Time-Differentiated Monopolies

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Abstract

We consider sequential competition among sellers, who recognize future sellers as potential competitors and therefore do not necessarily set their monopoly prices, which are the prices they would set if consumers could only buy from the first arriving seller. We show, however, that whether an equilibrium price is indeed lower than the sellers' monopoly prices depends on the form of consumers' impatience. With time discounting, this is so. But when impatience reflects decreasing valuations, the equilibrium price may coincide with the sellers' monopoly prices, which means that their market power is not diminished by competition with future sellers.

JEL Classifications C73, D91, L13

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1 Introduction

Firms or sellers that are alone in a particular market at a particular time have greater market power than they would in the presence of close competitors. This market power, however, may be constrained when consumers believe that future competitors will offer the good at a lower price and so defer buying it.[1] In this respect, competition between present and future sellers resembles that between producers of partially substitutable products. However, temporal separation between physically identical goods differs from product differentiation along other dimensions in several important ways. For example, if the current seller is later replaced by another seller, then a consumer who chooses not to buy from the current seller cannot later reverse himself even if the price set by a later seller turns out to be higher than he expected. In addition, that price may be affected by the demand that the later seller faces, which in turn depends on the number of consumers who deferred buying the good. Moreover, sequential competition often involves uncertainty—both the current seller and the current consumers may be uncertain about when the good will again be on the market.[2] Thus sequential competition creates unique strategic interdependencies between sellers and consumers, and also involves issues of information and beliefs that do not always arise for other forms of competition.

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[1] Zeithammer (2006) gives evidence that consumers' decisions are affected by expectations of future sales of a good. He finds that consumers on eBay bidding in a current auction and expecting a future auction for a similar good reduce their bids.

[2] The appearance of sellers at uncertain times can apply when supply depends on the vagaries of nature. An historically important example relates to sailing ships, which were subject to the whim of the winds, making the length of a voyage over a given route uncertain, and so creating much uncertainty about when the next ship will be in port and be available for another sea voyage (Marvin, 1902). For another example, given the risks of rockets failing or of the space shuttle needing repair, a firm wishing to launch a satellite is unsure about the times at which it would be able to launch.

As Diamond (1971) demonstrated, competing identical firms may enjoy effective monopoly power if each consumer bears an arbitrarily small cost in switching from one firm to another. This is because the switching cost makes consumers buy from a firm at a price that leaves them positive payoff even if it is slightly higher than the competitors' price. Hence, each firm can raise its price slightly above what other firms charge, and will do so as long as the others charge less than the monopoly price.

The analog of a firm's monopoly price in our case of sequential competition is the price it would set if every consumer could only buy from the first seller he encounters. Switching costs correspond to the loss of utility due to the waiting time to the next seller, which represents the consumers' impatience. It turns out, however, that whether these costs indeed result in the sellers enjoying complete monopoly power depends on the exact nature of the consumers' impatience. If the consumers' impatience reflects declining valuations, meaning that the item's value to them is lower the later it is purchased,[3] then Diamond's conclusion essentially holds: a common monopoly price for the sellers can be an equilibrium price. However, if consumers simply discount the future, then an equilibrium price will generally be lower than the monopoly price. These two forms of impatience are not equivalent. With time discounting, a consumer prefers buying the good sooner rather than later, but the maximum price he is willing to pay is the same in both cases. Declining valuation, by contrast, means that the consumer's willingness to pay depends on the time of the transaction. As we show in this paper, this difference has a dramatic effect on the equilibrium prices and thus on the sellers' gain from the absence of contemporary competitors.

Another important determinant of the effect of competition on the sellers' market power is the information they have about the history of sales or the current market conditions. As this paper shows, if the sellers are completely uninformed and the consumers have declining valuations, a common monopoly price is the *only* possible equilibrium price. But in the more realistic case of informed sellers, who in particular know if their predecessor raised the price and consequently lost some customers, lower equilibrium prices may also exist. Thus, somewhat paradoxically, sellers may suffer from being better informed.

Information about the history of sales may harm sellers because it adds credibility to a threat by consumers not to buy at a price higher than some specified maximum. The threat is implicit in the following strategy. If a seller sets a high price, then some of the consumers who would get positive payoff from buying the good refrain from doing so. Their number, which is determined by the strategy, induces the next seller to lower the price precisely to the point which equalizes the payoff of the consumers who did and did not buy from the first seller. Hence, no consumer is harmed by following the strategy. Thus, the credibility of the consumers' threat depends on the next seller's ability to react optimally to his predecessor's price, and in particular to set a lower price if old consumers, with relatively low valuations of the good, are sufficiently numerous.

Related literature

Models of dynamic competition have yielded two extreme, opposing, results. When a buyer faces a cost of searching for a seller, each seller may be able to charge the monopoly price, as if there were no other sellers (Diamond, 1971). At the other extreme, if a monopoly sells the good in later periods, it creates its own competition, and with rational expectations by buyers the price will be the same as under perfect competition (Coase, 1972). The reasoning is that after selling the good at some price, the firm faces a new demand curve from remaining consumers. The firm will want to sell to them, at a lower price, and the process continues until each consumer buys the good. The initial consumers, realizing that the seller has such incentives, would then be willing to wait, to buy at the lower price.

[3]For example, a parent may value a crib more just before a baby's birth than months afterward, when she needs it for little more time.

Much of the work considering dynamic competition has a consumer buying one unit of the good (as we do) and a seller selling only one unit (whereas we have elastic supply by the seller). Thus Said (2011) considers sequential auctions with randomly arriving buyers and sellers. A key result is that the seller will almost surely trade with the first buyer at a price close to the lowest possible valuation. Satterthwaite and Shneyerov (2007) study a first-price auction with a reservation price. A buyer or seller who fails to trade remains in the market, is rematched in the next period, and tries again to trade. We, in contrast, have the seller post a price, and new sellers appear in future periods. Bargaining, rather than auctions, with the appearance of new buyers or sellers, is modelled by Taylor (1995), Inderst (2008), and Fuchs and Skrzypacz (2010).

Dynamic considerations appear in Fudenberg et al. (1987), who model a seller of an indivisible good. They have the seller uncertain about the buyer's valuation, and becoming more pessimistic over time. Multiple equilibria can appear, because of an externality between the different buyers: if future buyers are likely to accept high prices, the seller is likely to switch buyers, and thus the current buyer should also accept a high price. The opposite holds if future buyers are expected to reject a high price. This general idea also appears in our paper, but for a different reason—if buyers with declining valuations do not buy now, the future price a profit-maximizing seller will post will be low.

The assumption of declining valuation for a good relates to work on transportation, where each consumer valuation of a trip depends on when it is provided. In particular, work on schedule delay in airlines (see Douglas and Miller, 1974, and Devany, 1975) considers consumers who want to depart at a particular time, and are willing to pay less for an airline ticket when the flight is earlier or later than the most preferred departure time. An airline, recognizing such behavior, then realizes that it can charge a higher price the more frequent the flights it offers, so that the schedule delay is small. Related ideas appear in the bottleneck model of commuting to work, where each person has a desired time of arriving to work, incurring a cost if he arrives too early or too late (Vickrey, 1969, and Arnott et al., 1993).

2 The Model

A tractable model of sequential competition requires making some assumptions. In our model, a consumer buys one or no unit of a particular good, and sellers have an unlimited supply of it. This rules out, for example, auctions in which sellers only own a single unit. Both sides of the market stay there only for a short time: a mere instant for the sellers, who therefore do not overlap, and a fixed, finite period for the buyers, which we take as the unit of time. The time axis extends from $-\infty$ to ∞. The origin, $t = 0$, has no special significance.

Consumers arrive at the market in a steady flow. The total mass, or "number", of consumers that arrive in a unit of time is taken as the unit of mass. Sellers arrive one at a time. We model their arrival as a simple point process $\boldsymbol{T}$ on the time axis (Daley and Vere-Jones, 1988). A realization of the process is a finite or infinite collection of distinct points, which represent the sellers' arrival times. This excludes simultaneous arrivals of sellers but includes deterministic arrival times and, in particular, arrival at regular intervals. Other special cases of our model are a predetermined number of sellers with random arrival times, and arrival only on positive t's. In general, however, both the sellers' arrival times and their number may be uncertain, and a first and a last seller may or may not exist. To exclude certain artificial, unreasonable cases, we assume that the total number of sellers arriving in any finite time interval has a finite expectation (which defines the so-called *mean measure* of the process $\boldsymbol{T}$) and that the probability of very short inter-arrival times is small. Specifically, the latter means that for every $\delta < 1$ there is $\epsilon > 0$ such that, at every time t at which a seller arrives and for all possible histories of previous arrivals, the (conditional) probability that the waiting time to the next seller is greater than ϵ (or that no more sellers arrive) is greater than δ. We also assume that there is non-zero probability that *some* seller arrives less than one unit of time after his predecessor's arrival. This assumption excludes the uninteresting case in which no seller ever competes with another.

A consumer enters the market, or is "born", at a certain time c and leaves it when he buys the good or at time $c + 1$,whichever comes first. At every moment, each consumer has a certain valuation of the good, $0 \leq v \leq 1$, which is the maximum price he may be willing to pay for it. The consumer's payoff from buying at any price $0 \leq p \leq v$ is $v - p$, and if he never buys the good, the payoff is zero. For any acceptable price, the consumer would prefer buying the good right away rather than later. We consider two alternative sources for this impatience: *declining valuations* and *time discounting*. In the first model, the value v that a consumer assigns to the good decreases has his "age" x, or the time the consumer has been in the market, increases. We model this dependence by setting $v = 1 - x$. In the second model, a consumer's valuation v is fixed throughout his lifetime. Because this means that we cannot rely on age differences to create demand heterogeneity, we assume that the consumers' valuations are varied already at birth, specifically, that v is uniformly distributed between 0 and 1. The consumers' (identical) time preferences take the form of an exponential discount function $e^{-\rho t}$, where ρ is the (common) discount rate. At any time t, the payoff that a consumer with valuation v attaches to buying the good at price $0 \leq p \leq v$ at a later time $t + \tau$ is $e^{-\rho\tau}(v - p)$.

A seller sets and announces a price $0 \leq p \leq 1$ for a unit of the good, sells the demanded quantity, which he produces at zero cost, and then immediately leaves the market. His payoff is the profit from selling the good, which equals the revenue.

Information and strategies

With sequential competition, the sellers' and buyers' possible strategies depend on the information they have. To simplify the analysis, this paper considers only the two extreme cases of sellers with either perfect information about the history of prior sales or no information at all. Information or lack thereof turns out to greatly affect the sellers' market power, so allowing only one of these possibilities would be overly restrictive. In both cases, we suppose buyers are perfectly informed about the past.

The (possible) randomness of the sellers' arrivals makes our model a variant of a random-player game (Milchtaich, 2004). In such a game, strategies are ascribed not to individual agents but to agent types.

A seller's type is his arrival time t. A strategy for type t is a rule that assigns an asking price $0 \leq p \leq 1$ to each possible history at time t. Such a history H_t is a complete description of all relevant past events: the arrival times of the previous sellers, the prices they set, and the total mass and age distribution of the consumers who bought the good from them. A strategy is *feasible* for a seller if it depends only on information about the history that the seller actually possesses. Hence, the better informed sellers are about the past, the larger are their sets of feasible strategies. When sellers are (perfectly) *informed*, their feasible strategies are all the strategies of their respective types. With (completely) *uninformed* sellers, the feasible strategies for each type of seller are simply specifications of an asking price $0 \leq p \leq 1$. Other conceivable settings are that some sellers are informed and some are uninformed, or that sellers are only partially informed, for example, informed about the previous sellers' asking prices but not about the consumers' reaction to them. For tractability, however, we consider only the two extreme cases described above.

Each seller has certain (posterior) *beliefs* about the other sellers' arrival times. Even for an uninformed seller, these beliefs are not necessarily identical to those derived from the *common prior*, which is the distribution of the point process $\boldsymbol{T}$. The difference arises because an uninformed seller who arrives at time t knows something that was not necessarily known in advance, namely, that a seller arrived at time t. This information may give an indication about the other arrival times. If the seller is uninformed, no additional information is available to him, so that his posterior about the arrival times is the conditional distribution of $\boldsymbol{T}$, given that a seller arrived at time t. This conditional distribution is called the *Palm distribution* (Kallenberg, 1986). For an informed seller, the posterior is obtained by further conditioning the Palm distribution on the actual arrival times of the previous sellers, that is, by taking into consideration both the seller's own arrival time and the history. For both kinds of sellers, the posterior induces a probability distribution for each variable that can be expressed as a function of (some or all of) the arrival times, such as the total number of sellers, the time σ from the last seller's appearance, and the waiting time τ to the next seller (with σ or τ defined as ∞ if there are no earlier or later sellers, respectively). If the distribution is degenerate, meaning that it assigns probability 1 to a particular value, then we say that the seller *knows* the variable. For example, an informed seller by definition knows σ, whereas an uninformed seller may or may not know it. Note that, regardless of whether or not sellers are informed, the probability that they *all* know that $\sigma \geq 1$ is less than 1. This is because, by assumption, the (prior) probability that the sellers' arrivals are all one or more units of time apart is zero.

A consumer's type is specified by his time of birth c and his valuation of the good at that time (which determines the valuation at any later time). A strategy for a consumer of type c is a rule that assigns either the decision Buy or Wait to each buying opportunity he may encounter. A buying opportunity is specified by the arrival time t of the seller (with $c \leq t \leq c+1$), the posted price $0 \leq p \leq 1$ and the history H_t. To simplify the analysis, we assume that all consumers are informed, that is, they know the history, so that all the strategies of their respective types are feasible. This assumption entails that whenever a seller arrives at the market, all the consumers have identical posterior beliefs about the future arrival times, which coincide with those of the arriving seller if he is also informed.

The information, beliefs and strategies of the sellers and consumers together determine each agent's expectation at each moment regarding his gain from unilaterally switching to any feasible strategy different from that specified for his type. For an uninformed agent, "unilateral" means that all the other agents' actions accord with their strategies. For an informed agent, the meaning is similar, but only concerning future actions; the history may or may not be consistent with the strategy profile. If the expected gain is positive, the deviation is profitable for the agent. A strategy profile is a (Bayesian perfect) *equilibrium* if profitable deviations do not exist. Note that this requirement concerns also histories that are not consistent with the strategy profile, which means that it excludes irrational off-equilibrium behavior by informed agents.

3 Monopoly Prices

Our main concern in this paper is the effect of competition from future sellers on the sellers' market power. Competition has no effect if the prices sellers set and the profits they earn are the same as they would be if each consumer could only buy from the first seller he encounters, in other words, if sellers had complete monopoly power.

A seller may face both young consumers, who were born after the previous seller appeared and so did not yet have a chance to buy the good, and older consumers, who could buy from the previous seller but chose not to do so. For $0 \leq p \leq 1$, denote by $\pi_M(p)$ the seller's expected profit from selling the good at that price to young consumers only, assuming that each such consumer who values the good at more than p buys it. This defines the seller's *monopoly profit function* $\pi_M : [0,1] \to [0,1]$, the form of which depends on the kind of consumers' impatience (see Figure 1). With declining valuations,

$$\pi_M(p) = p\,\mathbb{E}[\min(\sigma, 1-p)], \tag{1}$$

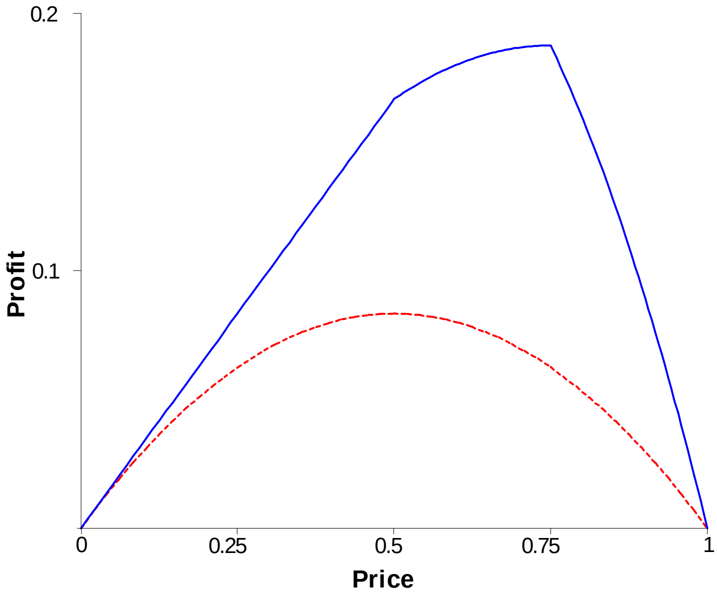

Figure 1: The monopoly profit function π_M of a seller who is uncertain about the time σ since the last seller appeared: with probability $\frac{2}{3}$, $\sigma = \frac{1}{4}$, and with probability $\frac{1}{3}$, $\sigma = \frac{1}{2}$. The consumers are impatient, either because of declining valuations (solid line) or time discounting (dashed line). The monopoly profit function peaks at the monopoly price, which is $p^M = \frac{3}{4}$ in the first case and $p^M = \frac{1}{2}$ in the second case.

and with time discounting,

$$\pi_M(p) = \mathbb{E}[\min(\sigma, 1)]\, p\,(1-p). \tag{2}$$

In these equations, σ is the time from the appearance of the previous seller and the expectation is with respect to the seller's beliefs. Thus, one seller's π_M may be different from that of another. The unique (see Lemma 1 in Appendix A) maximum point $0 < p^M < 1$ of π_M is the seller's *monopoly price*, and $\pi_M(p^M)$ is his *monopoly profit*. With declining valuations, different sellers generally have different monopoly prices, which are all greater than or equal to $\frac{1}{2}$. With time discounting, $p^M = \frac{1}{2}$ always.

The counterpart on the consumers' side of monopolistic pricing is the *monopoly strategy*. This strategy, which is oblivious to the current market conditions and the history, simply instructs the consumers to buy the good whenever doing so yields them positive payoff (with the decision in the borderline case of zero payoff from buying left unspecified, as it is inconsequential to both the consumer and the other agents). The monopoly strategy would be optimal if the possibility of buying from a later seller were absent.

4 Uninformed Sellers

The case of uninformed sellers may be viewed as a benchmark. In the next section, we compare it to the arguably more realistic, but more involved, case of informed sellers, and thus gain appreciation of the effect of the sellers' information on their profits.

By definition, uninformed sellers do not know the previous sellers' prices and the consumers' response to them. In addition, they may know the previous sellers' arrival times only if this information can be inferred from their own arrival time. Therefore, if the time of arrival does not provide any information about how long ago the previous sellers appeared, all

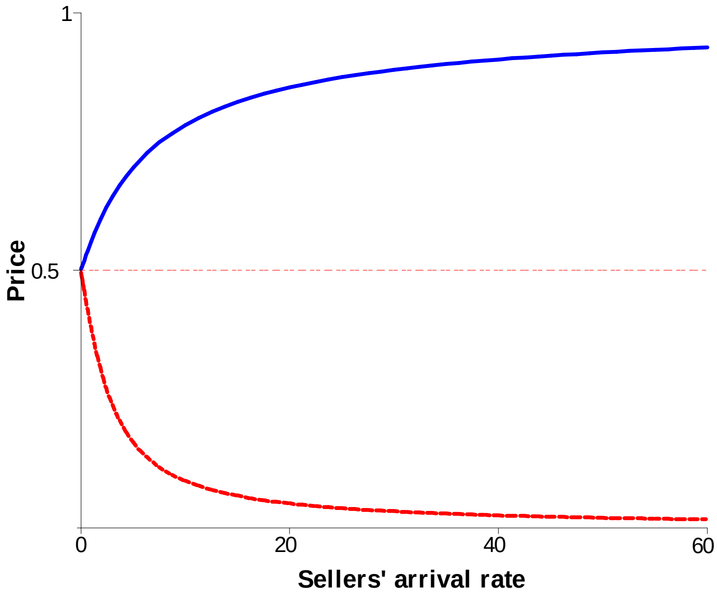

Figure 2: The highest equilibrium price in Example 1 as a function of the sellers' arrival rate λ. When players have declining valuations, the equilbrium price is in fact unique (thick solid line), and coincides with the seller' monopoly price. With time discounting, the highest equilibrium price (thick dashed line) is lower than the (constant) monopoly price (thin dashed line).

sellers have the same monopoly price, and it is reasonable to expect that the prices they will set are identical as well. The next two examples look at whether these two prices are equal. All the assertions made concerning these and the other examples in this paper are proved in Appendix B.

Example 1 *Uninformed sellers arrive according to a Poisson process. That is, the time τ from one seller to the next is independent of past arrivals and has an exponential distribution with parameter λ, which is the sellers' arrival rate. If the consumers have declining valuations, there is an equilibrium in which all sellers set the same price, and the same is true with time discounting if the discount rate ρ is sufficiently low. In the first case, the equilibrium price is unique, and is equal to the sellers' common monopoly price. In the second case, there are multiple equilibrium prices, which are all lower than the monopoly price.*

An additional difference between the two forms of consumers' impatience in Example 1 is that, as Figure 2 shows, with declining valuations, the equilibrium price increases rather than decreases as the sellers' arrival rate increases and competition thus intensifies. This seemingly paradoxical finding is actually a necessary consequence of declining valuations.[4] The price rises because when sellers arrive soon after each other, they face relatively young potential customers, who are willing to pay more. Parenthetically, the negative effect of this price increase on social welfare is not strong enough to negate the positive effect of the shorter waiting times to the next seller. As the sellers' arrival rate increases, so does the expected total equilibrium payoff (to sellers and consumers combined) in any time interval.

The next example differs from the previous one in that, even with consumers having declining valuations, the sellers' identical monopoly price is not necessarily an equilibrium

[4]Chen and Frank (2004) observe a somewhat similar phenomenon in a queuing system. In their model, a monopolistic server charges a profit-maximizing service fee. Because an increase in the number of customers admitted increases the expected queuing time, this fee generally declines with demand.

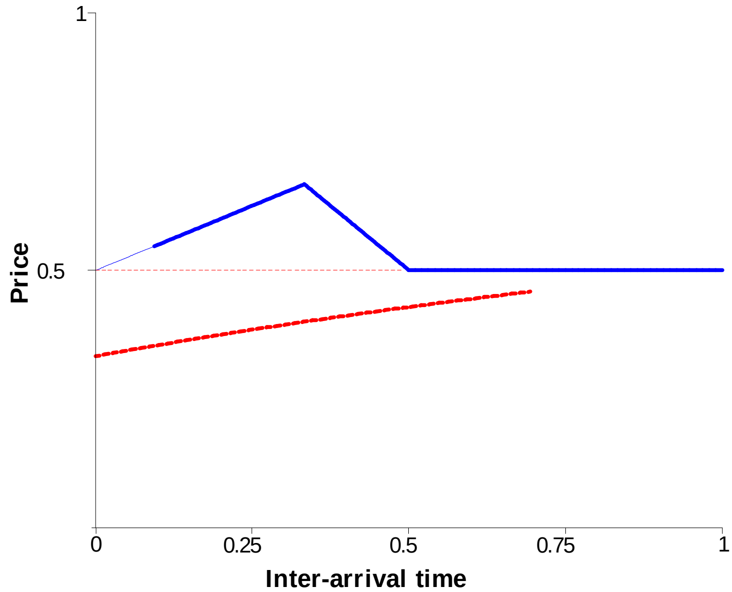

Figure 3: The monopoly price and the highest equilibrim price in Example 2 as a function of the time s between the two sellers' arrivals. In the case in which consumers have declining valuations, the equibrlium price is in fact unique, when it exists (thick solid line), and coincides with the sellers' monopoly price (thin solid line). In the case of time discounting, with unit discount rate $\rho = 1$, the highest equilibrium price (thick dashed line) is lower than the monopoly price (thin dashed line).

price. Indeed, for both forms of consumer impatience, an equilibrium with a common equilibrium price may not exist (see Figure 3). The basic reason for the possible nonexistence of an equilibrium price is that such a price is required to leave sellers with no incentive to either increase or decrease their prices. These two requirements may well be contradictory, as they involve different kinds of considerations. The profitability of a price decrease depends on the existence in the market of old consumers, who had a chance to buy at the equilibrium price but did not do so, whereas the profitability of a price increase depends on the consumers' willingness to buy also at the higher price rather than wait to the next seller who will sell at the equilibrium price.

Example 2 *There are two, uninformed sellers, one arriving* $0 < s < 1$ *units of time after the other. An arriving seller does not know whether he is first or second but the consumers do know that.*[5] *If the consumers have declining valuations, then an equilibrium in which both sellers set the same price exists if and only if* $s \geq \frac{1}{5+\sqrt{32}}$ (≈ 0.094)*, and that price is unique and is equal to the sellers' monopoly price. With time discounting, equilibria in which both sellers set the same price exist if and only if* $e^{-\rho s} \geq \frac{1}{2}$*, and these equilibrium prices are all lower than the monopoly price.*

The relations between the monopoly and equilibrium prices seen in the above examples hold very generally. This is shown by the following theorem, which, as proved in Appendix A, holds even under more general assumptions than those in Section 1. In particular, it holds

[5]The sellers' ignorance of their relative position can be modeled as follows. Let t be a random variable that is uniformly distributed on $[0, K]$, for some fixed large number K. Suppose that the sellers' arrival times are $\ldots, -K+t, -K+t+s, t, t+s, K+t, K+t+S, 2K+t, 2K+t+s, \ldots$. Thus, there are infinitely many pairs of sellers as above, which do not influence one another because of the long time periods separating them. As the consumers know the history of previous arrivals, (only) they can tell when the next seller will arrive.

with any discount function, not only an exponential one, and for certain kinds of non-linearly declining valuations. The same applies to Theorem 2 below.

Theorem 1 *Suppose that the sellers are uninformed and that there is an equilibrium in which they all set the same price p^E. If the consumers have declining valuations, then p^E necessarily coincides with the monopoly price of each of the sellers. By contrast, with time discounting, p^E is necessarily lower than the sellers' (common) monopoly price.*

As Theorem 1 shows, whether competition with future sellers reduces the monopoly power of uninformed sellers depends on the form of the consumers' impatience. The proof of the theorem in Appendix A reveals the reason for this dependence.

To understand the reason, recall that a seller's monopoly profit function specifies the connection between the price the seller sets and the profit under the (counterfactual) assumption that consumers are only allowed to buy the good from the first seller they meet. Suppose that the monopoly price, which maximizes the function, is the same for all sellers and that they all set that price. A seller who unilaterally deviates by slightly reducing his price decreases his monopoly profit, but on the other hand, attracts potential consumers of the previous sellers for whom the monopoly price was just a little high. As for every differentiable function, at the maximum point of the monopoly profit function the *first-order* effect of changing the price is zero. With time discounting, this means that the second, positive effect of attracting old consumers dominates, so that the price reduction is profitable. This explains why, in this case, an equilibrium price must be lower than the monopoly price. With declining valuations, this is not so. In this case, the positive effect of the price reduction is also of low order. This is because the time that passed since the previous sellers' visits means that, to attract their potential customer who are still in the market, a more drastic price reduction is needed.

A similar difference between the two forms of consumers' impatience applies to price increases. The effect of raising the price to slightly above the competitors' price can again be decomposed into an effect on the monopoly profit and an additional effect, which in this case is negative and expresses the loss that results from some consumers' decision to wait to the next seller. The second effect is again of low order in the case of declining valuations, which means that the condition for profit maximization coincides with the first-order condition of zero marginal monopoly profit. Therefore, with declining valuations, *only* the monopoly price can be an equilibrium price. By contrast, with time discounting, the loss of customers to the competition is of the same order as (in other words, roughly proportional to) the price increase. Herein lies the crucial difference from Diamond's (1971) model, where a seller can increase his price with impunity to above the competitors' price as long as the price gap remains below the consumers' switching cost. This difference explains why, with time discounting, the conclusion that an equilibrium price must be the monopoly price does not hold.

Different equilibrium prices

When sellers differ in their beliefs about how long ago the previous sellers arrived, they may well differ also in the prices they set in equilibrium. In this case, even if the consumers have declining valuations, each seller may have a unique equilibrium price that is different from his monopoly price. This is demonstrated by the following example, in which the two sellers are uninformed but they know whether they are first or second as well as the length of the time interval between their arrivals. Depending on the latter, the two sellers' equilibrium prices may be lower than their respective monopoly prices, may coincide with them, or may not exist (see Figure 4).

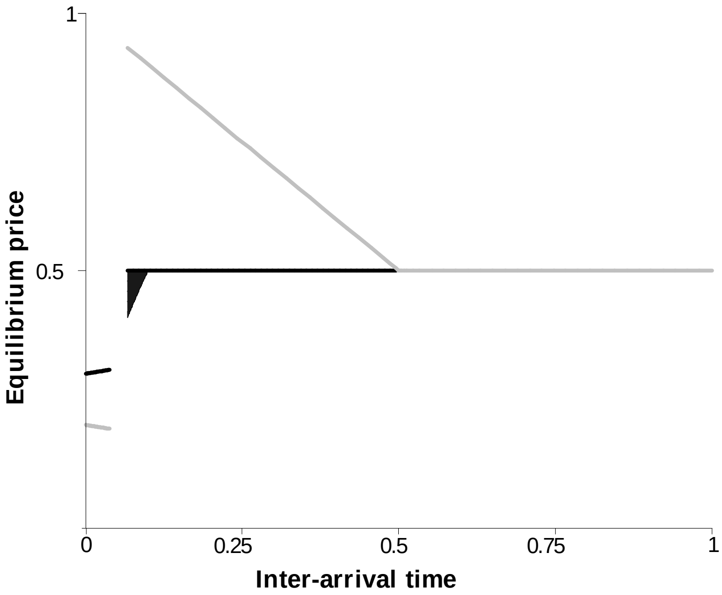

Figure 4: The equilbrium prices for the first and second seller (black and gray lines, respectively) in Example 3 as a function of the time s between their arrivals. Prices in the shaded area that are lower than the first seller's monoploy price of $\frac{1}{2}$ are not equilibrium prices for him in this example, but they are so in Example 4 (in the case where the probability that the second seller arrives is $\frac{1}{2}$).

Example 3 *One seller arrives for sure at time* 0. *A second, uninformed seller arrives with probability* $\frac{1}{2}$ *at (a fixed) time* $0 < s < 1$, *and with probability* $\frac{1}{2}$ *he never appears. The consumers have declining valuations. If* $s \leq \frac{1}{26}$ *or* $s \geq \frac{1}{8+\sqrt{48}}$, *there is a unique equilibrium price for each seller, which is respectively lower than or equal to his monopoly price, but if* $\frac{1}{26} < s < \frac{1}{8+\sqrt{48}}$, *an equilibrium does not exist.*

The reason the second seller in Example 3 sets a price lower than his monopoly price (and than the first seller's price) if he comes almost immediately after the first seller (specifically, if $s \leq \frac{1}{26} \approx 0.038$) is that he has in this case only very few potential customers who are young and willing to pay a high price for the good. The seller can therefore profit from setting a low price, which attracts some of the old consumers born before the first seller appeared. Anticipating this price reduction, some consumers who would gain little by buying from the first seller wait for the second seller. Therefore, the first seller cannot assume that all the consumers with positive payoff from buying will do so, which forces him to lower his price and receive a profit lower than his monopoly one. The profit for the second seller exceeds his monopoly profit, which he can always get regardless of what the first seller does.

5 Informed Sellers

If the inter-arrival time in Example 3 is not very short, specifically, if $s \geq \frac{1}{8+\sqrt{48}}$ (≈ 0.067), then the above argument does not apply and the monopoly prices are in fact equilibrium prices. The same conclusion holds much more generally. In particular, it holds with informed as well as uninformed sellers, as long as sellers never appear in very short succession and (as in Example 3) they know how long ago the previous seller arrived. Moreover, the exact meaning of "very short" here is the same as in the example. We interpret this result as saying that, with

Figure 5: The consumers' equilibrium strategy in Example 4. A consumer buys the good from the first seller if and only if his age and the price fall within the shaded area. The depicted strategy corresponds to an equilibrium price of $p^* = \frac{23}{53}$ and to the case where the second seller is equally likely to arrive $\frac{4}{53}$ units of time after the first seller or not at all.

declining valuations, competition from future sellers does not necessarily reduce the sellers' monopoly power.

Theorem 2 *Regardless of whether the sellers are informed or uninformed, if each of them knows the time since the appearance of the previous seller and that time is never shorter than* $\frac{1}{8+\sqrt{48}}$*, and if the consumers have declining valuations, there is an equilibrium in which every seller sells at his monopoly price and every consumer that would get positive payoff from buying at that price does so.*

Where the cases of informed and uninformed sellers fundamentally differ, even under the special circumstances considered in Theorem 2, is the existence of equilibrium prices lower than the monopoly ones. Getting such prices requires the consumers to play an active role, so to speak, in the determination of the prices, and punish a seller who sets a high price by reducing his sells. Crucially, the next seller must be able to know that not all the consumers who could buy from his predecessor did buy, so that he can set his own price accordingly.

The following example demonstrates this possibility. It differs from Example 3 mainly in that the second seller is informed (and also in only considering inter-arrival times that fall within a specific, narrow range, but on the other hand, allowing the second seller to appear with high probability, and even with certainty). The difference results in additional equilibria, in which the first seller sets lower prices than his monopoly price of $\frac{1}{2}$ (see Figure 4).

Example 4 *One seller arrives for sure at time* 0*. A second, informed seller arrives with probability* $\frac{1}{2} \le \alpha \le 1$ *at (a fixed) time* $\frac{1}{8+\sqrt{48}} \le s \le \frac{1}{10}$*, and with probability* $1-\alpha$ *he never appears. The consumers have declining valuations. For any given price* $\alpha s + (2-\alpha)\sqrt{s(1-s)} \le p^* \le \frac{1}{2}$*, there is an equilibrium in which the first seller's price is* p^* *and the second seller's price is* $1-s$*.*

The consumers' equilibrium strategy in this example is to buy the good from the second seller at any price that gives them positive payoff, but to buy from the first seller only if the price also does not exceed some threshold, which depends on their age. Specifically, older consumers who would receive low payoff from buying the good from the first seller at the equilibrium price p^* do not buy it at *any* higher price, while younger consumers may buy also at prices that are slightly higher than p^* (see Figure 5). This strategy entails that the punishment of a seller who sets a higher price than p^* reflects the severity of the deviation; it is not an all-or-nothing punishment. An all-or-nothing punishment strategy would not be credible here. This is because, if a price p that is not much higher than p^* prompted a boycott of the first seller by all the consumers, including the younger ones, who are willing to pay more, the second seller would have less incentive to lower his price. That price would then remain relatively high, leaving the older consumers with lower payoff than they would get from buying from the first seller at price p, which means that joining the others in severely punishing that seller was against these consumers' own interests.

Credibility, that is, rationality of the off-equilibrium behavior specified by the agents' strategies, is an important consideration. Even with a single seller, equilibria exist in which a price lower than the monopoly price is supported by a consumers' threat not to buy at any higher price. However, these equilibria are not subgame perfect. If the seller sets a moderately high price, it may be optimal for individual consumers to buy the good after all. A credible threat is possible only with sequential competition, and only if the sellers are sufficiently informed about the previous sellers' behavior or about the consumers' demand.

As all the equilibria identified in Example 4 satisfy the credibility requirement, and both sellers and consumers are informed, none of the equilibria is eliminated by any obvious notion of equilibrium refinement. Multiplicity of equilibrium prices in this example thus appears to be a robust inherent property.

A Proofs: Theorems

The following proofs of the papers' two theorems actually concern more general forms of consumers' impatience than specified in Section 2.

For the case of time discounting, the exponential discount function may be replaced by any continuous and strictly decreasing function $D : [0,1] \to [0,1]$ with $D(0) = 1$. In addition, the assumption that the distribution of the consumers' valuations is uniform is generalized by allowing any continuous distribution on the unit interval $[0,1]$, with cumulative distribution function F and density f, such that there is a unique value of p maximizing

$$p\,(1 - F(p)).$$

That maximum point $0 < p^M < 1$ is the sellers' (common) monopoly price. Correspondingly, the equation for the monopoly profit function is generalized from (2) to

$$\pi_M(p) = \mathbb{E}[\min(\sigma, 1)]\, p\,(1 - F(p)).$$

For the case of declining valuations, the assumption in Section 2 is that the *valuation function* $v : [0,1] \to [0,1]$ specifying the consumers' valuation $v(x)$ at each age x decreases *linearly* from 1 to 0, which implies the same for the function $a : [0,1] \to [0,1]$ that specifies the maximum age of the consumers who may buy at each price p, which is given by

$$a(p) = \max\{0 \le x \le 1 \mid v(x) \ge p\}. \tag{3}$$

Consequently, the function $\pi_S : [0,1] \to [0,1]$ defined by

$$\pi_S(p) = p\,a(p), \tag{4}$$

which is the monopoly profit function of a seller who knows that no other seller arrived in the previous unit of time, is quadratic and strictly concave and has a maximum at $p = \frac{1}{2}$.

That assumption is now generalized by allowing the valuation function v to be any continuous and strictly decreasing function with $v(0) = 1$ (but possibly $v(1) > 0$[6]) such that the corresponding function π_S defined by ((3) and) (4) is concave, has a unique maximum point $0 < p^S < 1$, and is *strictly* concave (and, necessarily, strictly decreasing) in $[p^S, 1]$.[7] As the following lemma shows, the monopoly profit function of any seller automatically have similar properties. This function $\pi_M : [0,1] \to [0,1]$, which generalizes (1), is given by

$$\pi_M(p) = p\,\mathbb{E}[\min(\sigma, a(p))] = \mathbb{E}[\min(\sigma p, \pi_S(p))], \tag{5}$$

where the expectation is with respect to the seller's beliefs about the time σ from the previous seller's appearance. If the seller *knows* σ,the expectation symbol $\mathbb{E}$ may be dropped.

Lemma 1 *If the consumers have declining valuations, then every seler's monopoly profit function π_M is continuous and concave in $[0,1]$, has a unique maximum point $0 < p^M < 1$, and is strictly concave in $[p^M, 1]$. If the seller knows the time σ from the previous seller's appearance, then $p^M = \max(p^S, v(\sigma))$.*

Proof. As the function π_S is assumed concave, and is easily seen to be continuous, the functions

$$\{\min(sp, \pi_S(p))\}_{s>0}$$

are equicontinuous and concave in $[0,1]$, and therefore π_M is also continuous and concave. Let p^M be a maximum point of π_M. As the assumption concerning p^S implies that $\min(sp, \pi_S(p)) < \min(sp^S, \pi_S(p^S))$ for all $p < p^S$ and $s > 0$, necessarily $p^M \geq p^S$. To prove that the maximum point is unique, it suffices to show that π_M has no other maximum point in the interval $[p^M, 1]$. For this, it suffices to show that π_M is strictly concave there. Strict concavity follows from the claim below. This is because it follows from the claim and from the fact that the function a is nonincreasing that, with positive probability, $\min(\sigma p, \pi_S(p)) = \pi_S(p)$ for all $p^M \leq p \leq 1$. As π_S is strictly concave on the interval $[p^M, 1]$, this implies that π_M is also strictly concave (rather than just concave) there.

Claim With positive probability, $\sigma \geq a(p^M)$.

Suppose otherwise, that $\sigma < a(p^M)$ almost surely, so that

$$\pi_M(p^M) = p^M \mathbb{E}[\sigma]\,. \tag{6}$$

The concavity of π_S implies that this function, and hence also a, have one-sided derivatives at p^M. Therefore, there exists some $\delta > 0$ such that, for sufficiently small $\epsilon > 0$,

$$\frac{a(p^M) - a(p^M + \epsilon)}{\epsilon} < \frac{\mathbb{E}[\sigma]}{\delta}. \tag{7}$$

The continuity of a implies that $\Pr(\sigma > a(p^M + \epsilon)) < \delta$ for sufficiently small $0 < \epsilon < 1 - p^M$.

[6]It is occasionally convenient to view v as defined on the whole nonnegative ray, with $v(x) = 0$ for $x > 1$. This means that the function may have a discontinuity point at 1.

[7]The conditions on π_S hold, for example, if v is a concave piece-wise linear function. They also hold for the power functions $v(x) = (1-x)^\alpha$ ($\alpha > 0$), the normalized exponential functions $v(x) = \frac{\beta^x - \beta}{1-\beta}$ ($0 < \beta \neq 1$), and more generally, whenever $(\frac{1}{v})'' > 0$ in $(0,1)$ and $v(1) = 0$. (This result follows from the identity $\pi_S'' \circ v = (\frac{v}{v'})^3\,(\frac{1}{v})''$.)

However, this inequality and (7) lead to a contradiction:

$$\begin{aligned}
0 &> \delta\left(a(p^M) - a(p^M+\epsilon)\right) - \epsilon\mathbb{E}[\sigma] \\
&\geq \Pr(\sigma > a(p^M+\epsilon))\left(a(p^M) - a(p^M+\epsilon)\right) - \epsilon\mathbb{E}[\sigma] \\
&\geq \left(p^M+\epsilon\right)\mathbb{E}\left[\sigma - \min(\sigma, a(p^M+\epsilon))\right] - \epsilon\mathbb{E}[\sigma] \\
&= p^M\mathbb{E}[\sigma] - \left(p^M+\epsilon\right)\mathbb{E}\left[\min(\sigma, a(p^M+\epsilon))\right] \\
&= \pi_M(p^M) - \pi_M(p^M+\epsilon) \\
&\geq 0,
\end{aligned}$$

where the third inequality holds because $a(p^M) > \sigma$ almost surely and $p^M + \epsilon < 1$, the second equality holds by (5) and (6), and the last inequality holds because p^M is a maximum point of π_M. This contradiction proves the claim.

Consider now the case where the seller knows σ. By definition of a, a price p satisfies $a(p) \geq \sigma$ if and only if p lies in $[0, v(\sigma)]$. Hence, by (5), $\pi_M(p) = \sigma p$ for all p in that interval, so that π_M is strictly increasing there, and $\pi_M(p) = \pi_S(p)$ for all p outside it. If $v(\sigma) \geq p^S$, then π_S, hence also π_M, are strictly decreasing in the closed interval $[v(\sigma), 1]$, so that they attain their maximum there at the point $v(\sigma)$, at which $\pi_M = \sigma v(\sigma)$. If $v(\sigma) < p^S$, then both π_S and π_M attain their maximum at p^S, where $\pi_M = \pi_S(p^S)$. This proves that $p^M = \max(p^S, v(\sigma))$. ■

A.1 Theorem 1

First case: declining valuations.

The equilibrium condition requires the expected profit $\pi(p^E)$ for any single seller from setting the equilibrium price p^E to be greater than or equal to the expected profit $\pi(p)$ from setting any other price $0 \leq p \leq 1$. We have to show that this condition cannot in fact hold if p^E is not equal to the seller's monopoly price p^M.

If $p^E > p^M$, then

$$\pi(p^E) = \pi_M(p^E) < \pi_M(p^M) \leq \pi(p^M). \tag{8}$$

The equality holds because the assumption of declining valuations implies that at equilibrium consumers never wait to the next seller, and therefore any consumer who was born after the arrival of the previous seller will buy the good at price p^E if this gives him positive payoff, while any consumer born before the arrival of the previous seller but did not buy from him will also not buy now. By the definition of the monopoly profit function, this means that the seller's expected profit from selling at the equilibrium price p^E is $\pi_M(p^E)$. The strict inequality holds because p^M is the unique maximizer of π_M. The weak inequality holds because the lower price p^M may also attract consumers who were born before the arrival of the previous seller but did not buy from him (at price p^E). It follows from (8) that, if $p^E > p^M$,then the seller would gain from reducing the price to p^M.

Suppose now that $p^E < p^M$. Consider a price $p^E < p \leq 1$ and a consumer for whom buying the good is an optimal decision if the price is p^E and waiting is an optimal decision if the price is p. Thus, the consumer's age x is such that (i) $v(x) - p^E \geq 0$ and (ii) $v(x) - p$ is less than or equal to the consumer's expected payoff if he defers buying the good. Condition (ii) holds for two kinds of consumers: those for whom the difference $v(x) - p$ is negative, and consumers for whom it is nonnegative but not greater than the expected payoff from waiting for the next seller. In the rest of the proof, the main idea is to show that, for p sufficiently close to p^E, consumers of the first kind greatly outnumber those of the second kind, so that the anticipated arrival of future sellers has a vanishingly small effect on consumers' decisions.

For a consumer of age x who defers buying, the payoff is $v(x+\tau) - p^E$ if the waiting time τ to the next seller makes this expression positive; otherwise the payoff is zero. Condition (ii) above is therefore equivalent to $v(x) - p \leq \mathbb{E}[\max(v(x+\tau) - p^E, 0)]$, or

$$\mathbb{E}\left[\min\left(v(x) - p^E, v(x) - v(x+\tau)\right)\right] \leq p - p^E, \tag{9}$$

where the expectation is with respect to the consumers' beliefs about τ, that is, about when the next seller will arrive. (As, by assumption, all the consumers are informed, their beliefs about τ are identical.) If the consumers know when the next seller will arrive, the distribution of (the random variable) τ is degenerate.

Fix $\delta > 0$. By the assumption in Section 2, there is some (small) $0 < \epsilon < 1$ that makes $\Pr(\tau > \epsilon) > \frac{1}{1+\delta}$. Since, by assumption, v is continuous and strictly decreasing in the unit interval, there is some price $p^E < p < \frac{p^M + \delta p^E}{1+\delta}$ that is sufficiently close to p^E to make

$$v(x) - v(x+\epsilon) \geq (1+\delta)\left(p - p^E\right) \tag{10}$$

for all $0 \leq x \leq 1 - \epsilon$. It follows from (10) that for any consumer whose age y satisfies $v(y) \geq p^E + (1+\delta)\left(p - p^E\right)$, if the waiting time τ satisfies $\tau > \epsilon$ (which means that either $\epsilon < \tau \leq 1 - y$ or $1 - y < \tau$, and in the latter case, trivially $v(y+\tau) = 0$), then

$$\min\left(v(y) - p^E, v(y) - v(y+\tau)\right) \geq (1+\delta)\left(p - p^E\right).$$

Therefore, for a consumer of such an age y,

$$\mathbb{E}\left[\min\left(v(y) - p^E, v(y) - v(y+\tau)\right)\right] \geq \Pr(\tau > \epsilon)\,(1+\delta)\left(p - p^E\right) > p - p^E.$$

Thus, (9) does not hold for $x = y$, and therefore a consumer of this age prefers buying the good at price p over waiting for the next seller. This conclusion shows that the age x of any consumer who *is* willing to wait satisfies $v(x) < p^E + (1+\delta)\left(p - p^E\right)$. If, in addition, $v(x) \geq p$ (that is, if the consumer is of the second kind considered above), then

$$a(p^E + (1+\delta)\left(p - p^E\right)) < x \leq a(p). \tag{11}$$

If the seller charges p^E, he sells to all the consumers who were born after the previous seller appeared and have positive payoff from buying at p^E. The expected profit is then $\pi_M(p^E)$. Raising the price to p changes the profit to some other value, $\pi(p)$. The consumers' response to the price increase may be thought of as having two stages. In the first stage, all the consumers with positive payoff from buying at price p still do so, giving the seller a profit of $\pi_M(p)$. In the second stage, the consumers who are better off waiting to the next seller drop out. The resulting reduction in the number of customers is constrained by (11), which gives the following upper bound on the second-stage loss of profit:

$$\pi_M(p) - \pi(p) \leq p\left(a(p) - a(p^E + (1+\delta)\left(p - p^E\right))\right). \tag{12}$$

Because, by Lemma 1, π_M is concave and has a maximum only at p^M, and $p^E < p < p^M$,

$$\frac{\pi_M(p) - \pi_M(p^E)}{p - p^E} \geq R, \tag{13}$$

where

$$R = \frac{\pi_M(p^M) - \pi_M(p^E)}{p^M - p^E} > 0.$$

Inequalities (12) and (13) give

$$\frac{\pi(p) - \pi_M(p^E)}{p - p^E} \geq R + p\left(\frac{a(p^E + (1+\delta)\left(p - p^E\right)) - a(p^E)}{p - p^E} - \frac{a(p) - a(p^E)}{p - p^E}\right). \tag{14}$$

If $p^E = 0$, the right-hand side of (14) tends to R as p tends to p^E. If $p^E > 0$, it tends to $R + \delta p^E a'(p^E_+)$, where $a'(p^E_+)$ is the right derivative of a at p^E. (The existence of this one-sided derivative follows from its necessary existence for the convex function π_S.) Therefore, choosing sufficiently small δ guarantees that in both cases the limit is positive, so that $\pi(p) > \pi_M(p^E)$ for p sufficiently close to (but greater than) p^E. Thus, raising the price by a small amount increases the seller's profit. This conclusion contradicts the assumption that p^E is an equilibrium price.

Second case: time discounting.

The proof that $p^E > p^M$ cannot hold is identical to that in the first case. Suppose that $p^E = p^M$. The equilibrium profit of a seller is given by

$$\pi(p^E) = \pi_M(p^E) = \mathbb{E}[\min(\sigma, 1)]\, p^E\, (1 - F(p^E)),$$

where σ is the time since the last seller's visit. Choosing a lower price, $p < p^E$, would change the profit to

$$\pi(p) = \mathbb{E}\left[\min(\sigma, 1)\right]\, p\,(1 - F(p)) + \mathbb{E}\left[\max(1-\sigma, 0)\right]\, p\, \left(F(p^E) - F(p)\right),$$

where the first term, which is equal to $\pi_M(p)$, is the revenue from consumers born after the arrival of the previous seller and the second term represents older consumers with $p < v < p^E$. It is not difficult to check that the above two equalities give:

$$\frac{\pi(p^E) - \pi(p)}{p^E - p} = \frac{1}{\mathbb{E}[\min(\sigma, 1)]}\,\frac{\pi_M(p^E) - \pi_M(p)}{p^E - p} - \mathbb{E}\left[\max(1-\sigma, 0)\right](1 - F(p^E)).$$

The first term on the right tends to zero as p tends to $p^E = p^M$, because, at the maximum point, $\pi'_M = 0$. Therefore, if the probability that $\sigma < 1$ is greater than zero, the right-, and hence also the left-hand side is negative for p sufficiently close to (and smaller than) p^E. Thus, p^E can be a profit-maximizing price only for a seller who *knows* that $\sigma \geq 1$. However, as indicated (see Section 2), there is nonzero probability that this is not the case for at least some sellers. This contradiction proves that p^M cannot in fact be an equilibrium price.

A.2 Theorem 2

Suppose that the assumptions in the theorem hold, and the consumers have a valuation function of the general form specified above. The lower bound on the time from one seller to the next stated in the theorem is correspondingly generalized by replacing it with the smallest solution s_1 of the equation

$$s\, v(s) = \max_{0 \leq p \leq 1} \left[p\left(a(p) - a(p^S)\right)\right]. \tag{15}$$

This equation has at least one solution s in the interval $[0, a(p^S)]$, because the valuation function is continuous and $0 \cdot v(0) \leq \max_p \left[p\left(a(p) - a(p^S)\right)\right] \leq \max_p \left[p\, a(p)\right] = p^S\, a(p^S) \leq a(p^S)\, v(a(p^S))$. For the originally considered special case of linear valuation function ($v(x) = 1 - x$), the smallest solution is $s_1 = \frac{1}{8+\sqrt{48}} \approx 0.067$, because $a(p) = 1 - p$ and $p^S = \arg\max_p \left[p\, a(p)\right] = \frac{1}{2}$, so that (15) is the quadratic equation $s\,(1-s) = \frac{1}{16}$.[8]

Suppose that each seller sets his monopoly price, which by Lemma 1 is given by

$$p^M = \max(p^S, v(\sigma)) \geq v(\sigma),$$

[8]The linear valuation function is a limit case of the one-parameter family of exponentially-decreasing valuation functions $v(x) = \frac{\beta^x - \beta}{1 - \beta}$, with $0 < \beta \neq 1$. It can be shown numerically that, for every such β, s_1 is less than about 0.082.

where σ is the time since the appearance of the previous seller. Any consumers born earlier would not buy at that price, which means that for him the price is not worth waiting to. Therefore, it is optimal for a consumer to buy at the first opportunity, if the requested price is not higher than his valuation at that moment. Note that, by the above characterization of the monopoly prices, the last condition may *not* hold only if the requested price is p^S.

It follows from this analysis of the consumers' behavior that, if all the sellers choose their monopoly prices p^M, each of them receives his monopoly profit $\pi_M(p^M)$. It remains to prove that no single seller would get a higher profit by choosing any price $p \neq p^M$.

If $p > p^M$, then, as explained above, no consumer born before the arrival of the previous seller will buy at that price, which means that the seller's profit is at most $\pi_M(p)$ ($\leq \pi_M(p^M)$, by definition of p^M). If $p < p^M$, then some consumers born before the arrival of the previous seller may buy. However, these consumers are rather old, specifically, older by at least $a(p^S)$ than any of the consumers born after the previous seller's arrival. This is because, as indicated, a consumer does not buy from a seller (at the seller's monopoly price) only if the price the seller sets is p^S and his valuation is less than or equal to that, which means that the consumer's age is at least $a(p^S)$. Therefore, if p is low enough to appeal to some old consumers (as well as to the young ones), the sellers' profit is at most $p\,\left(a(p) - a(p^S)\right)$. By definition of s_1, the last expression does not exceed $s_1 v(s_1)$, which by the assumption $s_1 \leq \sigma$ is equal to $\pi_M(v(s_1))$ ($\leq \pi_M(p^M)$).

B Proofs: Examples

This appendix proves the assertions made in the paper's four examples.

B.1 Example 1

Suppose that all the sellers set the same price p^E. Any equilibrium strategy for the consumers must specify that they buy the good at that price or lower if doing so gives them positive payoff. It must also specify that they buy or do not buy at any higher price if the corresponding payoff is higher or lower, respectively, than that expected from waiting to and (optionally) buying from the next seller at price p^E. Therefore, p^E is an equilibrium price if and only if, when all the sellers sell at that price and the consumers' strategy satisfies the above conditions, no seller can gain from setting a price $p \neq p^E$. Below, we investigate this equilibrium condition in the two cases of consumers' impatience described in Section 2.

First case: declining valuations.

We show below that a necessary and sufficient condition for p^E to be an equilibrium price is that $p = p^E$ is a solution of the equation

$$p + \frac{1}{\lambda}\ln(1 + \lambda p) = 1. \tag{16}$$

Consider first a seller who deviates by setting a higher price, $p > p^E$. The payoff of a consumer of age $x < 1 - p$ from buying at price p is $1 - x - p$. The payoff from waiting to the next seller (who will sell at price p^E) is $1 - (x + \tau) - p^E$ if this expression is positive and 0 otherwise, where τ is the (random) waiting time to the next seller. Hence, the expected payoff from waiting is

$$\int_0^{1-x-p^E} \left(1 - (x+\tau) - p^E\right) \lambda e^{-\lambda\tau} d\tau = 1 - x - (p^E + \frac{1}{\lambda}) + \frac{1}{\lambda} e^{-\lambda\left(1-x-p^E\right)}.$$

Comparison of the two payoffs shows that a necessary condition for selling any units at all at price p is

$$p < p^E + \frac{1}{\lambda},$$

which means that the price difference must be lower than the expected waiting time to the next seller, $\frac{1}{\lambda}$. If this condition holds, then a consumer of age x is better off buying immediately (at price p) than waiting to the next seller or not buying at all if and only if $x < x_p$, where

$$x_p = 1 - p^E + \frac{1}{\lambda} \ln\left(1 - \lambda\left(p - p^E\right)\right). \tag{17}$$

It is not difficult to see that the threshold value x_p is lower than $1-p$. If positive, it is the age at which a consumer is indifferent between buying and waiting. If negative, no consumer will buy at price p.

The seller's expected profit $\pi(p)$ from setting a price $p^E < p < p^E + \frac{1}{\lambda}$ with $x_p > 0$ can be computed as follows. From the seller's perspective, the time σ since the previous seller appeared is exponentially distributed with parameter λ. A consumer who did not buy from that seller at price p^E will certainly not buy at the higher price p. Therefore,

$$\begin{aligned}
\pi(p) &= p\,\mathbb{E}\left[\min(\sigma, x_p)\right] \\
&= p\left[\int_0^{x_p} \lambda s e^{-\lambda s} ds + \int_{x_p}^{\infty} \lambda x_p e^{-\lambda s} ds\right] \\
&= \frac{p}{\lambda}\left(1 - e^{-\lambda x_p}\right) \\
&= \frac{p}{\lambda}\left(1 - \frac{e^{-\lambda(1-p^E)}}{1 - \lambda\left(p - p^E\right)}\right),
\end{aligned}$$

where the last equality follows from (17). Differentiation gives

$$\pi'(p) = \frac{1}{\lambda} - \frac{e^{-\lambda(1-p^E)}\left(1 + \lambda p^E\right)}{\lambda(1 - \lambda\left(p - p^E\right))^2}.$$

The derivative is decreasing for $p^E < p < p^E + \frac{1}{\lambda}$, and therefore $\pi(p) \leq \pi(p^E)$ for *all* such p if and only if $\pi'(p^E) \leq 0$, or

$$e^{\lambda(1-p^E)} \leq 1 + \lambda p^E. \tag{18}$$

Consider next a price $p \leq p_e$. Any consumer of age $x < 1 - p$ will buy at that price. However, such a consumer is still in the market only if no previous seller arrived s units of time earlier, for every $x - (1 - p^E) < s < x$ (as the consumer would have bought the good from such a seller). The probability of this is $e^{-\lambda x}$ if $x < 1 - p^E$, and $e^{-\lambda(1-p^E)}$ if $x \geq 1 - p^E$. Therefore, the expected profit of a seller selling at price p is

$$\begin{aligned}
\pi(p) &= p\left[\int_0^{1-p^E} e^{-\lambda x} dx + \int_{1-p^E}^{1-p} e^{-\lambda(1-p^E)} dx\right] \\
&= \frac{p}{\lambda}\left(1 - \left(1 + \lambda\left(p - p^E\right)\right) e^{-\lambda(1-p^E)}\right).
\end{aligned}$$

Since this is a quadratic, concave function, the inequality $\pi(p) \leq \pi(p^E)$ holds for all $p \leq p^E$ if and only if $\pi'(p^E) \geq 0$, or

$$e^{\lambda(1-p^E)} \geq 1 + \lambda p^E.$$

The price p^E satisfies both the last inequality and the reverse one (18) if and only if it solves (16). This proves that the latter is indeed a necessary and sufficient condition for an equilibrium price.

Equation (16) has a unique solution, because the expression on the left-hand side is strictly increasing and is less or greater than 1 for $p = 0$ or $p = 1$, respectively. The solution is equal to the sellers' monopoly price p^M. This is because the exponential distribution (with parameter λ) of the time σ since the arrival of the previous seller implies that the monopoly profit function is

$$\begin{aligned}\pi_M(p) &= p\mathbb{E}\left[\min(\sigma, 1-p)\right] \\ &= p\left[\int_0^{1-p} \lambda s e^{-\lambda s} ds + \int_{1-p}^{\infty} (1-p)\, \lambda e^{-\lambda s} ds\right] \\ &= \frac{p}{\lambda}\left(1 - e^{-\lambda(1-p)}\right).\end{aligned}$$

The monopoly price p^M maximizes π_M, and hence satisfies the first-order condition

$$\frac{d\pi_M}{dp} = \frac{1}{\lambda}\left(1 - (1+\lambda p)\, e^{-\lambda(1-p)}\right) = 0.$$

This equation is equivalent to (16).

Equation (16) gives the equilibrium price as an implicit function of the sellers' arrival rate λ. This function tends to $\frac{1}{2}$ as λ tends to 0, and it is strictly increasing (see Figure 2). The latter can be verified by implicitly differentiating (16), which gives the equality

$$\left(1 + \frac{1}{1+\lambda p}\right)\frac{dp}{d\lambda} + \frac{1}{\lambda^2}\left(1 - \frac{1}{1+\lambda p} - \ln(1+\lambda p)\right) = 0.$$

Because $1 - \frac{1}{1+x} - \ln(1+x) < 0$ for all $x > 0$, the equality proves that $\frac{dp^E}{d\lambda} > 0$. Together with (16), it also gives the upper bound $\frac{dp^E}{d\lambda} < (p^E)^2$, because it follows from the two equalities that

$$p^2 - \frac{dp}{d\lambda} = (\frac{1}{\lambda} + p)(p - \frac{1}{2+\lambda p}),$$

and the right-hand side is positive for $p = p^E > \frac{1}{2}$.

Consider now the effect of the sellers' arrival rate on social welfare. The latter is expressed by the expectation of a consumer's valuation of the good at the time he buys it, which is 1 minus his age x (and is zero if the consumer leaves the market without buying). As only consumers younger than $1 - p^E$ buy the good, the expectation in question is

$$\int_0^{1-p^E} (1-x)\lambda e^{-\lambda x} dx = 1 - \frac{1 + (\lambda p^E - 1)e^{-\lambda(1-p^E)}}{\lambda} = 1 - \frac{2}{\lambda + \frac{1}{p^E}},$$

where the second equality uses (16). The expression in the last denominator increases with increasing λ, because, as shown above,

$$\frac{d}{d\lambda}\left(\lambda + \frac{1}{p^E}\right) = 1 - \frac{1}{(p^E)^2}\frac{dp^E}{d\lambda} > 0.$$

This proves that social welfare also increases.

Second case: time discounting.

Each seller's expected profit is equal to the monopoly profit from setting the price p^E:

$$\pi(p^E) = \pi_M(p^E) = \mathbb{E}[\,\min(\sigma, 1)]\; p^E\; (1 - p^E),$$

with

$$\mathbb{E}[\min(\sigma,1)] = \int_0^1 \mathbb{P}(\sigma > x)\,dx = \int_0^1 e^{-\lambda x}\,dx = \frac{1-e^{-\lambda}}{\lambda}. \tag{19}$$

Setting a lower price, $p < p^E$, would change a seller's profit to

$$\pi(p) = \mathbb{E}[\min(\sigma,1)]\; p\,(1-p^E) + p\,(p^E - p),$$

where the decomposition reflects the fact that all the consumers with valuations $p < v < p^E$ born during the last unit of time are still in the market and will buy at price p. The equilibrium condition $\pi(p) \le \pi(p^E)$ can therefore be written as

$$0 \ge (p - p^E)\left(\mathbb{E}[\min(\sigma,1)]\,(1-p^E) - p\right).$$

This condition holds (indeed, holds as strict inequality) for all $0 < p < p^E$ if and only if

$$\mathbb{E}[\min(\sigma,1)]\,(1-p^E) - p^E \ge 0.$$

By (19), the last inequality is equivalent to

$$p^E \le \frac{1}{1+\frac{\lambda}{1-e^{-\lambda}}}. \tag{20}$$

In particular, an equilibrium price is necessarily lower than the monopoly price $p^M = \frac{1}{2}$.

Consider now a seller who increases his price, to some $p > p^E$. Only customers born after the last seller's visit may have valuations higher than p, which means that the seller's profit $\pi(p)$ cannot be greater than $\pi_M(p)$. As π_M is the quadratic function (2), $\pi_M(p) < \pi_M(p^E)$ if $p > 1 - p^E$. Therefore, we only need to consider prices $p \le 1 - p^E$.

The profit $\pi(p)$ may actually be less than $\pi_M(p)$. This is because for some potential customers it may be better to wait to the next seller, who will sell at price p^E. The payoff this alternative yields depends on the waiting time τ to the next seller, and in particular, for a consumer of age x, on whether $x + \tau \le 1$. Specifically, waiting is *not* an optimal option for the consumer if and only if his valuation v satisfies

$$v - p > \mathbb{E}[e^{-\rho\tau} 1_{x+\tau\le 1}]\,(v - p^E),$$

or

$$v > p^E + \frac{p - p^E}{1 - \mathbb{E}[e^{-\rho\tau}\,1_{\tau \le 1-x}]}.$$

The probability that such a consumer is still in the market (because he did not have a chance to buy at price p^E) is $e^{-\lambda x}$. Therefore, the expected profit from setting price p is

$$\pi(p) = p\int_0^1 \max\left(1 - p^E - \frac{p-p^E}{1-\mathbb{E}[e^{-\rho\tau}\,1_{\tau\le 1-x}]}, 0\right) e^{-\lambda x} dx. \tag{21}$$

If p is sufficiently close to p^E to make

$$1 - p^E - \frac{p - p^E}{1 - \mathbb{E}[\,1_{\tau \le 1}]} \ge 0, \tag{22}$$

then by (19) and (21)

$$\begin{aligned}
\pi(p) - \pi(p^E) &= p\int_0^1 \left(1 - p^E - \frac{p-p^E}{1-\mathbb{E}[e^{-\rho\tau}\,1_{\tau\le 1-x}]}\right) e^{-\lambda x} dx - p^E \int_0^1 (1-p^E) e^{-\lambda x}\,dx \\
&= (p - p^E)\int_0^1 \left(1 - p^E - \frac{p}{1-\mathbb{E}[e^{-\rho\tau}\,1_{\tau\le 1-x}]}\right) e^{-\lambda x} dx.
\end{aligned}$$

A necessary and sufficient condition for this difference to be nonpositive for all $p > p^E$ satisfying (22) (which moreover implies that the difference is in fact negative) is that the integral on the right-hand side is nonpositive for $p = p^E$. This is so if and only if

$$p^E \geq \frac{1}{1 + \frac{\lambda}{1-e^{-\lambda}} \int_0^1 \frac{e^{-\lambda x}}{1-\mathbb{E}[e^{-\rho\tau}\, 1_{\tau \leq 1-x}]} dx}. \tag{23}$$

This inequality is consistent with the previous condition (20) if and only if the discount rate ρ satisfies

$$\int_0^1 \frac{e^{-\lambda x}}{1 - \mathbb{E}[e^{-\rho\tau}\, 1_{\tau \leq 1-x}]} dx \geq 1. \tag{24}$$

The integral depends on ρ as a strictly decreasing function. This is because the denominator is strictly increasing, and tends to 1 or $e^{-\lambda(1-x)}$ when ρ tends to ∞ or 0, respectively. At the first limit, (24) does not hold, and at the second limit, it holds as strict inequality, because

$$\int_0^1 \frac{e^{-\lambda x}}{e^{-\lambda(1-x)}} dx = \frac{e^{\lambda} - e^{-\lambda}}{2\lambda} > 1.$$

Therefore, for any discount rate above the threshold defined by equality in (24), an equilibrium where all sellers set the same equilibrium price does not exist. For discount rates sufficiently far below the threshold, there are many equilibria. In fact, as we show below, any

$$\frac{1}{1 + \frac{1}{2}\frac{e^{\lambda}-e^{-\lambda}}{1-e^{-\lambda}}} < p^E \leq \frac{1}{1 + \frac{\lambda}{1-e^{-\lambda}}}$$

is an equilibrium price for sufficiently low discount rate.

Fix any such price p^E. As shown above, a seller would only decrease his profit by deviating to any price p that is lower than p^E or higher than $1-p^E$, and the same holds for all p satisfying (22) if the discount rate ρ is low enough to give (23). Consider then a price

$$p^E + e^{-\lambda}(1 - p^E) \leq p \leq 1 - p^E, \tag{25}$$

and let x_0 be the unique point in $[0, 1]$ satisfying

$$e^{\lambda(1-x_0)}(p - p^E) = 1 - p^E.$$

As Equation (21) shows, a seller's profit from setting the price p is determined by ρ as a nondecreasing function, which in the limit $\rho \to 0$ becomes

$$\begin{aligned} p\int_{x_0}^1 \left(1 - p^E - e^{\lambda(1-x)}(p - p^E)\right) e^{-\lambda x} dx &= \frac{e^{-\lambda}}{2\lambda} p \left(p + p^E - 2 + \left(2(1-p^E) - e^{\lambda(1-x_0)}(p-p^E)\right) e^{\lambda(1-x_0)}\right) \\ &= \frac{e^{-\lambda}}{2\lambda} p \left(p + p^E - 2 + (1-p^E)\frac{1-p^E}{p-p^E}\right) \\ &= \frac{e^{-\lambda}}{2\lambda} \frac{p(1-p)^2}{p-p^E}. \end{aligned}$$

The last expression is determined by p as a strictly decreasing function in the interval specified by (25), and it therefore attains its maximum at the left endpoint. At that point, inequality (22) holds (as equality, and $x_0 = 0$), which as indicated implies that $\pi(p) < \pi(p^E)$ if ρ is sufficiently close to 0. It therefore follows from the joint continuity of the expression in (21) in p and ρ that, for sufficiently low ρ, the same inequality holds simultaneously for all p satisfying (25).

B.2 Example 2

First case: declining valuations.

The monopoly profit function in this case is given by

$$\pi_M(p) = \frac{1}{2} p\,(1-p) + \frac{1}{2} p \min(s, 1-p),$$

because a seller is equally likely to be first or second. The maximum point of this function, which is the monopoly price p^M, is given by

$$p^M = \begin{cases} \frac{1+s}{2}, & 0 < s \le \frac{1}{3} \\ 1-s, & \frac{1}{3} \le s \le \frac{1}{2} \\ \frac{1}{2}, & s \ge \frac{1}{2} \end{cases} \tag{26}$$

(see Figure 3). By Theorem 1, in any equilibrium with a single equilibrium price, that price is p^M. To check whether such an equilibrium actually exists, suppose that both sellers set the price to p^M but one of them contemplates setting a different price p.

If $p > p^M - s$, then the seller's profit from setting this price will not be greater than $\pi_M(p)$, because consumers who already had a chance to buy the good at the lower price p^M but chose not to do so will also not buy now at price p. Therefore, the profit from setting price p will be less than $\pi_M(p^M)$, the (monopoly) profit from selling at price p^M. If $0 < p \le p^M - s$ (which by (26) is possible only if $s < \frac{1}{2}$), then the profit will be $\frac{1}{2}p\,(1-p) + \frac{1}{2}p(p^M - p)$. This is because, if the seller is the second one, some consumers who did not buy from the first seller at price p^M, namely, those older than $1 - p^M + s$ but younger than $1 - p$, *will* buy at price p. If $\frac{1}{3} \le s < \frac{1}{2}$, then by (26) $p^M = 1 - s$, and therefore $\pi_M(p^M) = s\,(1-s)$, whereas $\max_p \left[\frac{1}{2}p\,(1-p) + \frac{1}{2}p(p^M - p)\right] = \frac{1}{16}\,(2-s)^2 \le \frac{1}{16}\left(2 - \frac{1}{3}\right)^2 < \frac{1}{3}\left(1 - \frac{1}{3}\right) \le s\,(1-s)$. This proves that p^M is indeed an equilibrium price if $s \ge \frac{1}{3}$. If $s < \frac{1}{3}$, then $p^M = \frac{1+s}{2}$, and therefore $\pi_M(p^M) = \frac{1}{8}\,(1+s)^2$ and $\max_p \left[\frac{1}{2}p\,(1-p) + \frac{1}{2}p(p^M - p)\right] = \frac{1}{64}\,(3+s)^2$. The maximum is greater than $\frac{1}{8}\,(1+s)^2$ if and only if $s < \frac{1}{5+\sqrt{32}}$, and for such s, the maximum point p satisfies $p = \frac{3+s}{8} < \frac{1-s}{2} = p^M - s$. Therefore, p^M is not an equilibrium price if $s < \frac{1}{5+\sqrt{32}}$ but it is so if $s \ge \frac{1}{5+\sqrt{32}}$.

Second case: time discounting.

If both sellers set the same price p^E, their expected profit is

$$\pi(p^E) = \frac{1+s}{2} p^E (1 - p^E).$$

A unilateral change of price to $p < p^E$ changes the seller's profit to

$$\pi(p) = \frac{1+s}{2}\, p\,(1 - p^E) + p\,(p^E - p),$$

where the second term reflects the fact that all consumers with valuations $p < v < p^E$ born during the last unit of time are still in the market. The difference

$$\pi(p) - \pi(p^E) = (p - p^E)\left(\frac{1+s}{2}(1 - p^E) - p\right)$$

is nonpositive for all $0 < p < p^E$ if and only if

$$p^E \le \frac{1}{2 + \frac{1-s}{1+s}}. \tag{27}$$

Therefore, this inequality presents an upper bound, which is lower than the monopoly price $p^M = \frac{1}{2}$, to any equilibrium price. In the following, we assume that (27) holds.

If a seller changes his price to $p > p^E$, the profit changes to

$$\pi(p) = \frac{1+s}{2}\, p\,(1-p) - \frac{1-s}{2} p\, \min(\frac{p-p^E}{e^{\rho s}-1}, 1-p). \tag{28}$$

The second term represents the loss of consumers to the other seller, and it is computed as follows. If the seller charging p arrives first, some of the consumers younger than $1-s$ will find it beneficial to wait and buy from the next seller, even though their valuation satisfies $v > p$. Specifically, these are the young consumers for whom

$$0 < v - p < e^{-\rho s}(v - p^E),$$

or

$$p < v < p + \frac{p - p^E}{e^{\rho s}-1}.$$

This gives (28). By that equation,

$$\pi(p) - \pi(p^E) = \max\left((p-p^E)\left(\frac{1+s}{2}(1-p^E-p) - \frac{1-s}{2}\frac{p}{e^{\rho s}-1}\right), s\,p\,(1-p) - \frac{1+s}{2}p^E(1-p^E)\right).$$

Therefore, the payoff difference is nonpositive for all $p > p^E$ if and only if this is so for both the first and second expression on the right-hand side. The first condition holds if and only if

$$\frac{1+s}{2}(1-p^E-p^E) - \frac{1-s}{2}\frac{p^E}{e^{\rho s}-1} \leq 0,$$

that is,

$$p^E \geq \frac{1}{2 + \frac{1-s}{1+s}\frac{1}{e^{\rho s}-1}}, \tag{29}$$

and the second condition holds if and only if

$$p^E \geq \frac{1 - \sqrt{\frac{1-s}{1+s}}}{2}.$$

Therefore, $p^E$is an equilibrium price if and only if it solves both inequalities as well as (27). It is not difficult to check that the set (specifically, interval) of all such solutions is nonempty if and only if $e^{-\rho s} \geq \frac{1}{2}$.

B.3 Example 3

Consider a generalized version of the example, where the probability that the second seller arrives is $0 < \alpha < 1$. We prove below that there is a number $0 < s_0(\alpha) < \frac{1}{8+\sqrt{48}}$ (≈ 0.067) such that the following holds:

1. If $0 < s \leq s_0(\alpha)$, the unique equilibrium prices are $p_1 = \frac{1-\alpha}{4-3\alpha}(2-\alpha+2\alpha s)$ for the first seller and $p_2 = \frac{1-\alpha(1+s)}{4-3\alpha}$ for the second seller, and $p_1 > p_2$.
2. If $s \geq \frac{1}{8+\sqrt{48}}$, the unique equilibrium prices are $p_1 = \frac{1}{2}$ and $p_2 = \max(1-s, \frac{1}{2})$, so that $p_1 \leq p_2$.
3. If $s_0(\alpha) < s < \frac{1}{8+\sqrt{48}}$, no equilibrium exists.

In Case 1, the equilibrium price p_1 is lower than the first seller's monopoly price, which is $\frac{1}{2}$, and p_2 is lower than the second seller's monopoly price, which is $\max(1-s, \frac{1}{2})$. In particular, for $\alpha = \frac{1}{2}$ and $s < s_0(\frac{1}{2}) = \frac{1}{26}$, the equilibrium prices satisfy $p_1 = \frac{3}{10} + \frac{1}{5}s < \frac{4}{13}$ and $p_2 = \frac{1}{5}(1-s)$. The probability α that the second seller will arrive determines the critical value $s_0(\alpha)$ as a continuous and strictly decreasing function (see (36) below), which tends to 0 or $\frac{1}{8+\sqrt{48}}$ as α tends to 1 or 0, respectively.

To prove the assertions made in Cases 1–3, fix the second seller's price p_2 and consider the consumers' reaction to any price $0 < p_1 < 1$ the first seller may set. A consumer whose valuation v of the good at time 0 exceeds p_1 receives the payoff $v - p_1$ if he buys it at that time and the expected payoff $\max(\alpha(v - p_2 - s), 0)$ if he chooses to wait. The first option is better if and only if v exceeds the critical value v^c given by

$$v^c = \max\left(\frac{p_1 - \alpha(p_2 + s)}{1-\alpha}, p_1\right) = p_1 + \frac{\alpha}{1-\alpha}\max(p_1 - p_2 - s, 0). \tag{30}$$

Therefore, if $p_1 \le p_2 + s$, all the consumers who would get positive payoff from buying at time 0 do so. But if $p_1 > p_2 + s$, then the consumers with $v < v^c$, who are the ones older than $\frac{1-p_1-\alpha(1-p_2-s)}{1-\alpha}$, do not buy. We conclude that the first seller's profit π_1 depends on the price p_1 he sets as follows:

$$\pi_1(p_1) = \begin{cases} p_1(1-p_1), & p_1 \le p_2 + s \\ p_1\frac{1-p_1-\alpha(1-p_2-s)}{1-\alpha}, & p_2 + s < p_1 \le 1 - \alpha(1 - p_2 - s) \\ 0, & p_1 > 1 - \alpha(1 - p_2 - s) \end{cases}. \tag{31}$$

Therefore, price p_1 is profit-maximizing if and only if

$$p_1 = \begin{cases} \frac{1-\alpha}{2} + \frac{\alpha}{2}(p_2 + s), & p_2 + s \le \frac{1-\alpha}{2-\alpha} \\ p_2 + s, & \frac{1-\alpha}{2-\alpha} \le p_2 + s \le \frac{1}{2} \\ \frac{1}{2}, & p_2 + s \ge \frac{1}{2} \end{cases}. \tag{32}$$

Plugging this into (30) gives $v^c < 1$.

If $s \ge \frac{1}{2}$ (which falls in Case 2), then (32) and (30) give $p_1 = v^c = \frac{1}{2}$ regardless of p_2, which means that all the consumers younger than $\frac{1}{2}$ buy the good from the first seller. It follows that the second seller's price p_2 is profit-maximizing if and only if it is also $\frac{1}{2}$. In the rest of the proof, we assume that $0 < s < \frac{1}{2}$ and examine the optimally of p_2 by looking a the profit the second seller would make if he changed it unilaterally to any other price p.

One option is to sell only to young consumers, who were born after time 0, by setting a price $p \ge 1 - s$ ($> \frac{1}{2}$, by assumption). The corresponding profit is given by the expression $p(1-p)$, which attains its maximum of $s(1-s)$ at the unique maximum point $p = 1 - s$. The alternative is to set a lower price that will appeal also to older consumers, who were born before time 0 but did not buy the good then because their valuations were lower than the threshold v^c specified by (30). As the value of the good decreases linearly with time, selling to such consumers requires setting a price $0 \le p \le v^c - s$ ($< 1 - s$). The corresponding profit, which comes from selling to both young and old consumers, is $p(v^c - p)$ .The maximum of this expression, which is attained at the unique maximum point $p = \frac{1}{2}v^c$, is $(\frac{1}{2}v^c)^2$. Therefore, if $(\frac{1}{2}v^c)^2 > s(1-s)$ (which is possible only if $\frac{1}{2}v^c < v^c - s$), then $p_2 = \frac{1}{2}v^c$ is the second seller's unique profit-maximizing price, and if $(\frac{1}{2}v^c)^2 < s(1-s)$, then $p_2 = 1 - s$ has this property. If an equality holds, both prices, and only them, are profit-maximizing. In the following, we examine these conditions more closely.

Suppose first that $p_2 > \frac{1}{2} - s$. By (32) and (30), $p_1 = v^c = \frac{1}{2}$, so that $(\frac{1}{2}v^c)^2 \le s(1-s)$ if and only if $s \ge \frac{1}{8+\sqrt{48}}$ (≈ 0.067). This proves that, for $\frac{1}{8+\sqrt{48}} \le s < \frac{1}{2}$, the prices $p_2 = 1 - s$ and $p_1 = \frac{1}{2}$ are equilibrium prices. There are no other equilibrium prices with $p_2 > \frac{1}{2} - s$ for any $0 < s < \frac{1}{2}$, because $p_2 = \frac{1}{2}v^c = \frac{1}{4}$ could be so only if s were less than 0.067, which would contradict the assumption $p_2 > \frac{1}{2} - s$.

Suppose now that $p_2 \leq \frac{1}{2} - s$. By (32),

$$p_1 - p_2 - s = \frac{1-\alpha}{2} - \frac{2-\alpha}{2} \min\left(p_2 + s, \frac{1-\alpha}{2-\alpha}\right) \geq 0, \tag{33}$$

and therefore $v^c = \frac{p_1 - \alpha(p_2+s)}{1-\alpha}$. As shown, p_2 (which is less than $1-s$) is an equilibrium price if and only if $p_2 = \frac{1}{2}v^c$ and

$$p_2^2 \geq s\,(1-s)\,. \tag{34}$$

The first condition, which is equivalent to

$$p_2 = \frac{p_1 - \alpha s}{2-\alpha}, \tag{35}$$

and (33) together imply $p_1 > p_2 \geq s$. If $\frac{1-\alpha}{2-\alpha} \leq p_2 + s \leq \frac{1}{2}$, their unique solution is $p_1 = 2s$ and $p_2 = s$ ($< \frac{1}{2}$), which does not satisfy (34). If $p_2 + s < \frac{1-\alpha}{2-\alpha}$, the solution is

$$\begin{aligned} p_1 &= \frac{1-\alpha}{4-3\alpha}\,(2-\alpha+2\alpha s) \\ p_2 &= \frac{1-\alpha(1+s)}{4-3\alpha}\;(<\frac{1}{4}). \end{aligned}$$

These prices satisfy (34) and the inequality $p_2 + s < \frac{1-\alpha}{2-\alpha}$ if and only if $s \leq s_0(\alpha)$, where

$$s_0(\alpha) = \frac{2\,(1-\alpha)^2}{(8-7\alpha)\,(2-\alpha) + (4-3\alpha)\sqrt{\alpha^2 + 12\,(1-\alpha)}}, \tag{36}$$

and in this case, they are the unique equilibrium prices. By (31), the first seller's equilibrium profit is $p_1 \frac{1-p_1-\alpha(1-p_2-s)}{1-\alpha}$, which equals $p_1\,(1-p_2)$,and the second seller's profit is p_2^2. For $s > s_0(\alpha)$,there are no equilibrium prices with $p_2 + s \leq \frac{1}{2}$.

B.4 Example 4

The parameters of this example are similar to the generalized version of Example 3, except that the second seller arrives with probably $\frac{1}{2} \leq \alpha \leq 1$ and $\frac{1}{8+\sqrt{48}} \leq s \leq \frac{1}{10}$. The second pair of inequalities is equivalent to $\frac{1}{4} \leq \sqrt{s\,(1-s)} \leq \frac{3}{10}$.

For any $\alpha s + (2-\alpha)\sqrt{s(1-s)} \leq p^* \leq \frac{1}{2}$, we show that there is an equilibrium in which the first seller sells at price p^*. For example, if the second seller arrives with probability $\alpha = \frac{1}{2}$, and he does so $s = \frac{4}{53}$ (≈ 0.075) units of time after the first seller, then for every price between $\frac{23}{53}$ and $\frac{1}{2}$ there is an equilibrium in which the first seller sets this price. If the second seller arrives with certainty $\frac{1}{10}$ units of time after the first seller, then every price between $\frac{2}{5}$ and $\frac{1}{2}$ is an equilibrium price for the first seller.

The consumers' equilibrium strategy specifies that a consumer buys from the second seller at any price lower than his valuation (at time s), but buys from the first seller at a requested price $0 \leq p \leq 1$ if and only if his age (at time 0) is less than x_p, where

$$x_p = \begin{cases} 1-p, & p \leq p^* \\ 1 - \frac{2}{2-\alpha}\,(p-\alpha s)\,, & p^* < p < 1 - \alpha\left(\frac{1}{2} - s\right) \\ 0, & p \geq 1 - \alpha\left(\frac{1}{2} - s\right) \end{cases}. \tag{37}$$

In other words, if $p \leq p^*$, a consumer buys the good from the first seller if and only if he values it at more than p, but if $p > p^*$, he buys if and only if the value to him exceeds $\frac{2}{2-\alpha}\,(p-\alpha s)$.

The second threshold is higher than p, because, for $p > p^* \ (\geq \alpha s + (2-\alpha)\sqrt{s(1-s)})$,

$$\begin{aligned}
\frac{2}{2-\alpha}(p-\alpha s) - p &> \frac{2}{2-\alpha}(p^*-\alpha s) - p^* \qquad (38)\\
&= \frac{\alpha}{2-\alpha}(p^*-2s)\\
&\geq \frac{\alpha}{2-\alpha}\left(\alpha s + (2-\alpha)\sqrt{s(1-s)} - 2s\right)\\
&= \alpha\left(\sqrt{s(1-s)} - s\right)\\
&> 0.
\end{aligned}$$

The second seller's price p_2 is determined as follows. If the price p set by the first seller is such that $1 - x_p \leq 2\sqrt{s(1-s)}$, then $p_2 = 1 - s$, which yields the profit $s(1-s)$. Only consumers who were not yet born at time 0 buy at that price. Selling also to those who were then older than x_p, and for this reason did not buy from the first seller, would require setting a price $p_2 \leq 1 - x_p - s$, for which the profit would be $p_2(1 - p_2 - x_p)$. However, $p_2(1-p_2-x_p) \leq \frac{1}{4}(1-x_p)^2 \leq s(1-s)$ for any p_2. If $1 - x_p > 2\sqrt{s(1-s)}$, then $p_2 = \frac{1-x_p}{2}$ ($< 1 - x_p - s$, as $1 - x_p > 2s$), and the corresponding profit is $p_2(1-p_2-x_p) = \frac{1}{4}(1-x_p)^2$ ($> s(1-s)$). Therefore, by (37) and the assumption $\frac{1}{4} \leq \sqrt{s(1-s)} \leq \frac{3}{10}$: (i) if $p \leq p^*$ ($\leq \frac{1}{2}$), then $1 - x_p = p \leq \frac{1}{2} \leq 2\sqrt{s(1-s)}$ and $p_2 = 1 - s$, and (ii) if $p > p^*$ ($\geq \alpha s + (2-\alpha)\sqrt{s(1-s)}$), then $1 - x_p = \min\left(\frac{2}{2-\alpha}(p-\alpha s), 1\right) > 2\sqrt{s(1-s)}$ and

$$p_2 = \frac{1-x_p}{2} = \min\left(\frac{p-\alpha s}{2-\alpha}, \frac{1}{2}\right). \qquad (39)$$

In case (i), $p \leq p^* < p_2$, and so it is optimal for consumers who value the good at more than p at time 0 to buy at that price, as their strategy indeed instructs them to do. In case (ii), buying the good at time 0 (at price p) is optimal for a consumer of age $x < 1 - p$ if and only if $1 - x - p \geq \alpha(1 - x - p_2 - s)$, or $(1-\alpha)x \leq 1 - p - \alpha(1 - p_2 - s)$, and waiting for the second seller is optimal if and only if the reverse inequality holds. By (39),

$$1 - p - \alpha(1 - p_2 - s) \leq (1-\alpha)\left(1 - \frac{2}{2-\alpha}(p-\alpha s)\right), \qquad (40)$$

with equality if $x_p > 0$. Thus, if $p > p^*$ and $x_p > 0$, the consumers' strategy (37) prescribes optimal actions for them. The same holds if $x_p = 0$, or $p \geq 1 - \alpha\left(\frac{1}{2} - s\right)$, as this inequality means that the right-hand side of (40) is nonpositive, and hence not buying the good at time 0 is an optimal action for all consumers.

It remains to show that the given p^* is the profit-maximizing price for the first seller. If the seller sets a price $0 < p \leq p^*$, his profit will be $p(1-p)$, which is less than or equal to $p^*(1-p^*)$, as $p^* \leq \frac{1}{2}$. By (37), the profit for any price $p > p^*$ is the maximum of $p\left(1 - \frac{2}{2-\alpha}(p-\alpha s)\right)$ and zero. The former depends on p as a quadratic, concave function, with a maximum at $\frac{1}{2} - \frac{\alpha}{2}\left(\frac{1}{2} - s\right)$. This maximum point lies to the left of p^*, since

$$\begin{aligned}
p^* - \left(\frac{1}{2} - \frac{\alpha}{2}\left(\frac{1}{2} - s\right)\right) &\geq \alpha s + (2-\alpha)\sqrt{s(1-s)} - \left(\frac{1}{2} - \frac{\alpha}{2}\left(\frac{1}{2} - s\right)\right)\\
&= (2-\alpha)\left(\sqrt{s(1-s)} - \frac{1}{4}\right) + \frac{1}{2}\alpha s\\
&> 0.
\end{aligned}$$

Therefore, for any price $p > p^*$, $p\left(1 - \frac{2}{2-\alpha}(p-\alpha s)\right) \leq p^*\left(1 - \frac{2}{2-\alpha}(p^*-\alpha s)\right) \leq p^*(1-p^*)$, where the last inequality follows from (38). Thus, no such price gives the first seller a higher profit than p^* does.

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