\|\eta\|_\infty^l\,\|\eta_{\mathcal{F}} - \eta\|_{BV} = k4^k\,\|E(d\eta/dm|\mathcal{F}) - d\eta/dm\|_{L_1(m)}$. By Theorem IV.8.18 of Dunford and Schwartz (1958), the limit of the last expression as $\mathcal{F}$ varies over the finite subfields of $\mathcal{C}$, directed by inclusion, is zero. $\square$

Topological lemma

The relative weak topology on the extended range of a nonatomic vector measure of bounded variation μ is the strongest topology on that set with respect to which $\mu(\cdot)$ is continuous on $\mathcal{I}$. This result constitutes the first part of the following lemma.

LEMMA 5. *If μ is a nonatomic vector measure of bounded variation with values in a Banach space X, then a set $A \subseteq \mu(\mathcal{I})$ is open with respect to the relative weak topology on $\mu(\mathcal{I})$ if and only if $\{h \in \mathcal{I}|\mu(h) \in A\}$ is open (with respect to the NA-topology on $\mathcal{I}$). In this case, a function f from A to some topological space Y is weakly continuous if and only if $f(\mu(\cdot)) : \{h \in \mathcal{I}|\mu(h) \in A\} \to Y$ is continuous.*

Proof. We have to show that if $h_\alpha \to h$ is a converging net in $\mathcal{I}$ then $\mu(h_\alpha) \to \mu(h)$ weakly, that is, $x^*(\mu(h_\alpha)) \to x^*(\mu(h))$ for every continuous linear functional $x^* \in X^*$. But this follows immediately from the fact that $x^* \circ \mu \in NA$. Conversely, we have to show that if $x_\alpha \to x$ is a weakly converging net in $\mu(\mathcal{I})$ then there exists $h \in \mathcal{I}$ such that $\mu(h) = x$ and in every neighborhood of h there is some h' such that $\mu(h') \in \{x_\alpha\}$. Let $\{h_\alpha\} \subseteq \mathcal{I}$ be such that $\mu(h_\alpha) = x_\alpha$ for every α. It follows from Alaoglu's theorem that, by passing to a subnet if necessary, we may assume that there is some $h \in \mathcal{I}$ such that $m(h_\alpha) \to m(h)$ for every $m \in NA$ that is absolutely continuous with respect to $|\mu|$. In particular, $x^*(x_\alpha) = x^*(\mu(h_\alpha)) \to x^*(\mu(h))$, and therefore $x^*(\mu(h)) = x^*(x)$, for every $x^* \in X^*$. This proves that $\mu(h) = x$. Every neighborhood of h contains an open neighborhood of the form $\{g \in \mathcal{I}|\max_{1\leq i\leq l}|m_i(g-h)| < \epsilon\}$, where $\epsilon > 0$, $m_1, m_2, \ldots, m_k \in NA$ are absolutely continuous with respect to $|\mu|$, and $m_{k+1}, \ldots, m_l \in NA$

are singular with respect to $|\mu|$. This follows from the fact that every nonatomic measure can be written as the sum of two nonatomic measures, one absolutely continuous with respect to $|\mu|$ and the other singular with respect to $|\mu|$. Let α be such that $|m_i(h_\alpha - h)| < \epsilon$ for every $i \leq k$. Let $h' \in \mathcal{I}$ be equal to h in some subset of I of $|\mu|$-measure zero in which $m_{k+1}, \ldots, m_l$ are supported and equal to h_α elsewhere. Then $\mu(h') = \mu(h_\alpha) = x_\alpha$, and $|m_i(h' - h)| < \epsilon$ for every $k < i \leq l$ as well as for every $i \leq k$.

The second part of the lemma follows from that fact that, for every set B in Y, $f^{-1}(B)$ is weakly open in $\mu(\mathcal{I})$ if and only if $\{h \in \mathcal{I} | \mu(h) \in f^{-1}(B)\}$ is open. $\square$

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