Social Optimality and Cooperation in Nonatomic Congestion Games[1]

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Congestion externalities may result in non-optimal equilibria. For these to occur, it suffices that facilities differ in their fixed utilities or costs. As this paper shows, the only case in which equilibria are always socially optimal, regardless of the fixed components, in that in which the costs increase logarithmically with the size of the set of users. Therefore, achieving a socially optimal choice of facilities generally requires some form of external intervention or cooperation. For heterogeneous populations (in which the fixed utilities or costs vary across users as well as across facilities), this raises the question of utility or cost sharing. The sharing rule proposed in this paper is the Harsanyi transferable-utility value of the game—which is based on the users' marginal contributions to the bargaining power of coalitions. *Journal of Economic Literature Classification Numbers*: C71, C72, D62.

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INTRODUCTION

Nonatomic strategic (or noncooperative) games ([20]) model interactions involving a large number of individuals, each with a negligible ability to affect the others. Only coalitions, comprising many individuals, can significantly affect the utility of those outside them. Congestion games ([14], [17]) are strategic games in which the contribution to a player's utility from choosing a particular action or facility varies with the number of other players making the same choice. Such games aim at modeling the congestion externalities occurring in many real-life situations in which several, or many, independent decision-makers interact by using the same facilities. In the nonatomic congestion games studied in this paper, each player only chooses one action or facility, and his utility strictly decreases as the size of the set of other players choosing the same action or facility increases. Different players do not necessarily achieve the same utility when making the same choice. In this respect, the population of players is heterogeneous. However, the manner in which utility decreases with increased congestion is the same for all. The following example (adapted from [4]) illustrates these assumptions. People in a professional meeting may prefer to go to different sessions, since the intrinsic quality they assign to each session varies. As more people crowd into a room, it becomes more difficult to see and hear. The actual quality each person assigns to a session therefore depends on both the intrinsic quality and the number of other people present. The delays experienced by clients in a computer network when many of them simultaneously try to access the same server are another example.

In nonatomic congestion games of the kind considered here, the equilibrium payoffs are always unique. Moreover, the equilibria are Pareto efficient in the sense that it is not possible to modify an equilibrium in such a way that some players become better off without making some of the others worse off. However, the equilibria need not be socially optimal. That is, they may all be inferior to some non-equilibrium assignment of facilities in terms of the aggregate, or equivalently average, utility or cost. (Note that, unlike Pareto efficiency, the notion of social optimality involves interpersonal comparisons of utilities.) For the equilibria to maximize social welfare, a player's utility from choosing a facility should reflect the external effects of his choice on the other players. In other words, the cost or benefit for the individual must mirror the social cost or benefit. The first main result of this paper is that, under the assumption that the marginal social costs of congestion are increasing and there are at least three facilities, a necessary and sufficient condition for always reaching maximum aggregate utility at the equilibria of the game is that the players' utility from choosing a particular facility decreases logarithmically as the size of the set of other players choosing the same facility increases. For non-logarithmic cost functions, there are always fixed utilities or costs for which none of the equilibria is socially optimal. If there are only two facilities, the class of cost functions for which equilibria are always socially optimal is somewhat larger.

An alternative way of demonstrating the connection between social optimality of the equilibria and a logarithmic relation between congestion and utility is the so-called potential approach. For a given nonatomic congestion game, there is always some function attaining its maximum at the equilibria of the game. Indeed, there are always cost functions such that the aggregate utility with respect to them is maximized at the equilibria of the original game. If, up to an additive constant, these cost functions are equal to the original ones, then, clearly, all the equilibria in the original game are socially optimal. For logarithmic congestion externalities, this is, indeed, the case.

The result that social optimality of the equilibria is guaranteed if and only if the cost functions are logarithmic may very well be interpreted as a negative one. Unless the congestion externalities have this special form, maximum aggregate utility or minimum aggregate cost cannot generally be achieved without some form of external intervention or, alternatively, cooperation among the players. This raises the question of how this utility or cost should be shared among them. This may be viewed either in a normative light, as a question of each player's "appropriate" or "just" share, or in a positive light, as a question of the likely outcome of negotiations among the players. The utility or cost-sharing rule proposed in this paper is derived from a very general solution concept, the Harsanyi transferable-utility value of a strategic game ([2], [9], [21]). This solution concept is based on the players' marginal contributions to the bargaining power of various coalitions, each bargaining with its complement about its share of the maximum aggregate utility. More specifically, the Harsanyi TU value is defined as the Aumann-Shapley value of the coalitional game in which the worth of each coalition and that of its complement add up to the maximum aggregate utility in the strategic game, and are determined as Nash's solution to the corresponding bargaining problem with threats. The second main result of this paper is that, if the marginal social costs of congestion are increasing and the cost functions themselves are not "too" convex, then optimal threat strategies for this bargaining problem exist and the resulting coalitional game has a well-defined Aumann-Shapley value. The Harsanyi TU value of the game can then be expressed by an explicit formula, which specifies the share of the maximum aggregate utility that should be allocated to each group of players.

THREE MOTIVATING EXAMPLES

An example of how congestion externalities may lead to an inefficient use of facilities is shown in Fig. 1. It involves a simple network of four arcs, with cost functions giving the time it takes to traverse each arc as a function of the flow through it. A continuum of identical players simultaneously choose one route leading from the common point of origin *o* to the common destination *d*. Each player's goal is to find the fastest such route. If all players succeed in doing this, their choice of routes constitutes an equilibrium. Since, in this example, choosing the left route from *o* to *d*, comprising the two left arcs, is a dominant strategy, every player making this choice is the unique equilibrium. However, this equilibrium is not <u>Pareto efficient</u>. If half the players were taking the left arc from *o* to the intermediate point *v* and then the right arc from there to *d*, and the other half were taking right, then left, <u>everyone's</u> travel time would go down from 10 minutes to 9.5. This is still not <u>socially optimal</u>; it is possible to reduce the players' <u>average</u> travel time even further. Indeed, if 0.7 of the players take the equilibrium (i.e., left) route and 0.3 take the right one, the average travel time is only 9.1 minutes. This <u>is</u> a social optimum, since the minimum of $x_1 (1 + 4x_1) + x_2 (6 + x_2)$ under the constraint $x_1 + x_2 = 1$ is attained at $x_1 = 0.7$, $x_2 = 0.3$. Note that, at a social optimum, some players are <u>worse off</u> than at the equilibrium. In fact, the travel time of any player not taking the equilibrium route is at least 10.1 minutes. Whereas, by definition, a Pareto improvement is beneficial to <u>all</u> the players, moving from an equilibrium to a social optimum may be beneficial only <u>on average</u>. Any socially optimal equilibrium (or other arrangement) is Pareto efficient, but the converse is not true.

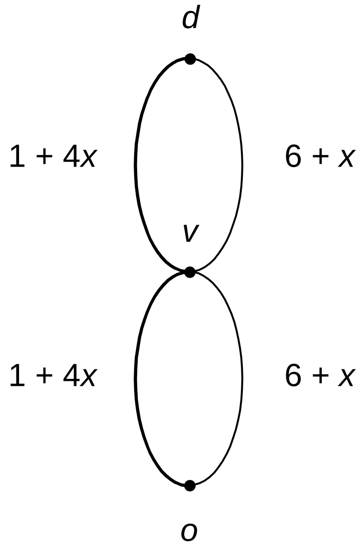

Fig. 1. A Pareto inefficient equilibrium. The time (in minutes) it takes to transverse each of the four arcs is shown as a function of the fraction of the population *x* whose route from *o* to *d* passes through the arc. At equilibrium, all the players take the route marked in bold, which takes 10 minutes to complete. However, this is not Pareto efficient. If half the players were taking the left–right route and the other half the right–left one, each play's travel time would be only 9.5 minutes.

An example in which the equilibrium is Pareto efficient but not socially optimal is shown in Fig. 2. In this example, the population of players is heterogeneous. For three-quarters of the population (type I players), the left route from *o* to *d* is faster than the right one when both routes are equally congested. For the rest (type II players), the opposite is true. (If the two routes represent, for example, two parallel bridges over a river, the different travel times may reflect the distance each user has to travel to get to the bridge.) When all the type I players take the left route and all the type II players take the right one, their travel times are 7 and 3 minutes, respectively. For each player, this is less than the travel time on the alternative route—which is greater than 7 minutes as long as that route is used by some other players. This shows both that the above arrangement is an equilibrium and that it is Pareto efficient. However, it is not socially optimal. To achieve the social optimum, 1/16 of the population (all type I) should shift from the left to the right route. This reduces the average travel time from 6 minutes to 5.9375. The social optimum is not an equilibrium, since the travel time of type I players taking the right route is 3 minutes longer than of identical players taking the left route.

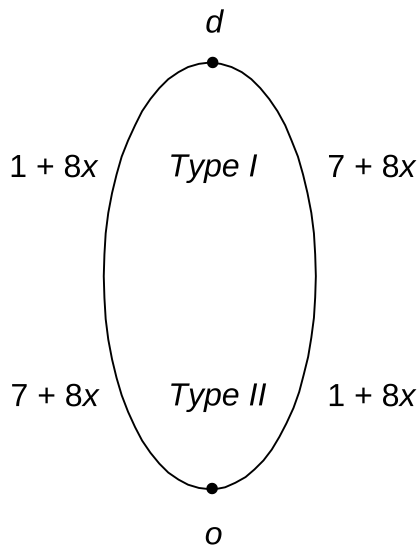

Fig. 2. A socially non-optimal equilibrium. The travel times of type I and type II players (3/4 and 1/4 of the population, respectively) on each route from *o* to *d* are shown as functions of the fraction of the population *x* taking the route. At equilibrium, each type of player takes a different route. This is not socially optimal: The players' average travel time at equilibrium is 1/16 of a minute longer than the minimum average travel time.

Non-optimality of the equilibrium in the last example can be attributed to overuse of the left route (by type I players). It is possible to reduce the use of that route to the socially optimal level by charging a toll equivalent to 3 minutes of travel time for the use of the left route. This toll and the resulting increased congestion in the right route would make everyone worse off in comparison with the equilibrium. However, if toll revenues are returned to the players, for instance, in the form of a lump sum transfer to each player, then the toll's net effect is positive, at least on average. If the population of players were homogeneous (as it is in the first example), there would be little question that the transfers to players should be equal, and bring their net costs to the social optimum level. However, in the present heterogeneous case, the question of whether and to what extent the transfers to type I and type II players should differ arises.

There are a number of alternative principles that may be evoked to answer this question, each prescribing different lump sum transfers to players and different net costs. Some of the possibilities are shown in Table 1. One option is simply equal net costs to all players, regardless of type. However, this option, which involves negative transfers to type II players, may be difficult to justify in view of these players' much lower equilibrium costs. Another option is equal transfers, regardless of type. This has the opposite effect of favoring type II players to a degree that may be hard to justify. A third conceivable principle is that everybody should be equally better off compared with the equilibrium, i.e., equal distribution of the surplus arising from the shift from the (non-cooperative) equilibrium to the social optimum. A possible objection to this arrangement is that type II players, who have no active role in this shift, would benefit from it as mush as type I players. A fourth scheme for sharing the gains from shifting to the social optimum among the players, described later in this paper, is the Harsanyi transferable-utility value of the nonatomic congestion game. In the example at hand, this rule prescribes giving most of the toll revenues to the type I players and much less to type II, so that the former are better off while the latter are worse off compared to the equilibrium. This reflects the payoffs, at the social optimum, of those players still using their equilibrium strategies.

	Type I	Type II
Equilibrium	7	3
With toll (but no transfers)	9.5	3.5
Equal net costs	5.9375	5.9375
Equal transfers	7.4375	1.4375
Equal distribution of the surplus	6.9375	2.9375
Harsanyi TU value	6.8750	3.1250

Table 1. Schemes of cost sharing. For the example in Fig. 2, net costs (in minutes of travel time) of type I and type II players are shown: at equilibrium; with an optimal toll charged for the use of the left route, but without returning toll revenues to players; and with the toll revenues distributed according to each of the four alternatives described in the text.

An obvious difference between the networks in Figs. 1 and 2 is that, in the latter, different routes do not cross. Therefore, the negative externalities of the players' choice of routes only affect those making the same choice. The fact that the equilibrium in the second example, but not in the first, is Pareto efficient can be attributed to this difference in network topology. Specifically, the class of all two-terminal networks in which, regardless of the cost functions, all equilibria are Pareto efficient includes (but is not limited to) the networks with parallel routes. A complete characterization of this class of networks is given in [13, Theorem 3]. As the first example makes clear, the network in Fig. 1 does not belong to this class. The nonatomic congestion games studied in this paper are such that each player's payoff is only affected by the measure of the set of players whose choice of action or facility is the same as his. This corresponds to a network with parallel routes and excludes the one in Fig. 1. Therefore, for these games, Pareto efficiency of the equilibria is guaranteed. The only issue is their social optimality.

As a final example, consider the situation in Fig. 3. Here, for each type of player, choosing each of the facilities brings a certain utility (that may be positive or negative), which depends on the size of the set of other players making the same choice. As the size of this set increases, the utility decreases. When it tends to zero, the utility tends to infinity. At equilibrium, 3/7 of the players, all type I, choose facility 1, and the rest choose facility 2. The payoff of type I players choosing either facility is then log 7/8, and that of type II is log 35/32. To find the social optimum, observe, first, that increasing the utility of type II players choosing facility 1 by a positive constant ε can only increase the maximum average utility, or leave it without a change. Setting $\varepsilon = \log 15/8$ makes the difference between the utility of type II and type I players choosing the same facility equal to the constant log 5/4. Therefore, with this ε, the maximum average utility is given by $\max_{0 \le x \le 1} [x (\log 3 - \log 8x) + (1 - x) (\log 4 - \log 8(1 - x))] + 1/4 \log 5/4$. This maximum (which is attained at $x = 3/7$) can easily be shown to be equal to the average utility at equilibrium in the original game. This proves that the equilibrium in that game is socially optimal. As it turns out, this finding is not a coincidence. It is shown below that, when the cost of congestion is given by a logarithmic function, the equilibria are always socially optimal.

	Facility 1	Facility 2
Type I	$\log 3 - \log 8x$	$\log 4 - \log 8x$
Type II	$\log 2 - \log 8x$	$\log 5 - \log 8x$

Fig. 3. A socially optimal equilibrium. The utility of type I and type II players (3/4 and 1/4 of the population, respectively) from choosing each of the two facilities is shown as a function of the fraction of the population x choosing that facility. At equilibrium, 3/7 of the players, all type I, choose facility 1, and the rest choose facility 2. This equilibrium is socially optimal.

The Setup

An infinite population I of players uses a finite number m of facilities. A nonatomic probability measure μ, the population measure, is defined on a σ-algebra $\mathcal{C}$ of subsets of I. The elements of $\mathcal{C}$ are called coalitions. For each coalition S, $\mu(S)$ is interpreted as the "size" of S (with the normalization $\mu(I) = 1$). Each player uses one, and only one, facility. The externality generated by the fact that the same facility j is shared by others is captured by a continuous and strictly increasing cost function $c_j : (0, \infty) \to \mathbb{R}$, with $\lim_{x\to 0} x\, c_j(x) = 0$.[3] If the size of the set of players using facility j is $x_j > 0$, the cost of congestion for each of them is $c_j(x_j)$. For different facilities j, the cost of congestion may take different functional forms. This reflects the fact that certain roads, for example, are more easily congested than others. The social cost of congestion is $\sum_j x_j\, c_j(x_j)$. Marginal social costs of congestion are said to be increasing if, for all facilities j, the derivative

$$MC_j(x) \overset{def}{=} \frac{d}{dx}\,[x\, c_j(x)] \qquad (1)$$

exists and is strictly increasing in $(0, \infty)$. It can be shown that, in this case, each cost function c_j is continuously differentiable in $(0, \infty)$, satisfies $dc_j/dx > 0$, and has a

[3] Note that the cost function c_j may take negative as well as positive values. Also, for technical convenience, the cost function is assumed to be defined for values of x_j both above and below unity. However, the values that c_j takes for $x_j > 1$ have no effect on the actual costs.

second derivative almost everywhere (with respect to Lebesgue measure); and each marginal social cost function MC_j is continuous in $(0, \infty)$ and (since $MC_j > c_j$) satisfies $\lim_{x \to 0} x\, MC_j(x) = 0$. The cost and (when defined and increasing) marginal social cost functions can be extended to continuous functions on $\mathbb{R}_+$ by setting $c_j(0) = \lim_{x \to 0} c_j(x)$ and $MC_j(0) = \lim_{x \to 0} MC_j(x)$. These two limits are, in fact, equal. They may, however, be $-\infty$. (In this paper, standard rules for the arithmetic of extended real numbers, e.g., $\pm\infty \cdot x = \pm\infty$ if $x > 0$, $= \mp\infty$ if $x < 0$, and $= 0$ if $x = 0$, are used.) For $(x_1, x_2, \dots, x_m) \in \mathbb{R}_+^m$, the vectors $(c_1(x_1), c_2(x_2), \dots, c_m(x_m))$ and $(MC_1(x_1), MC_2(x_2), \dots, MC_m(x_m))$ are written as $c(x_1, x_2, \dots, x_m)$ and $MC(x_1, x_2, \dots, x_m)$, respectively. A similar notation is also used for other vector-valued functions.

The utility each player i achieves is made up of two terms:

$$f_j(i) - c_j(x_j).$$

The first term $f_j(i)$ is the fixed utility player i gains from the facility j he uses. This does not depend on the other players' choices of facility, and may be positive or negative. In the latter case, it may be interpreted as a fixed cost. The second term $c_j(x_j)$ is the cost of congestion. Note that the heterogeneity of the population is assumed to involve only the fixed utility (or cost) and not the cost of congestion (the variable cost). For a more general model, in which different players may be affected to a different degree by congestion, see [12]. The fixed-utility assignment $f : I \to \mathbb{R}^m$, defined by $f(i) = (f_1(i), f_2(i), \dots, f_m(i))$, is assumed to be bounded and measurable (with respect to $\mathcal{C}$).

A (pure-) strategy profile is any measurable function $\sigma : I \to \{0, 1\}^m$ assigning each player i a binary vector $\sigma(i) = (\sigma_1(i), \sigma_2(i), \dots, \sigma_m(i))$ such that $\sigma_j(i)$ is 1 for some facility j and 0 for all the others. A value of 1 indicates that player i uses facility j. The size of the set of players using facility j equals the integral $\int \sigma_j(i)\, d\mu(i)$, henceforth written as $\mu(\sigma_j)$. The set of all strategy profiles is denoted by Σ. For a given strategy profile σ, the utility each player i achieves can be written as

$$u_i(\sigma) \overset{def}{=} (f(i) - c(\mu(\sigma))) \cdot \sigma(i),$$

where $\mu(\sigma)$ is the vector $(\mu(\sigma_1), \mu(\sigma_2), \dots, \mu(\sigma_m))$ and the dot denotes scalar product. For given cost functions c and fixed-utility assignment f, this defines a nonatomic congestion game $\Gamma(c, f)$, with utility functions u_i. A strategy profile σ is a (pure-strategy Nash) equilibrium in $\Gamma(c, f)$ if, for (μ-) almost all players i,

$$u_i(\sigma) = \max_j \,(f_j(i) - c_j(\mu(\sigma_j))). \tag{2}$$

In this case, the expression on the right-hand side of (2) gives player i's equilibrium payoff. The indefinite integral[4] (with respect to μ) of this expression is the equilibrium payoff distribution.

A strategy profile σ is Pareto efficient if, for every strategy profile τ such that $u_i(\tau) \geq u_i(\sigma)$ for almost all players i, an equality holds for almost all i. A strategy profile σ will be said to be hyper-efficient if it satisfies the following stronger condition:

(H) For every $\tau \in \Sigma$, if $u_i(\tau) \geq u_i(\sigma)$ for almost all players i with $\tau(i) \neq \sigma(i)$, then $u_i(\tau) = u_i(\sigma)$ for almost all i.

In other words, a strategy profile is hyper-efficient if any effective change of strategies is harmful to some of those whose strategies change.[5] The assumed continuity of the cost functions implies that such a strategy profile is an equilibrium. Indeed, it is a strong, and even strictly strong, equilibrium.[6] This means that deviations are unprofitable, not just for individuals but also for groups of players, or coalitions: Any deviation that makes some members of the coalition better off must leave some of the others worse off. From a social-welfare point of view, a strategy profile σ is socially optimal, or welfare maximizing, in $\Gamma(c, f)$ if it maximizes the aggregate utility, i.e., for all $\tau \in \Sigma$,

$$\int_I u_i(\tau)\, d\mu(i) \leq \int_I u_i(\sigma)\, d\mu(i). \tag{3}$$

Social optimality implies Pareto efficiency. However, it does not imply hyper-efficiency, or vice versa.

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[4] The indefinite integral of an integrable function $g : I \to \mathbb{R}$ is the measure η defined by $\eta(S) = \int_S g\, d\mu$ ($S \in \mathcal{C}$).

[5] Note that, in contrast to Pareto efficiency, the definition of hyper-efficiency involves both the players' utilities and their strategies. The term "hyper-efficiency," generalized in a straightforward manner, is also applicable to other situations in which utilities are determined by some map on the space of players, such as a strategy profile or an allocation of goods.

[6] A strategy profile σ is a strictly strong equilibrium ([23]) if the following is true: For every strategy profile τ, if $u_i(\tau) \geq u_i(\sigma)$ for almost all players i with $\tau(i) \neq \sigma(i)$, then $u_i(\tau) \leq u_i(\sigma)$ for almost all i.

EXISTENCE AND SOCIAL OPTIMALITY OF EQUILIBRIA

The existence of equilibrium in all nonatomic congestion games in the class considered in this paper is an immediate corollary of [12, Theorem 3.1].

Proposition 1. For every c and f, the nonatomic congestion game $\Gamma(c, f)$ has at least one equilibrium.

It can be shown (cf. [12]) that, in some precise sense, the equilibrium is generically unique. However, for present purposes, it suffices to establish the uniqueness of the equilibrium payoffs. The proof of the following proposition is given in Appendix B.

Proposition 2. For every c and f, a strategy profile is an equilibrium in $\Gamma(c, f)$ if and only if it is hyper-efficient (i.e., has the property H). For each facility j, the measure of the set of players using j is the same in all the equilibria in $\Gamma(c, f)$. Consequently, the equilibrium payoffs are unique.

By Proposition 2, in all nonatomic congestion games, all equilibria are hyper-, and hence Pareto, efficient. (As already mentioned, this result is, in fact, true in a much larger class of nonatomic congestion games than the one considered here. See [13, Theorem 3].)[7] The equilibria need not, however, be socially optimal. This is because players choosing their facilities do not take into consideration the negative external effects of their choice on the other players. As is well known, to guarantee social optimality, players should bear, not the cost c_j of using facility j, but rather the marginal social cost MC_j. The following proposition establishes and extends this fact. The proof of the proposition is given in Appendix B.

Proposition 3. Suppose that the marginal social costs of congestion are increasing. Then, for every fixed-utility assignment f, the nonatomic congestion game $\Gamma(c, f)$ has at least one socially optimal strategy profile. Moreover, the set of all socially optimal strategy profiles coincides with the set of equilibria in $\Gamma(MC, f)$. Either it also coincides with the set of equilibria in the original game $\Gamma(c, f)$, or the two sets are disjoint.

[7] It can easily be shown that, if there were only a few, non-identical, players, the equilibria would not have to be Pareto efficient. In this respect, finite congestion games and nonatomic ones differ significantly.

It is not difficult to see that, for an arbitrary socially optimal strategy profile σ, it is possible to replace the marginal social cost functions MC in Proposition 3 by any vector of cost functions $\tilde{c}$ satisfying $\tilde{c}_j(\mu(\sigma_j)) = MC_j(\mu(\sigma_j))$ for all j. (If this condition holds, then σ, which by Proposition 3 is an equilibrium in $\Gamma(MC, f)$, is also an equilibrium in $\Gamma(\tilde{c}, f)$, and it therefore follows from Proposition 2 that the sets of equilibria in these two games coincide.) One cost function satisfying this is given by $\tilde{c}_j(x) = c_j(x) + \mu(\sigma_j)\, dc_j/dx(\mu(\sigma_j))$. (If $\mu(\sigma_j) = 0$, the second term is understood as 0.) This observation establishes the following result.

Proposition 4. Suppose that the marginal social costs of congestion are increasing. Then, for every fixed-utility assignment f, there is a nonnegative vector $w \in \mathbb{R}_+^m$ such that a strategy profile is socially optimal in $\Gamma(c, f)$ if and only if it is an equilibrium in $\Gamma(c + w, f)$.

The vector w may be interpreted as follows: Its jth component w_j is a (Pigouvian) toll charged for the use of facility j.[8] Proposition 4 thus asserts that, with increasing marginal social costs of congestion, there is always a toll system guaranteeing socially optimal use of the facilities.[9] Clearly, for every $\varepsilon > 0$, subtracting ε from all the components of w does not change any of the players' behavior. Therefore, it is always possible to maximize social welfare and run a balanced budget by implementing a system of tolls and subsidies which, at equilibrium, cancel out one another. It also follows from these considerations that if for some (and, hence, every) socially optimal strategy profile σ the product $\mu(\sigma_j)\, dc_j/dx(\mu(\sigma_j))$ has the same value for all j, then no tolls or subsidies are required since all the equilibria are automatically socially optimal. If the cost functions (up to arbitrary additive constants) are logarithmic with a common base $a > 1$, then this condition clearly holds, and hence social optimality of the equilibria is guaranteed for every fixed-utility assignment f. The following

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[8] The idea of imposing tolls in order to increase social welfare was first proposed by Pigou. See also Knight's [10] discussion of it, and the much more detailed analysis in [5].

[9] This conclusion depends critically on the infiniteness of the set of players. With a finite number of players, for any toll system there may be at least one equilibrium that is not even Pareto efficient (see [18]). The difference in this respect between the finite- and infinite-player cases may be due to the effective discontinuity of the cost functions in the former case.

theorem shows that, if there are three or more facilities, then this is, in fact, the only case in which the equilibria are guaranteed to be socially optimal. Note that the proof of the theorem, which is given in Appendix B, does not rely on the potential heterogeneity of the population. Therefore, the theorem would also be true if, in the definition of nonatomic congestion game, players were assumed to be identical (i.e., only constant fixed-utility assignments were allowed).

Theorem 1. Suppose that the marginal social costs of congestion are increasing. If $m \geq 3$, then the following three conditions are equivalent:

(i) For every fixed-utility assignment f, the set of all socially optimal strategy profiles in $\Gamma(c, f)$ coincides with the set of equilibria in this game.[10]

(ii) For every strictly positive probability vector $(x_1, x_2, \ldots, x_m)$ (with $x_j > 0$ for all j and $\sum_j x_j = 1$), $x_j \dfrac{dc_j}{dx}(x_j) = x_k \dfrac{dc_k}{dx}(x_k)$ for all j and k.

(iii) For some $a > 1$, $c_j(x) = \log_a x + c_j(1)$ for all j and $0 < x < 1$.

If $m = 2$, then (i) and (ii) are still equivalent, and are implied by (iii), but the reverse implication need not hold.

As the last part of Theorem 1 asserts, if there are only two facilities, there exist certain non-logarithmic cost functions (which are, however, similar in some respect to the logarithmic functions; see the proof of Theorem 1) that satisfy the condition of increasing marginal social costs of congestion, for which equilibria are always socially optimal. For such cost functions, condition (ii) in the theorem holds. The following cost functions are an example of this:

$$(4) \qquad c_1(x) = \int_1^x \frac{1}{t} e^{\arctan(t-1/2)} dt \quad \text{and} \quad c_2(x) = \int_1^x \frac{1}{t} e^{\arctan(1/2-t)} dt \,.$$

Even with more than two facilities, there are certain non-logarithmic cost functions c for which social optimality of the equilibria in $\Gamma(c, f)$ holds for some fixed-utility assignments f. For example, if all the players are identical, with $f = 0$, then this is the case when the cost functions are homogeneous of the same degree, i.e., for some

[10] By Proposition 3, this is equivalent to social optimality of some equilibrium in $\Gamma(c, f)$.

$-1 < \beta \neq 0$, $c_j(x) = c_j(1)\, x^\beta$ for all j (see [4]). This is because, for such cost functions, the marginal social costs are proportional to the respective costs, and therefore the sets of equilibria in $\Gamma(c, 0)$ and $\Gamma(MC, 0)$ coincide. However, with more than two facilities, non-logarithmic cost functions cannot guarantee social optimality of the equilibria for all fixed-utility assignments.

THE POTENTIAL

By Proposition 3, if the marginal social costs of congestion are increasing, then, for every fixed-utility assignment f, the set of all socially optimal strategy profiles in $\Gamma(c, f)$ coincides with the set of equilibria in another nonatomic congestion game, namely, $\Gamma(MC, f)$. The same result implies that the set of equilibria in $\Gamma(c, f)$ coincides with the set of all socially optimal strategy profiles in any nonatomic congestion game $\Gamma(\hat{c}, f)$ such that the marginal social cost functions with respect to $\hat{c}$ are given by c, i.e., $M\hat{C}_j = c_j$ for all j (in which case the marginal social costs are automatically increasing, since c_j is increasing by definition of cost function). This condition is clearly satisfied by the following cost function:

$$\hat{c}_j(x) = \frac{1}{x} \int_0^x c_j(t)\, dt.$$

Aggregate utility in $\Gamma(\hat{c}, f)$ is given by the function $P : \Sigma \to \mathbb{R}$ defined by

$$P(\sigma) = \int_I f(i) \cdot \sigma(i)\, d\mu(i) - \sum_j \int_0^{\mu(\sigma_j)} c_j(x)\, dx.$$

Therefore, as a corollary of Proposition 3, we get the following result.

Proposition 5. Suppose that, for all facilities j, the integral $\int_0^1 c_j(x)\, dx$ is finite. Then, for every fixed-utility assignment f, a strategy profile σ is an equilibrium in $\Gamma(c, f)$ if and only if it maximizes P, i.e.,

$$P(\sigma) = \max_{\tau \in \Sigma} P(\tau).$$

In the transportation literature, the fact that the equilibrium assignment problem can be formulated as a maximization problem is well known (see, e.g., [22, p. 59]). The original formulation, in the case of a homogeneous population of players, is due to

Beckmann *et al.* [5]. As these authors remark, the function P is not to be interpreted as the consumers' surplus. This is because c_j is the average, rather than marginal, social cost. P can be interpreted as a potential for $\Gamma(c, f)$. Recall that, in a finite-player congestion game ([7], [14], [17]), the potential is defined as any real-valued function over the set of strategy profiles with the property that, for all strategy profiles and all single-player deviations from them, the gain or loss for the deviator equals the corresponding change in the potential. Any strategy profile maximizing the potential is clearly an equilibrium, and for certain finite-player congestion games with concave potentials ([16]), as well as certain symmetric ones ([23]), the converse is also known to be true. In the present infinite-player model, the potential P has similar properties. Intuitively, when a single player switches from one facility to another, the change in that player's utility has the same sign as the corresponding infinitesimal change in P. This provides an intuition for the "if" part of Proposition 5. The "only if" part can be demonstrated by a simple concavity argument. Proposition 5 can be used for giving an alternative proof for the existence of an equilibrium and the uniqueness of the equilibrium payoffs in $\Gamma(c, f)$. Indeed, the potential approach can also be used to establish these properties in more general models, in which each player chooses a combinations of facilities, rather than a single one (e.g., a number of road segments, constituting a particular route from his point of origin to the destination). See, e.g., [1].

Proposition 5 also sheds some new light on Theorem 1. Suppose that, for some constant b, $c_j(x) = \hat{c}_j(x) + b$ for all j and $0 < x < 1$. Then, for every strategy profile σ, aggregate utility in $\Gamma(c, f)$ is equal to $P(\sigma) + b$. Therefore, a strategy profile maximizes the aggregate utility if and only if it maximizes the potential, and hence, by Proposition 5, if and only if it is an equilibrium. This shows that a sufficient condition for (i) in Theorem 1 to hold is that a constant b as above exists. This condition can easily be shown to be equivalent to (iii) in that theorem.

COOPERATION

It follows from Theorem 1 (and the uniqueness of the equilibrium payoffs) that, in many nonatomic congestion games of the type considered here, none of the equilibria is socially optimal. To maximize social welfare in such games, outside intervention, or alternatively cooperation among the players, is required. This may, for example, take the concrete form of a toll system, since, by Proposition 4, social optimality can always be achieved by charging suitable tolls for the use of certain facilities. However, regardless of the way it is achieved, social optimality generally requires that some players choose their facilities in an individually non-optimal way, relative to the original fixed utilities and cost functions. Arguably, these players should be compensated, e.g., by transferring to them some toll revenues. Thus, achieving maximum aggregate utility or minimum aggregate cost involves both a mechanism and a predetermined rule for sharing this utility or cost among the players.

If all players are identical, and their fates vary only because social good dictates that some of them make different choices than others, then arguably the maximum aggregate utility should be shared equally among them. However, in a heterogeneous population, in which players' innate preferences differ, there is much less basis for arguing that everybody should be treated equally. For, even in the absence of externalities, players would differ in the choices they make and the utility they achieve. One alternative to equal distribution of the aggregate utility is equal distribution of the surplus. According to this alternative, all players' shares of the maximum aggregate utility should be higher than their equilibrium payoffs by the same amount. However, there are arguments against this idea, too. For example, if the players' contributions to achieving the social optimum differ, then it is not clear why their gains from it should be equal. Suppose, for example, that the population of players is made up of several sub-populations, favoring and using disjoint sets of facilities. It seems reasonable to argue that, since different sub-populations do not interact in any way with one another, the gains from cooperation within each sub-population should be shared among its members only, and not with the other sub-populations.

As already mentioned, one concrete tool for achieving social optimality in a nonatomic congestion game is a suitable system of tolls, which make players internalize the social effects of their choices. This suggests a third way of sharing the maximum aggregate utility achievable in such games, namely, equal distribution of toll revenues. This can be done either by making equal lump-sum transfers to all players or by lowering tolls uniformly for all facilities, so that some of them became negative (i.e., subsidies). Again, this raises the question of why players who are not affected in any way by the tolls should get the same share of the toll revenues as those who are affected.

The three schemes mentioned above for sharing the aggregate utility or cost of the social optimum among the players are illustrated by the example in Table 1. This table also gives the solution prescribed by a fourth sharing rule, which will now be described. The idea behind this utility or cost-sharing rule is that the players' shares should reflect their marginal contributions to the bargaining power of various coalitions of players to which they may belong. In the present context, a coalition is a potential alliance, capable of coordinating its members' actions in any way deemed right and negotiating with other coalitions. It is assumed that essentially any group of players may form a coalition. (More precisely, any group belonging to the collection $\mathcal{C}$ that defines the measurable structure on I.) The first step in formalizing this idea is to determine each coalition's bargaining power or, more concretely, the particular division of the aggregate utility between the coalition and its complement determined by their relative bargaining powers.

Denote the maximum aggregate utility that can be achieved in a given nonatomic congestion game $\Gamma(c, f)$ by $\upsilon(I)$. Arguably, if the division of $\upsilon(I)$ between a coalition of players S and its complement $I \setminus S$ is decided by bilateral bargaining, it depends on what each coalition would do if negotiations break down, i.e., on the two coalitions' threat strategies. If these are chosen so as to put each coalition in the best possible bargaining position vis-à-vis the other, the division of $\upsilon(I)$ between the two complementary coalitions may be viewed as an instance of the Nash bargaining problem with threats ([15]). The specification of this problem has two elements. The first is the set of all possible joint actions, which, in the present case, is the set of all divisions of the maximum aggregate utility between the two coalitions. The second is the two-person strategic game that determines the outcome if there is no agreement about the joint action. In the case under consideration, the set of strategies available to S is $\{1_S\, \sigma \mid \sigma \in \Sigma\}$, where 1_S denotes the indicator function of the set S, and the strategy set of $I \setminus S$ is $\{1_{I \setminus S}\, \sigma \mid \sigma \in \Sigma\}$. For each strategy σ^S of S and each strategy $\sigma^{I \setminus S}$ of $I \setminus S$, the payoffs of S and $I \setminus S$ are the respective aggregate utilities, $\int (f(i) - c(\mu(\sigma^S + \sigma^{I \setminus S}))) \cdot \sigma^S(i)\, d\mu(i)$ and $\int (f(i) - c(\mu(\sigma^S + \sigma^{I \setminus S}))) \cdot \sigma^{I \setminus S}(i)\, d\mu(i)$. The difference between these payoffs is

$$H^S(\sigma^S, \sigma^{I \setminus S}) \overset{def}{=} \int_I (f(i) - c(\mu(\sigma^S + \sigma^{I \setminus S}))) \cdot (\sigma^S(i) - \sigma^{I \setminus S}(i))\, d\mu(i). \tag{5}$$

According to Nash, with threat strategies σ^S and $\sigma^{I \setminus S}$, $\upsilon(I)$ is divided between S and

$I \setminus S$ in such a way that the difference between the two coalitions' shares equals $H^S(\sigma^S, \sigma^{I \setminus S})$. Therefore, S would like the right-hand side of (5) to be as large as possible, and $I \setminus S$ would like it to be as small as possible. Each coalition's threat strategy is optimal, given the other's, if and only if $(\sigma^S, \sigma^{I \setminus S})$ is a saddle point of H^S. In this case, S's share of $\upsilon(I)$ is given explicitly by

$$\upsilon(S) \stackrel{def}{=} \frac{1}{2}\,[H^S(\sigma^S, \sigma^{I \setminus S}) + \upsilon(I)]. \tag{6}$$

This will be referred to as the worth of coalition S. If, for each coalition S, the function H^S has a saddle point, then Eq. (6) defines a coalitional game υ, called the coalitional form of $\Gamma(c, f)$.[11] A sufficient condition for H^S to have a saddle point for each coalition S, and hence for every coalition to have a well-defined worth, is given below. It requires (i) increasing marginal social costs and (ii) cost functions that are not "too" convex. However, this condition, which involves the functional form of the congestion externalities, would not itself be sufficient to guarantee a well-defined υ. Also important is the assumption, made throughout this paper, that the cost of congestion is the same for all players. The fact that neither of these assumptions can be dispensed with is shown in Appendix A.

It is generally impossible to share the maximum aggregate utility among the players in such a way that each coalition S gets at least its worth $\upsilon(S)$. Indeed, when such a scheme of distribution exists, it belongs to the core of υ. However, since υ is a fixed-sum game, its core is nonempty only if υ is additive, or inessential (in which case the only core element is υ itself). Since this is not generally the case, the core is of little relevance here. An alternative solution concept, which is based on the players' marginal contributions to the worth of coalitions, is the Aumann-Shapley value. (If the core of υ is nonempty, its unique element coincides with the value.) The Aumann-Shapley value of υ, which is denoted by $\varphi\upsilon$, is referred to in this paper as the Harsanyi transferable-utility value of $\Gamma(c, f)$.[12] It is shown below that, under the condition

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[11] Aumann and Kurz [2] call υ the "Harsanyi coalitional form" of the strategic game. Selten [21] uses the term "1/2-characteristic" to refer to essentially the same thing.

[12] The Harsanyi TU value is essentially the same thing Harsanyi [9], Aumann and Kurz [2], and Selten [21] call the "modified Shapley value," "Harsanyi-Shapley TU value," and "1/2-value" of the game, respectively. The term "Harsanyi-Selten value" is also sometimes used.

mentioned in the previous paragraph, both υ and its Aumann-Shapley value are well defined. Specifically, υ belongs to a space of coalitional games, namely *pNA*, on which a unique value exists. For definitions of the Aumann-Shapley value of nonatomic coalitional games, the space *pNA*, and related terms, see [3].

Definition. The cost functions satisfy the convexity condition if, for all j and all $y \geq 0$, the partial derivative

$$MC_j^*(x, y) \stackrel{def}{=} \frac{\partial}{\partial x}[(x - y)\, c_j(x + y)]$$

exists and is strictly increasing as a function of x in $(0, \infty)$.

It can be shown that, if the cost functions satisfy the convexity condition, then each MC_j^* can be extended in a unique way to a continuous extended real-valued function on $\mathbb{R}^2_+$. This function, which will also be denoted by MC_j^*, may take an infinite value (namely, $-\infty$) only at $(0, 0)$. It satisfies $MC_j^*(x, 0) = MC_j(x)$ for all $x \geq 0$; $\lim_{x \to 0} x\, MC_j^*(x, y) = 0$ for all $y \geq 0$; and $MC_j^*(x, y) = c_j(x + y) + (x - y)\, dc_j/dx(x + y)$ for all $x \geq 0$ and $y \geq 0$, where the second term is understood as 0 if $(x, y) = (0, 0)$. Clearly, the convexity condition is stronger than increasing marginal social costs of congestion. The additional requirement it represents is explicitly spelt out in the following proposition, the proof of which is given in Appendix B.

Proposition 6. The convexity condition holds if and only if

(i) the marginal social costs of congestion are increasing, and

(ii) for each j, the function $c_j(\sqrt[3]{x})$ $(x > 0)$ is concave.

Condition (ii) in Proposition 6 is equivalent to the following: For each j, there is a convex function φ_j such that $\varphi_j(c_j(x)) = x^3$ for all $x > 0$. (This function is the inverse of that in (ii).) This can be viewed as a requirement that the cost functions be less, or as convex as, the cubic function x^3. Using Proposition 6, or directly from the definition, it is easy to show that each of the following cost functions satisfies the convexity condition: $c_j(x) = (x + b)^\beta$, with $0 < \beta \leq 3$ and $b \geq 0$; $c_j(x) = -(x + b)^\gamma$, with $-1 < \gamma < 0$ and $b \geq 0$, or $\gamma = -1$ and $b > 0$; and $c_j(x) = \log_a(x + b)$, with $a > 1$ and $b \geq 0$. (Note that the last function is concave.) The two cost functions in (4) also satisfy this condition. By contrast, for $c_j(x) = x^4$, the convexity condition does not hold: this cost function is "too" convex. This is consequential. As shown in Appendix A, with this cost function, the worth of coalitions may not be well defined. This shows the importance of assuming the convexity condition in the following theorem, the proof of which is given in Appendix B.

Theorem 2. Suppose that the cost functions satisfy the convexity condition. Then, for every fixed-utility assignment f, the coalitional form υ of $\Gamma(c, f)$ is well defined and is in pNA, and the Harsanyi TU value of $\Gamma(c, f)$ is given by the formula

$$(\varphi\upsilon)(S) = \int_0^1 \int_S \max_j \left(f_j(i) - MC_j{}^*(\mu(\sigma_j^t), \mu(\sigma_j^{1-t}))\right) d\mu(i)\, dt \qquad (S \in \mathcal{C}), \tag{7}$$

where, for every $0 \le t \le 1$, the inner integral is uniquely determined by the following condition: There exists a pair of strategy profiles σ and $\overline{\sigma}$ such that

$$\sigma^t = t\sigma \quad \text{and} \quad \sigma^{1-t} = (1-t)\overline{\sigma}$$

and, for almost all players i,

$$(f(i) - MC^*(\mu(\sigma^t), \mu(\sigma^{1-t}))) \cdot \sigma(i) = \max_j \left(f_j(i) - MC_j{}^*(\mu(\sigma_j^t), \mu(\sigma_j^{1-t}))\right)$$

$$\text{and} \tag{8}$$

$$(f(i) - MC^*(\mu(\sigma^{1-t}), \mu(\sigma^t))) \cdot \overline{\sigma}(i) = \max_j \left(f_j(i) - MC_j{}^*(\mu(\sigma_j^{1-t}), \mu(\sigma_j^t))\right).$$

To understand the last part of the theorem, note that, by (8), for every $0 < t \le 1$, the corresponding strategy profile σ is an equilibrium, and the inner integral in (7) gives the equilibrium payoff distribution, in the nonatomic congestion game $\Gamma(c^t, f)$ with cost functions $c_j^t(x) \overset{def}{=} MC_j{}^*(tx, \mu(\sigma_j^{1-t}))$. In particular, for $t = 1/2$, σ is an equilibrium, and the inner integral gives the equilibrium payoff distribution, in the original game $\Gamma(c, f)$. This follows from the fact that setting $t = 1/2$ and $\overline{\sigma} = \sigma$ reduces both equations in (8) to (2). For $t = 1$, σ is an equilibrium, and the inner integral in (7) gives the equilibrium payoff distribution, in $\Gamma(MC, f)$. By Proposition 3, this strategy profile σ is socially optimal in the original game $\Gamma(c, f)$.

In some cases, the inner integral in (7) can also be given a similar interpretation for other values of t. Specifically, consider the case of linear cost functions of the form $c_j(x) = c_j(1)\, x$. In this case, direct computation gives $c_j^t(x) = 2t\, c_j(x)$. Therefore, Eq. (7) implies that the Harsanyi TU value of $\Gamma(c, f)$ equals the integral mean of the equilibrium payoff distributions in all games of the form $\Gamma(2t\, c, f)$, with t varying between 0 and 1. At one end of this interval, the costs of congestion tend to zero, while at the other, they tend to the respective marginal social costs.

Example. Consider again the nonatomic congestion game $\Gamma(c, f)$ described in Fig. 2. At the unique equilibrium in this game, all the type I and type II players take the left and right routes, respectively. This separating equilibrium is common, in fact, to all games of the form $\Gamma(2t\,c, f)$, with $0 < t \leq 3/4$. However, for $3/4 < t \leq 1$, the equilibria in $\Gamma(2t\,c, f)$ involve some type I players joining the type II players in taking the right route, thereby decreasing the cost for the remaining type I players and increasing it for the type II players. It therefore follows from (7) that the value of $\Gamma(c, f)$ to type I players is greater than their equilibrium payoff and the value to type II players is less than their equilibrium payoff. Exact computation shows, in fact, that, compared to the equilibrium, type I and type II players are better off and worse off, respectively, by the equivalent of exactly 1/8 minute of travel time.

In the last example, some players' equilibrium, or noncooperative, payoffs, which are obtained when all the players seek to maximize their own utility or minimize their own cost, disregarding those of the others, are greater than their cooperative payoffs, which are given by the formula (7). For other players, the converse is true. This may also occur in nonatomic congestion games with socially optimal equilibria, and in this case, the Harsanyi TU value of the game is different not only from the equilibrium payoff distribution but also from the payoff distribution induced by any strategy profile; in other words, a value strategy profile does not exist. (Indeed, by Proposition 3, a strategy profile that is not an equilibrium is not even socially optimal.) A very different case, in which the equilibrium payoff distribution and the value always coincide, is that of logarithmic cost functions. The proof of the following proposition is given in Appendix B.

Proposition 7. Suppose that the cost functions are as in (iii) in Theorem 1. Then, for every fixed-utility assignment f, the equilibrium payoff distribution in $\Gamma(c, f)$ coincides with the Harsanyi TU value of the game.

It follows from Theorem 1 and Proposition 7 that, with three or more facilities, the logarithmic cost functions are the only ones satisfying the convexity condition for which the coincidence of the Harsanyi TU value and the equilibrium payoff distribution is guaranteed. Proposition 7 is illustrated by the following example.

Example. Consider again the nonatomic congestion game $\Gamma(c, f)$ described in Fig. 3. Clearly, for each coalition, the aggregate equilibrium payoff depends only on the fraction of the type I players and the fraction of the type II players belonging to the coalition. Similarly, in the coalitional form υ of $\Gamma(c, f)$, the worth of each coalition depends only on these fractions. The aggregate equilibrium payoff and the worth of a coalition as functions of the fraction of the players of each type belonging to it are shown in Fig. 4.[13] As seen in this figure, for coalitions S comprising most of the players of one type and only few of the other type, the worth $\upsilon(S)$ differs from the aggregate equilibrium payoff. For example, for the coalition represented in Fig. 4 by the point (1, 0), which consists of all the type I players and none of the type II players, the worth is greater than the aggregate equilibrium payoff. This reflects the greater bargaining power of this coalition relative to the complementary one, which consists of all the type II players. Specifically, the second coalition would suffer a greater loss than the first if negotiations would break down and both coalitions would carry out their optimal threat strategies. Because of this, the first coalition is able to extract from its rival some of the latter coalition's equilibrium aggregate utility.[14] However, for less homogeneous coalitions, whose composition is closer to that of the whole population, the worth and the aggregate equilibrium payoff are equal. Such coalitions consist of a certain fraction of the type I subpopulation and roughly the same fraction of the type II subpopulation. In Fig. 4, they are represented by points lying close to the diagonal joining the (0, 0) and (1, 1) vertices. Now, the computation of the Aumann-Shapley value of a coalitional game is based on the so-called <u>diagonal formula</u>. This entails that it only takes into consideration the worth of coalitions that are more-or-less scaled down versions of the whole population. (For a discussion of this property of the Aumann-Shapley value, see [3].) Because of this, coalitions of the kind mentioned before, in which the ratio between the two types is highly biased, are

[13] Note that the former function is linear. This is so by definition. The latter function is obviously <u>non</u>linear, which implies that the core of υ is empty. For further discussion of this nonlinearity, see below.

[14] Another aspect of the difference in bargaining power between the two coalitions is the fact that the marginal contribution of type I players to the worth of either coalition is greater than that of type II players. In fact, the marginal contribution of the former is positive while that of the latter is negative. This difference stems from the different abilities of these two types of players to help the coalition they join while simultaneously causing maximum harm to the rival coalition. For coalitions lying closer to the diagonal (see below), the marginal contributions have the reverse signs: negative for type I players and positive for type II. The reason, as explained below, is that the marginal contributions to the worth of such coalitions are equal to the players' equilibrium payoffs.

irrelevant to the computation of the value. In fact, the value is completely determined by any open set of coalitions that contains the relative interior of diagonal (that is, the diagonal without its endpoints). As seen in Fig. 4, there are some (indeed, many) such sets in which, for all coalitions, the worth and the aggregate equilibrium payoff are equal. The existence of such sets, which in some sense (that will not be explained here) is a generic property of nonatomic congestion games with logarithmic cost functions, implies that, close to the diagonal, the marginal contributions of players to the worth of coalitions are equal to their equilibrium payoffs. Therefore, the Harsanyi TU value, which reflects the players' marginal contributions to the worth of coalitions lying along the diagonal, equals the equilibrium payoff distribution. This may help understand why, in games with logarithmic cost functions, the players' noncooperative (i.e., equilibrium) and cooperative (i.e., value) payoffs coincide.

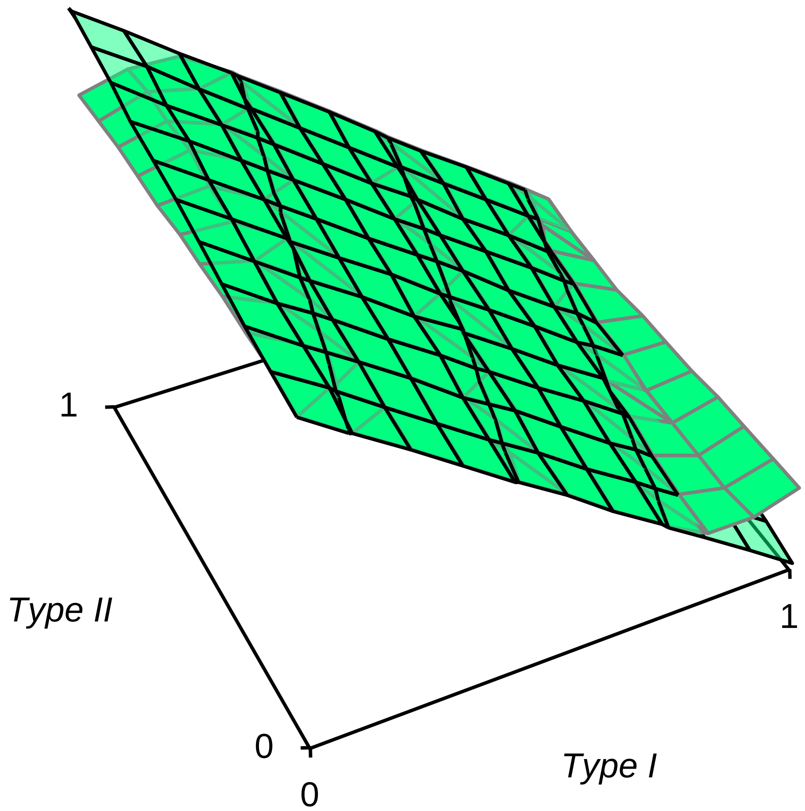

Fig. 4. For the nonatomic congestion game described in Fig. 3, each coalition's aggregate equilibrium payoff (semi-transparent, black meshed, surface) and worth (opaque, gray meshed, surface) are shown as functions of the fractions of the type I and type II players who are members of the coalition.

CONCLUDING REMARKS

The efficiency of noncooperative equilibria, and comparison between noncooperative and cooperative payoffs, are much-studied themes in economics. Market economies are one context in which such studies are particularly prevalent. For a transferable-utility market economy (in which the utility functions are quasilinear with respect to a numeraire commodity, which may appear in positive as well as negative quantities), the first fundamental theorem of welfare economics asserts that every competitive equilibrium is Pareto efficient. Pareto efficiency is equivalent, in this case, to social optimality. The value equivalence theorem for TU market economies with a continuum of traders ([3, Proposition 32.3]) asserts that, under certain conditions, the competitive payoff distribution is unique and coincides with the Aumann-Shapley value of the corresponding market game. In this (coalitional) game, a coalition's worth is the maximum aggregate utility it can obtain for its members when they are only allowed to trade among themselves. The Aumann-Shapley value of a market game is directly affected by the value of each trader's initial endowment to the other traders. The competitive payoff, by contrast, is affected by this only indirectly, through the market price of the initial endowment. Nevertheless, the value equivalence theorem tells us that these two are, in fact, equal.

The present paper represents an attempt to make comparisons similar to those for market economies in the context of strategic games. While questions similar to those considered here can be raised for any strategic game with, or even without, transferable utility, this study is only concerned with one specific class of nonatomic congestion games. As it shows, even in this restricted context it is not possible to obtain results as general as the first fundamental theorem of welfare economics or the value equivalence theorem. In fact, equilibria may or may not be socially optimal, and, even when they are, the players' equilibrium, or noncooperative payoffs need not coincide with their cooperative ones.

There are two exceptions to these findings. The most striking are logarithmic cost functions. For these, not only are the equilibria always socially optimal, but also the equilibrium payoff distribution coincides with the value. The other special case is that of linear cost functions. For these, another connection between the value and certain equilibrium payoffs holds. Namely, the value can be computed by first finding the equilibrium payoff distribution in all games differing from the original one only in that the slopes of the cost functions are multiplied by a positive constant, which is less than two, and then taking the average of these payoffs.

The fact that in nonatomic congestion games equilibria may fail to be socially optimal raises the question of how far these equilibria can be from the social optimum. A recent paper addressing this question in a model related to, but not identical with, the present one is [19]. For example, for a homogeneous population of users and linear, nonnegative costs (such as those in Fig. 1), the average cost in equilibrium is shown in [19] not to exceed 4/3 of that at the social optimum. For cost functions that are given by higher-degree polynomials, the corresponding upper bound is higher.

APPENDIX A: NONCONVEXITIES

In this appendix, the convexity condition and the assumption that the cost of congestion is the same for all players are both shown to be crucial for a well-defined coalitional form of nonatomic congestion games. If the convexity condition does not hold, there may be coalitions S for which the function H^S, which measures the difference between the aggregate utilities of S and $I \setminus S$, does not have a saddle point. The worth of such coalitions is not well defined. A similar phenomenon might occur if the cost of congestion were allowed to vary across players.

The importance of the convexity condition. There are two facilities, with identical cost functions $c_1(x) = c_2(x) = x^4$. (It follows from Proposition 6 that this cost function does not satisfy the convexity condition.) All players are identical, with $f_1 = f_2 = 0$. Coalition S consists of 1/10 of the population, and its complement $I \setminus S$ consists of the rest. For each strategy σ^S of S and strategy $\sigma^{I \setminus S}$ of $I \setminus S$, there is a corresponding pair of numbers $0 \leq x \leq 0.1$ and $0 \leq y \leq 0.9$ such that $\mu(\sigma^S) = (x, 0.1 - x)$ and $\mu(\sigma^{I \setminus S}) = (y, 0.9 - y)$. Without loss of generality, it may be assumed that $x > 0$. Computation of the right-hand side of (5) gives

$$(9) \qquad H^S(\sigma^S, \sigma^{I \setminus S}) = (y - x)\,(x + y)^4 + (0.8 + x - y)\,(1 - x - y)^4.$$

A necessary condition for $(\sigma^S, \sigma^{I \setminus S})$ to be a saddle point of H^S is that the members of S cannot increase H^S by moving from facility 1 to facility 2, i.e., by decreasing x. A necessary condition for this is that the partial derivative of the expression on the right-hand side of (9) with respect to x is nonnegative:

$$(10)\quad -(x+y)^4 + 4\,(y-x)\,(x+y)^3 + (1-x-y)^4 - 4\,(0.8+x-y)\,(1-x-y)^3 \geq 0.$$

A second necessary condition for $(\sigma^S, \sigma^{I \setminus S})$ to be a saddle point of H^S is that members of $I \setminus S$ cannot decrease H^S by moving between facilities. If all the members of $I \setminus S$ use the same facility, then this condition is not satisfied, for it is easy to see that, in such a case, moving a few members of $I \setminus S$ to the other facility would increase this coalition's aggregate utility more than it would increase the aggregate utility of S (if at all). Therefore, a necessary condition for a saddle point is that members of $I \setminus S$ use both facilities, i.e., $0 < y < 0.9$. This implies that, for $(\sigma^S, \sigma^{I \setminus S})$ to be a saddle point, the partial derivative of the expression on the right-hand side of (9) with respect to y must vanish:

$$(11)\quad (x+y)^4 + 4\,(y-x)\,(x+y)^3 - (1-x-y)^4 - 4\,(0.8+x-y)\,(1-x-y)^3 = 0.$$

Subtracting (11) from (10) gives $(1 - x - y)^4 \geq (x + y)^4$, or equivalently $x + y \leq 0.5$. Adding these equations gives $(y - x)\,(x + y)^3 \geq (0.8 + x - y)\,(1 - x - y)^3$. Now, if $x = 0.1$, then the right-hand side of the last inequality is equal to $(1 - x - y)^4$, and therefore $(y - x)\,(x + y)^3 \geq (x + y)^4$, which contradicts the assumption that $x > 0$. Therefore, it must be that $0 < x < 0.1$, i.e., members of S use both facilities. This implies that a necessary condition for $(\sigma^S, \sigma^{I \setminus S})$ to be a saddle point is that (10), as well as all the weak inequalities that follow from it, hold as equalities. In particular, $x + y = 0.5$ and $(y - x)\,0.5^3 = (0.8 + x - y)\,0.5^3$. The unique solution of these two linear equations is $x = 0.05$ and $y = 0.45$. Thus, the members of both coalitions are equally divided between the two facilities. This condition, which implies $H^S(\sigma^S, \sigma^{I \setminus S}) = 0.05$, is a necessary condition for $(\sigma^S, \sigma^{I \setminus S})$ to be a saddle point. However, with such x and y, $(\sigma^S, \sigma^{I \setminus S})$ is not a saddle point. In fact, with respect to coalition S's strategies, it is a minimum rather than a maximum point. For example, if all the members of S move to facility 1, H^S increases to 0.05048. This proves that a saddle point does not exist.

The importance of the assumption that all players have the same cost functions. There are three facilities, with linear cost functions, and two types of players. For type I (half of all players), $f_1 = 1/4$, $f_2 = 1/8$, $f_3 = 1/2$, and $c_1(x) = c_2(x) = c_3(x) = x$. For type II (the other half), $f_1 = 1/8$, $f_2 = 1/4$, $f_3 = 4$, and $c_1(x) = c_2(x) = x$, but $c_3(x) = 8x$. Thus, the third cost function is type-specific. For the coalition S consisting of half the type I players and half the type II players, and for a given strategy σ^S of S and a given strategy $\sigma^{I \setminus S}$ of $I \setminus S$, consider the difference between the aggregate utilities of S and $I \setminus S$. This is given by a function H^S which is a straightforward generalization of (5) to type-specific cost functions. It is not difficult to see that, because S and its complement are identical in composition, a necessary condition for H^S to have some saddle point is that is has a symmetric one, in which the facility choice of each type of player is the same in S and $I \setminus S$. The symmetry of the saddle point implies that, if a small group of players in S switches from one facility to another, the first-order change in H^S equals the change in these players' aggregate utility. By definition, at a saddle point the change in H^S cannot be positive. Therefore, a symmetric saddle point must correspond to an equilibrium in the nonatomic congestion game. When half of the type I players use facility 1, half the type II players use facility 2, and the rest of the players use facility 3, the game is at equilibrium. In fact, this is the only kind of equilibrium in the game. It follows that, in every symmetric saddle point of H^S, the players in both S and $I \setminus S$ choose their facilities in the manner just indicated. However, suppose that, at such a point, a small group of type II members of S using facility 2 switch to facility 3, and at the same time a group of type I members of S who use facility 3 switch to facility 1. Brief computation shows that, if these groups have measures ε and 2ε, respectively, then H^S increases by ε^2. (Incidentally, this shows that, at the point under consideration, H^S is not concave in its first argument.) The fact that the change in H^S is positive contradicts the assumption that the original point was a saddle point. This contradiction proves that H^S does not have a saddle point.

APPENDIX B: PROOFS

This appendix contains the proofs of the two theorems in this paper, as well as those of Propositions 2, 3, 6, and 7. First, several new definitions, and two lemmas, are given.

An ideal coalition is a measurable function $h : I \to [0, 1]$. The set of all ideal coalitions is denoted by $\mathcal{J}$. In the following, $\mathcal{J}$ is seen as a subset of the Banach space $L_\infty(\mu)$ endowed with the relative weak* topology.[15] Thus, ideal coalitions that are equal almost everywhere are identified. By Alaoglu's theorem, the space $\mathcal{J}$ is compact.

[15] This is equivalent to seeing $\mathcal{J}$ as a subset of $L_1(\mu)$ endowed with the relative weak topology.

For $h \in \mathcal{I}$, let Σ^h denote the set $\{\sigma^h = (\sigma_1{}^h, \sigma_2{}^h, \dots, \sigma_m{}^h) \in \mathcal{I}^m \mid \sum_j \sigma_j{}^h = h\}$. For $g, h \in \mathcal{I}$ and $(\sigma^h, \sigma^g) \in \Sigma^h \times \Sigma^g$, define

$$H(\sigma^h, \sigma^g) \stackrel{def}{=} \int_I (f - c(\mu(\sigma^h + \sigma^g))) \cdot (\sigma^h - \sigma^g)\, d\mu$$

$$= \int_I f \cdot (\sigma^h - \sigma^g)\, d\mu - c(\mu(\sigma^h) + \mu(\sigma^g)) \cdot (\mu(\sigma^h) - \mu(\sigma^g)).$$

This is a generalization of (the two-person game) H^S defined in (5). Note that $H(\sigma^h, \sigma^g) = -H(\sigma^g, \sigma^h)$. For given $h \in \mathcal{I}$, a pair $(\sigma^h, \sigma^{1-h}) \in \Sigma^h \times \Sigma^{1-h}$ will be said to be a <u>saddle point</u> if, for all $(\tau^h, \tau^{1-h}) \in \Sigma^h \times \Sigma^{1-h}$,

(12) $$H(\tau^h, \sigma^{1-h}) \le H(\sigma^h, \sigma^{1-h}) \le H(\sigma^h, \tau^{1-h}).$$

It is easy to see that if the pair (τ^h, τ^{1-h}) is also a saddle point, then both inequalities in (12) are, in fact, equalities.

Lemma 1. Suppose that the cost functions satisfy the convexity condition. Then, for every $h \in \mathcal{I}$, there is a saddle point in $\Sigma^h \times \Sigma^{1-h}$. Moreover, there is a saddle point of the form $(h\sigma, (1-h)\overline{\sigma})$, where $\sigma, \overline{\sigma} \in \Sigma$ (i.e., strategy profiles). For each j, there is a continuous function $\Theta_j : \mathcal{I} \to [0, 1]^2$ such that, for every $h \in \mathcal{I}$, $\Theta_j(h) = (\mu(\sigma_j{}^h), \mu(\sigma_j{}^{1-h}))$ for all saddle points (σ^h, σ^{1-h}) in $\Sigma^h \times \Sigma^{1-h}$.

Proof. For given $h \in \mathcal{I}$, both Σ^h and Σ^{1-h} can be seen as compact convex sets in a locally convex Hausdorff linear topological space (namely, $L_\infty(\mu)^m$ with the product weak* topology). The function $H(\cdot, \cdot)$ is easily seen to be continuous on $\Sigma^h \times \Sigma^{1-h}$. By the convexity condition, $H(\cdot, \sigma^{1-h})$ is concave for all $\sigma^{1-h} \in \Sigma^{1-h}$ and $H(\sigma^h, \cdot)$ is convex for all $\sigma^h \in \Sigma^h$. Therefore, by the minimax theorem in locally convex Hausdorff linear topological spaces ([8, Theorem 3]), there is a saddle point in $\Sigma^h \times \Sigma^{1-h}$.

It follows from [6, Theorem 4] that, for every $(\sigma^h, \sigma^{1-h}) \in \Sigma^h \times \Sigma^{1-h}$, there is some $\sigma \in \Sigma$ such that, for all j, $\int \sigma_j{}^h\, d\mu = \int h\, \sigma_j\, d\mu$ and $\int f\, \sigma_j{}^h\, d\mu = \int f\, h\, \sigma_j\, d\mu$. Clearly, $H(\sigma^h, \tau^{1-h}) = H(h\sigma, \tau^{1-h})$ for all $\tau^{1-h} \in \Sigma^{1-h}$. Similarly, there is some $\overline{\sigma} \in \Sigma$ such that $H(\tau^h, \sigma^{1-h}) = H(\tau^h, (1-h)\overline{\sigma})$ for all $\tau^h \in \Sigma^h$. These equations imply that, if (σ^h, σ^{1-h}) is a saddle point, then so is $(h\sigma, (1-h)\overline{\sigma})$.

Let (σ^h, σ^{1-h}) and (τ^h, τ^{1-h}) be two saddle points in $\Sigma^h \times \Sigma^{1-h}$. If $\mu(\sigma^h) \neq \mu(\tau^h)$ then, by the convexity condition and the remark that follows Eq. (12), $H(1/2\, \sigma^h + 1/2\, \tau^h, \sigma^{1-h}) > 1/2\, H(\sigma^h, \sigma^{1-h}) + 1/2\, H(\tau^h, \sigma^{1-h}) = H(\sigma^h, \sigma^{1-h})$. This, however, contradicts the definition of saddle point. Therefore, $\mu(\sigma^h) = \mu(\tau^h)$. By a similar argument, $\mu(\sigma^{1-h}) = \mu(\tau^{1-h})$. To complete the proof of the lemma it remains to show that, for every j, the function $\Theta_j : \mathcal{I} \to [0, 1]^2$ (well-) defined by $\Theta_j(h) = (\mu(\sigma_j^h), \mu(\sigma_j^{1-h}))$, where (σ^h, σ^{1-h}) is an arbitrary saddle point in $\Sigma^h \times \Sigma^{1-h}$, is continuous. Consider the set $\mathcal{S}$ of all triplets $(h, \rho, \bar{\rho})$ in $\mathcal{I} \times \mathcal{I}^m \times \mathcal{I}^m$ such that $\sum_j \rho_j = h$, $\sum_j \bar{\rho}_j = 1 - h$, and $H(h\sigma^1, \bar{\rho}) \leq H(\rho, \bar{\rho}) \leq H(\rho, (1-h)\sigma^1)$ for all $\sigma^1 \in \Sigma^1$. (By definition, $\sum_j \sigma_j^1$ is the constant function 1.) Since the set $\mathcal{S}$ is easily seen to be closed, and hence compact, the range of the continuous function $\Xi_j : \mathcal{S} \to \mathcal{I} \times [0, 1]^2$ defined by $\Xi_j(h, \rho, \bar{\rho}) = (h, (\mu(\rho_j), \mu(\bar{\rho}_j)))$ is also compact. The range of Ξ_j coincides with the graph of Θ_j. Therefore, the latter function has a compact graph, and hence is continuous. ■

Lemma 2. Suppose that the cost functions satisfy the convexity condition. For every $h \in \mathcal{I}$, a pair (σ^h, σ^{1-h}) in $\Sigma^h \times \Sigma^{1-h}$ is a saddle point if and only if it satisfies the following equations:

$$(13) \qquad (f - MC^*(\mu(\sigma^h), \mu(\sigma^{1-h}))) \cdot \sigma^h = h \max_j (f_j - MC_j^*(\mu(\sigma_j^h), \mu(\sigma_j^{1-h})))$$

and

$$(14) \qquad (f - MC^*(\mu(\sigma^{1-h}), \mu(\sigma^h))) \cdot \sigma^{1-h} = (1-h) \max_j (f_j - MC_j^*(\mu(\sigma_j^{1-h}), \mu(\sigma_j^h))).$$

In this case, $MC_j^*(\mu(\sigma_j^h), \mu(\sigma_j^{1-h}))$ is finite (i.e., $> -\infty$) for all j.

The intuitive content of (13) is that the facility choice of each "member" of the ideal coalition h maximizes his contribution to the difference between the aggregate utilities of the complementary ideal coalitions h and $1 - h$. If the player chooses facility j, his contribution is equal to his fixed utility from using j minus the effect of his choice on the difference between the aggregate costs of using the facility to the members of h and $1 - h$. The intuitive content of (14) is similar.

Proof of Lemma 2. Fix $h \in \mathcal{I}$, and $(\sigma^h, \sigma^{1-h}) \in \Sigma^h \times \Sigma^{1-h}$. For every $g \in \mathcal{I}$ and every $\tau^g \in \Sigma^g$,

$$H(\tau^g, \sigma^{1-h}) - H(\sigma^h, \sigma^{1-h}) = \int_I f \cdot (\tau^g - \sigma^h)\, d\mu$$

$$- \left[c(\mu(\tau^g) + \mu(\sigma^{1-h})) \cdot (\mu(\tau^g) - \mu(\sigma^{1-h})) - c(\mu(\sigma^h) + \mu(\sigma^{1-h})) \cdot (\mu(\sigma^h) - \mu(\sigma^{1-h}))\right].$$

If $\mu(\tau^g) \neq \mu(\sigma^h)$, then, by the convexity condition, the expression in square brackets is strictly greater than $MC^*(\mu(\sigma^h), \mu(\sigma^{1-h})) \cdot (\mu(\tau^g) - \mu(\sigma^h))$. Therefore, if (13) holds, then

$$(15) \quad H(\tau^g, \sigma^{1-h}) - H(\sigma^h, \sigma^{1-h}) \leq \int_I (f - MC^*(\mu(\sigma^h),\mu(\sigma^{1-h}))) \cdot (\tau^g - \sigma^h)\, d\mu$$

$$\leq \int_I \max_j \,(f_j - MC_j^*(\mu(\sigma_j^h), \mu(\sigma_j^{1-h})))\,(g - h)\, d\mu.$$

In the special case $g = h$, this gives the left inequality in (12). (The general case is only required later.) The right inequality is similarly implied by (14). Therefore, (13) and (14) together imply that (σ^h, σ^{1-h}) is a saddle point.

Conversely, for $h \in \mathcal{J}$, suppose that (σ^h, σ^{1-h}) is a saddle point. Suppose also that $\int h\, d\mu$ is strictly positive, say equal to ε. (If $h = 0$ almost everywhere, (13) holds trivially.) Let σ be an equilibrium in the nonatomic congestion game in which the population measure μ^h is defined by $\mu^h(S) = (1/\varepsilon) \int_S h\, d\mu$, the cost functions are $c_j^h(x) \overset{def}{=} MC_j^*(\varepsilon x, \mu(\sigma_j^{1-h}))$, and the fixed-utility assignment is f. By definition of equilibrium, Eq. (13) holds with $h\sigma$ substituted for σ^h. Therefore, by (15), $H(\tau^h, \sigma^{1-h}) \leq H(h\sigma, \sigma^{1-h})$ for all $\tau^h \in \Sigma^h$. Moreover, inspection of the proof of (15) shows that the last inequality holds as an equality only if $\mu(\tau^h) = \mu(h\sigma)$ and $(f - MC^*(\mu(h\sigma), \mu(\sigma^{1-h}))) \cdot \tau^h = h \max_j (f_j - MC_j^*(\mu(h\sigma_j), \mu(\sigma_j^{1-h})))$. It follows that, for every $\tau^h \in \Sigma^h$ such that $H(\tau^h, \sigma^{1-h}) \geq H(h\sigma, \sigma^{1-h})$, $MC^*(\mu(\tau^h), \mu(\sigma^{1-h})) \cdot \tau^h = h \max_j (f_j - MC_j^*(\mu(\tau^h_j), \mu(\sigma_j^{1-h})))$. Since $H(\sigma^h, \sigma^{1-h}) \geq H(h\sigma, \sigma^{1-h})$ by definition of saddle point, this proves (13). The proof of (14) is similar.

For every $h \in \mathcal{J}$, every $(\sigma^h, \sigma^{1-h}) \in \Sigma^h \times \Sigma^{1-h}$, and every j such that $\mu(\sigma_j^h) > 0$ or $\mu(\sigma_j^{1-h}) > 0$, $MC_j^*(\mu(\sigma_j^h), \mu(\sigma_j^{1-h})) > -\infty$. Therefore, the left-hand sides of (13) and (14) are finite almost everywhere. If (σ^h, σ^{1-h}) is a saddle point, then these two equations hold, and therefore their right-hand sides are also finite almost everywhere. Hence, in this case, $MC_j^*(\mu(\sigma_j^h), \mu(\sigma_j^{1-h})) > -\infty$ for <u>all</u> j. ■

Proof of Proposition 2. For given cost functions c and fixed-utility assignment f, let σ be an equilibrium in $\Gamma(c, f)$, and τ some strategy profile. If $\mu(\sigma) \neq \mu(\tau)$, then the set J of all facilities j such that $\mu(\sigma_j) < \mu(\tau_j)$ is nonempty, and therefore $\sum_{j\in J} \mu(\sigma_j) < \sum_{j\in J} \mu(\tau_j)$. This inequality implies that, for some $j \in J$ and $k \notin J$, the set of all players i such that $\sigma_k(i) = 1$ and $\tau_j(i) = 1$ has positive measure. Since $k \notin J$ means that $\mu(\sigma_k) \geq \mu(\tau_k)$, $u_i(\sigma) = f_k(i) - c_k(\mu(\sigma_k)) \leq f_k(i) - c_k(\mu(\tau_k))$ for all players i in this set. And, since $j \in J$ means that $\mu(\tau_j) > \mu(\sigma_j)$ and σ is an equilibrium, $u_i(\tau) = f_j(i) - c_j(\mu(\tau_j)) < f_j(i) - c_j(\mu(\sigma_j)) \leq u_i(\sigma)$ for almost all of them. Together, these equalities and inequalities give

$$u_i(\tau) < u_i(\sigma) \leq f_k(i) - c_k(\mu(\tau_k)). \tag{16}$$

Two conclusions can be drawn from this. First, if τ is such that $\mu(\sigma) \neq \mu(\tau)$, then (16) shows that τ is not an equilibrium. This proves the uniqueness of the measure of the set of players using each facility at equilibrium. Second, if τ is such that $u_i(\tau) \geq u_i(\sigma)$ (and hence (16) does not hold) for almost all players i with $\tau(i) \neq \sigma(i)$, then $\mu(\sigma) = \mu(\tau)$. Clearly, in this case, $u_i(\tau) = u_i(\sigma)$ for all players i with $\tau(i) = \sigma(i)$. And since σ is an equilibrium, $u_i(\tau) = (f(i) - c(\mu(\sigma))) \cdot \tau(i) \leq u_i(\sigma)$ for almost <u>all</u> i. For almost all players i with $\tau(i) \neq \sigma(i)$, the reverse inequality, $u_i(\tau) \geq u_i(\sigma)$, holds <u>by assumption</u>. It follows that, for almost all i, $u_i(\tau) = u_i(\sigma)$. This proves that the equilibrium σ satisfies the condition H.

It remains to show that every strategy profile σ that is <u>not</u> an equilibrium does not satisfy H. It follows from the assumed continuity of the cost functions that, for every strategy profile σ that is not an equilibrium, there is some facility j and some $\varepsilon > 0$ such that the set of all players i with $u_i(\sigma) < f_j(i) - c_j(\mu(\sigma_j) + \varepsilon)$ has positive measure. Let I_j be a subset of this set with $0 < \mu(I_j) < \varepsilon$, and τ the strategy profile defined by $\tau_j(i) = 1$ for $i \in I_j$ and $\tau(i) = \sigma(i)$ for $i \notin I_j$. Since, clearly, $u_i(\tau) > u_i(\sigma)$ for all players i with $\tau(i) \neq \sigma(i)$, the strategy profile σ does not satisfy condition H. ■

Proof of Proposition 3. Let σ be a strategy profile. Clearly, $\sigma \in \Sigma^1$ and $0 \in \Sigma^0$, where 0 and 1 denote the corresponding constant functions. Since, as shown in the proof of Lemma 1, for every $\tau^1 \in \Sigma^1$ there is some $\tau \in \Sigma$ such that $H(\tau^1, 0) = H(\tau, 0)$, the pair $(\sigma, 0)$ is a saddle point in $\Sigma^1 \times \Sigma^0$ if and only if σ is socially optimal in $\Gamma(c, f)$. On the other hand, by Lemma 2 and the identity $MC_j{*}(x, 0) = MC_j(x)$, if the convexity condition holds, then $(\sigma, 0)$ is a saddle point if and only if

(17) $$(f - MC(\mu(\sigma))) \cdot \sigma = \max_j (f_j - MC_j(\mu(\sigma_j))).$$

Quick inspection of the proof of Lemma 2 shows, in fact, that the last equivalence remains true if the convexity condition is replaced by the weaker condition of increasing marginal social costs. Therefore, if the latter condition holds, σ is socially optimal in $\Gamma(c, f)$ if and only if it satisfies (17), i.e., it is an equilibrium in $\Gamma(MC, f)$. This proves that the set of socially optimal strategy profiles in $\Gamma(c, f)$ coincides with the set of equilibria in $\Gamma(MC, f)$. Since, by Proposition 1, the latter set is nonempty, and, by Proposition 2, all its elements are the same in terms of the measure of the set of players using each facility, the same two properties hold for the set of all socially optimal strategy profiles in $\Gamma(c, f)$. It remains to show that if this set has at least one element in common with the set of equilibria in $\Gamma(c, f)$, then the two sets are, in fact, equal.

Suppose there is some equilibrium σ in $\Gamma(c, f)$ that is socially optimal. It then follows from the uniqueness of the equilibrium payoffs (Proposition 2) that <u>all</u> the equilibria in $\Gamma(c, f)$ are socially optimal. Conversely, let τ be a socially optimal strategy profile. As shown above, $\mu(\tau) = \mu(\sigma)$. It therefore follows from (2) that $u_i(\tau) \leq u_i(\sigma)$ for almost all i, and hence the inequality (3) holds. However, since τ is assumed to be socially optimal, (3) must, in fact, be an equality. Therefore, the equality $u_i(\tau) = u_i(\sigma)$ must hold for almost all i. This, together with (2) and the equality $\mu(\tau) = \mu(\sigma)$, implies that τ is an equilibrium. ■

Proof of Theorem 1. Clearly, (iii) implies (ii). And if $m \geq 3$, then the converse holds as well. For if $m \geq 3$ and (ii) holds, then, for every j and every $0 < x_j < 1$,

(18) $$x_j \frac{dc_j}{dx}(x_j) = x \frac{dc_k}{dx}(x)$$

for all $k \neq j$ and $0 < x < 1 - x_j$. This implies that the limit $\lim_{x \to 0} x\, c'_k(x)$ (where $c'_k = dc_k/dx$) exists, does not depend on k, and is positive. Denoting this limit by b, Eq. (18) gives $c'_j(x_j) = b/x_j$, for all j and $0 < x_j < 1$. Integrating both sides of this equality, we get $c_j(x_j) - c_j(1) = b \ln x_j$. This gives (iii), with $a = e^{1/b}$.

If $m = 2$, then, in general, (ii) $\not\Rightarrow$ (iii). For example, the cost functions in (4) also satisfy (ii). However, even if $m = 2$, (ii) does imply $c_j(0) = -\infty$, for all j. Indeed, if $x\,c'_1(x) = (1-x)\,c'_2(1-x)$ for all $0 < x < 1$, then $\lim_{x\to 0} x\,c'_1(x) = c'_2(1) > 0$. This implies that there are two constants b, $b' > 0$ such that $b < x\,c'_1(x) < b'$ for all $0 < x < 1$. Hence, $b \ln x > c_1(x) - c_1(1) > b' \ln x$ for all such x. Similar inequalities hold for $c_2(x)$. Therefore, $c_1(0) = c_2(0) = -\infty$. This result will help to prove the equivalence of (i) and (ii).

To prove that (ii) implies (i), suppose that (ii) holds, and let f be a fixed-utility assignment and σ a strategy profile. As shown above, $MC_j(0) = c_j(0) = -\infty$ for all j. Therefore, if $\mu(\sigma_j) = 0$ for some j, then $\max_j (f_j(i) - MC_j(\mu(\sigma_j))) = \max_j (f_j(i) - c_j(\mu(\sigma_j))) = \infty$ for all i. In this case, σ is not an equilibrium in either $\Gamma(MC, f)$ or $\Gamma(c, f)$. If $\mu(\sigma_j) > 0$ for all j, then, by (ii), there is some $b > 0$ such that $MC_j(\mu(\sigma_j)) = c_j(\mu(\sigma_j)) + \mu(\sigma_j)\,c'_j(\mu(\sigma_j)) = c_j(\mu(\sigma_j)) + b$ for all j, and it follows that σ is an equilibrium in $\Gamma(MC, f)$ if and only if it is an equilibrium in $\Gamma(c, f)$. Since, by Proposition 3, σ is an equilibrium in $\Gamma(MC, f)$ if and only if it is socially optimal in $\Gamma(c, f)$, this proves that (i) holds.

To prove that (i) implies (ii), suppose that (ii) does <u>not</u> hold, and let $(x_1, x_2, \dots, x_m)$ be a strictly positive probability vector such that $x_j\,c'_j(x_j)$ is not the same for all j. Since μ is nonatomic, there is a strategy profile σ such that $\mu(\sigma_j) = x_j$ for all j. Consider the constant fixed-utility assignment f defined by $f(i) = c(\mu(\sigma))$ for all i. Clearly, σ is an equilibrium in $\Gamma(c, f)$. Therefore, the aggregate equilibrium payoff (indeed, each player's equilibrium payoff) in this game is zero. A necessary condition for the vector $\mu(\sigma)$ to be a local maximum of the function $z \mapsto (c(\mu(\sigma)) - c(z)) \cdot z$, where $z = (z_1, z_2, \dots, z_m)$ ranges over the set of all strictly positive probability vectors, is that, at the point $z = \mu(\sigma)$, there is some λ (i.e., a Lagrange multiplier) such that $c_j(\mu(\sigma_j)) - c_j(z_j) - z_j\,c'_j(z_j) = \lambda$ for all j. However, since, by assumption, $\mu(\sigma_j)\,c'_j(\mu(\sigma_j))$ is not the same for all j, this necessary condition is not satisfied. Hence, there is some $\tau \in \Sigma$ such that $\int_I (f - c(\mu(\tau))) \cdot \tau\, d\mu = (c(\mu(\sigma)) - c(\mu(\tau))) \cdot \mu(\tau) > 0$. Therefore, the equilibrium σ is not socially optimal in $\Gamma(c, f)$. This proves that (i) does not hold. ■

Proof of Proposition 6. Setting $y = 0$ in the convexity condition gives (i). Assume, then, that the marginal social costs are increasing. As already remarked, this implies that each cost function c_j is continuously differentiable in $(0, \infty)$, its derivative satisfies $c'_j > 0$, and the second derivative c''_j exists almost everywhere. For each j, let the function $c_j(\sqrt[3]{x})$ $(x > 0)$ be denoted by ψ_j. To complete the proof, it suffices to show that, for each j, ψ_j is concave if and only if, for all $y > 0$, $MC_j{}^*(x, y)$ is strictly increasing as a function of x in $(0, \infty)$.

Suppose that $MC_j{}^*(x, y)$ satisfies the last condition. Then,

$$\text{(19)} \qquad \liminf_{\Delta x \to 0} \frac{MC_j{}^*(\Delta x, y) - MC_j{}^*(0, y)}{\Delta x} \geq 0 \text{ for all } y > 0.$$

Consider the nominator in (19). As Δx tends to zero, $MC_j{}^*(\Delta x, y) - MC_j{}^*(0, y) = (\Delta x - y)\, c'_j(y + \Delta x) + y\, c'_j(y) + c_j(y + \Delta x) - c_j(y) = [((\Delta x)^2 - y^2)/(y + \Delta x)]\, c'_j(y + \Delta x) + (y + \Delta x)\, c'_j(y) + o(\Delta x) = -\, y^2\,(y + \Delta x)\, [c'_j(y + \Delta x)/(y + \Delta x)^2 - c'_j(y)/y^2] + o(\Delta x)$. Therefore, for every $y > 0$, the inequality in (19) is equivalent to $\limsup_{\Delta x \to 0} (1/\Delta x)\, [c'_j(y + \Delta x)/(y + \Delta x)^2 - c'_j(y)/y^2] \leq 0$. It follows, in particular, from this equivalence that

$$\frac{d}{dy}\left[\frac{c'_j(y)}{y^2}\right] \leq 0,$$

provided that this derivative exists (which is the case for almost all $y > 0$, since c''_j exists almost everywhere). Since $c'_j(y)/y^2 = 3\,\psi'_j(y^3)$, this proves that the derivative of ψ is nonincreasing. Therefore, ψ is concave.

Conversely, suppose that ψ_j is concave, and its derivative is therefore nonincreasing. As shown above, this implies (19). For every $x \geq 0$, $y > 0$, and $\Delta x > 0$, a little algebra gives that

$$\text{(20)} \qquad MC_j{}^*(x + \Delta x, y) - MC_j{}^*(x, y) = \frac{x}{x + y}\,[MC_j(x + y + \Delta x) - MC_j(x + y)]$$
$$+ \frac{y}{x + y}\,[MC_j{}^*(\Delta x, x + y) - MC_j{}^*(0, x + y)].$$

Therefore, by the assumption of increasing marginal costs of congestion and (19), $\liminf_{\Delta x \to 0} (1/\Delta x)\, [MC_j{}^*(x + \Delta x, y) - MC_j{}^*(x, y)] \geq 0$. In particular,

$$\frac{\partial}{\partial x} MC_j^*(x, y) \geq 0,$$

provided that this derivative exists (which, for every $y > 0$, is the case for almost all $x \geq 0$). This proves that, for every $y > 0$, $MC_j^*(x, y)$ is nondecreasing as a function of x in $[0, \infty)$. By this and the assumption of increasing marginal costs of congestion, the first and second terms on the right-hand side of (20) are, respectively, positive and nonnegative, for every $x > 0$, $y > 0$, and $\Delta x > 0$. Therefore, the left-hand side is positive. This proves that, for every $y > 0$, $MC_j^*(x, y)$ is strictly increasing as a function of x in $(0, \infty)$. ■

Proof of Theorem 2. Fix a fixed-utility assignment f. To prove that υ is well defined, it has to be shown that, for every coalition S, the function H^S defined by (5) has a saddle point. The existence of such a saddle point is implied by Lemma 1. This lemma asserts that, for every ideal coalition h, there is a saddle point in $\Sigma^h \times \Sigma^{1-h}$ of the form $(h\sigma, (1-h)\bar{\sigma})$, where $\sigma, \bar{\sigma} \in \Sigma$. In particular, there is such a point for $h = 1_S$.

Next, it has to be shown that $\upsilon \in pNA$. As a first step, it will be shown that the ideal game $\upsilon^* : \mathcal{J} \to \mathbb{R}$ defined by

$$\upsilon^*(h) = \frac{1}{2}\,[\max_{\sigma^h \in \Sigma^h}\ \min_{\sigma^{1-h} \in \Sigma^{1-h}} H(\sigma^h, \sigma^{1-h}) + \max_{\sigma \in \Sigma^1} H(\sigma, 0)]$$

is differentiable in the sense of [11]. This ideal game "extends" the coalitional game υ in the sense that $\upsilon(S) = \upsilon^*(1_S)$ for all coalitions S.

For $g, h \in \mathcal{J}$, and for every pair of saddle points $(\sigma^h, \sigma^{1-h}) \in \Sigma^h \times \Sigma^{1-h}$ and $(\tau^g, \tau^{1-g}) \in \Sigma^g \times \Sigma^{1-g}$, a little algebra gives

$$2(\upsilon^*(g) - \upsilon^*(h)) = H(\tau^g, \tau^{1-g}) - H(\sigma^h, \sigma^{1-h})$$

$$= [H(\tau^g, \sigma^{1-h}) - H(\sigma^h, \sigma^{1-h})] + [H(\sigma^{1-h}, \tau^g) - H(\tau^{1-g}, \tau^g)].$$

Using (15) twice, first in its original form and then with τ^{1-g}, τ^g, and σ^{1-h} substituted for σ^h, σ^{1-h}, and τ^g, respectively, gives

(21) $$2(\upsilon^*(g) - \upsilon^*(h)) \leq \int_I \max_j\, (f_j - MC_j^*(\mu(\sigma_j^h), \mu(\sigma_j^{1-h})))\,(g-h)\, d\mu$$

$$+ \int_I \max_j\, (f_j - MC_j^*(\mu(\tau_j^{1-g}), \mu(\tau_j^g)))\,((1-h) - (1-g))\, d\mu$$

$$= \int_I \left[\max_j (f_j - MC_j^*(\Theta_j(h))) + \max_j (f_j - MC_j^*(\Theta_j(1-g)))\right] (g-h)\, d\mu,$$

where the function Θ_j is as in Lemma 1. Multiplying both sides of (21) by -1 and interchanging g and h, we get

$$2(v^*(g) - v^*(h)) \geq \int_I \left[\max_j (f_j - MC_j^*(\Theta_j(g))) + \max_j (f_j - MC_j^*(\Theta_j(1-h)))\right] (g-h)\, d\mu.$$

It follows from this inequality and (21) that, for some constant $0 \leq \theta \leq 1$,

$$v^*(g) - v^*(h) = \int_I \frac{1}{2}\left[\max_j (f_j - MC_j^*(\Theta_j(h))) + \max_j (f_j - MC_j^*(\Theta_j(1-h))) + \varepsilon_\theta^{g,h}\right] (g-h)\, d\mu,$$

where $\varepsilon_\theta^{g,h} : I \to \mathbb{R}$ is defined by

$$\varepsilon_\theta^{g,h}(i) = \theta\left[\max_j (f_j(i) - MC_j^*(\Theta_j(g))) - \max_j (f_j(i) - MC_j^*(\Theta_j(h)))\right]$$
$$+ (1-\theta)\left[\max_j (f_j(i) - MC_j^*(\Theta_j(1-g))) - \max_j (f_j(i) - MC_j^*(\Theta_j(1-h)))\right].$$

Since, for every $i \in I$ and $0 \leq \theta \leq 1$, $|\varepsilon_\theta^{g,h}(i)| \leq \max_j |MC_j^*(\Theta_j(g)) - MC_j^*(\Theta_j(h))| + \max_j |MC_j^*(\Theta_j(1-g)) - MC_j^*(\Theta_j(1-h))|$, the continuity of Θ_j and MC_j^* implies that $\varepsilon_\theta^{g,h}(i) \to 0$ uniformly in i and θ when $g \to h$. This proves (see [11]) that v^* is differentiable at the point h, its derivative there $Dv^*(h)$ is (a nonatomic measure which is) absolutely continuous with respect to μ, and

$$(22) \qquad \frac{d(Dv^*(h))}{d\mu} = \frac{1}{2}\left[\max_j (f_j - MC_j^*(\Theta_j(h))) + \max_j (f_j - MC_j^*(\Theta_j(1-h)))\right].$$

It follows from the last part of Lemma 2 that the right-hand side of (22) is in $L_\infty(\mu)$.

The function $d(Dv^*(\cdot))/d\mu : \mathcal{I} \to L_\infty(\mu)$ is continuous. Indeed, $\|d(Dv^*(g))/d\mu - d(Dv^*(h))/d\mu\|_\infty = \|\varepsilon_{1/2}^{g,h}\|_\infty \to 0$ when $g \to h$. It follows, by [11, Theorem 2], that the game v is in pNA (indeed, pNA_∞) and its Aumann-Shapley value φv is given by

$$(\varphi v)(S) = \int_0^1 Dv^*(t)(S)\, dt \qquad (S \in \mathcal{C})$$

(where t is identified with the corresponding ideal coalition). By (22), this formula gives (7). The two equations in (8), and the fact that the inner integral in (7) is uniquely determined by them, follow from Lemmas 1 and 2. ■

Proof of Proposition 7. To prove the conclusion of the proposition, it suffices to assume that condition (ii) in Theorem 1 holds (which is weaker than condition (iii)). As shown in the proof of that theorem, this condition implies that, for every fixed-utility assignment f and every equilibrium σ in $\Gamma(c, f)$, $\mu(\sigma_j) > 0$ for all j. Hence, by condition (ii), there is a constant b such that, for all j and $0 \leq t \leq 1$,

$$(23) \quad MC_j{}^*(\mu(t\sigma_j), \mu((1-t)\sigma_j)) = c_j(\mu(\sigma_j)) + (t\mu(\sigma_j) - (1-t)\mu(\sigma_j))\, c'_j(\mu(\sigma_j))$$
$$= c_j(\mu(\sigma_j)) + (2t-1)\, b.$$

Therefore, it follows from (2) that, for all $0 \leq t \leq 1$, (8) holds with $\bar{\sigma} = \sigma$, $\sigma^t = t\sigma$, and $\sigma^{1-t} = (1-t)\,\sigma$. Eqs. (7) and (23) then give $(\varphi v)(S) = \int_S (f - c(\mu(\sigma))) \cdot \sigma\, d\mu - b\,\mu(S) \int_0^1 (2t-1)\, dt$ $(S \in \mathcal{C})$. Since the second integral is zero, this shows that the Harsanyi UT value is equal to the equilibrium payoff distribution. ■

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