Weighted Congestion Games With Separable Preferences

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Players in a congestion game may differ from one another in their intrinsic preferences (e.g., the benefit they get from using a specific resource), their contribution to congestion, or both. In many cases of interest, intrinsic preferences and the negative effect of congestion are (additively or multiplicatively) separable. This paper considers the implications of separability for the existence of pure-strategy Nash equilibrium and the prospects of spontaneous convergence to equilibrium. It is shown that these properties may or may not be guaranteed, depending on the exact nature of player heterogeneity. ***JEL classification:*** C72.

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1 Introduction

Congestion games model the congestion externalities that arise when users compete for limited resources. The intensity of competition over resource j is measured by the number w_j of *standard users* of j. The weight, or congestion impact, of each physical user i of resource j is expressed as a (positive, but not necessarily whole) number w_j^i of standard users. Thus,

$$w_j = \sum_{i \in I_j} w_j^i, \tag{1}$$

where I_j is the set of all users of resource j. The cost c^i for each user i is affected by the intrinsic characteristics of i and the resource j he uses, and the intensity of competition for that resource, which is expressed by w_j. The subject of this paper is games in which these two effects can be separated. Player i's preferences may be multiplicatively separable, which means that the cost has the form

$$c^i = a_j^i l(w_j), \tag{2}$$

where a_j^i is a positive constant that represents the *base cost* for i of using j and $l\colon (0, \infty) \to (0, \infty)$ is a nondecreasing function that gives the *crowding cost*. Alternatively, the preferences may be additively separable (Konishi et al., 1997), which means that the cost has the form

$$c^i = c(w_j) + b_j^i, \tag{3}$$

where the constant b_j^i is the *fixed cost* for i of using j and the nondecreasing function $c: (0, \infty) \to \mathbb{R}$ gives the corresponding *variable cost*. Both terms in (3) can be positive or negative (although $c > 0$ could clearly be assumed without loss of generality). Negative b_j^i may be interpreted as representing a *gain* for i from using j. The multiplicative and additive formulations, (2) and (3), are interchangeable in that they represent the same ordinal utilities. It is possible to move from one formulation to the other by using the transformation

$$\begin{aligned} b_j^i &= \log a_j^i \\ c(\cdot) &= \log l(\cdot). \end{aligned} \tag{4}$$

Therefore, as long as only pure strategies are considered, the choice between the two formulations is merely a matter of convenience (Mavronicolas et al., 2007).

Definition 1. A *weighted congestion game with separable preferences* is a finite noncooperative game with n players (numbered from 1 to n) and m resources (numbered from 1 to m), such that each player i has a certain nonempty subset of *allowable resources*, of which he has to choose one. The resource j player i chooses and the set I_j of players who make the same choice together determine the cost (or the negative of the payoff) for the player. This cost c^i is given either by (2), for all players i, or by (3), for all players i.[1,2]

This kind of games is largely a special case of the weighted version of congestion games with player-specific payoff functions (Milchtaich, 1996), which however does not allow the weights to vary across resources. The main interest in games with separable preferences lies in the fact that many of the specific examples considered in the theoretical literature and in applications have this property, i.e., intrinsic preferences and the negative effects of congestion are separable. Three examples follow.

Job balancing (See Even-Dar et al., 2003). The n players represent independent, selfish jobs. Each job i can run on each of several parallel machines. The cost of using machine j is the machine's total running time, which is given by

$$\frac{1}{S_j} w_j, \tag{5}$$

where S_j is the speed of machine j and w_j is the total weight of the jobs assigned to it. The contribution of each job to the total weight can be the same for every machine it is assigned to or it can vary across machines. If the former holds for all jobs (i.e., the weights are machine-independent), the machines are said to be *related*; otherwise they are *unrelated*. Either way, the cost c^i is the same for all jobs i that use a particular machine j, and it is given by (5). Comparison with (2) shows that, in this example, the base cost is machine-specific but player-independent, and the crowding cost is given by the identity function.

[1] Since the additive and multiplicative formulations are connected by (4), whenever one is explicitly given the other is implicitly so.

[2] An alternative to 'separable preferences' might be 'separable costs'. However, using the latter term would be potentially confusing, since the literature on congestion games (e.g., Milchtaich, 2006a; Correa et al., 2008) already assigns it several meanings that are substantially different from the meaning of separability in this paper and others.

M/M/1 queuing (See Libman and Orda, 2001). Each of n users can send jobs to each of m servers. However, users have to make their choices of server once and for all; they cannot send different jobs to different servers. All jobs are identical, but the rates at which they are sent are player-specific and the service times are server-specific. Specifically, jobs from user i arrive at i's chosen server as a Poisson process with parameter λ^i (which means that the time from one job to the next is an exponentially distributed random variable), and the service time in each server j is an exponentially distributed random variable with parameter μ_j. Jobs are serviced on a first-in-first-out basis. The waiting time for each job is the time it waits in queue plus the service time. The goal of each user i is to minimize the expected waiting time for his jobs, T^i, which depends on the server j he chooses and the set of users I_j who choose j. It follows from above assumptions that

$$T^i = \frac{1}{\mu_j - \lambda_j},$$

where $\lambda_j = \sum_{i' \in I_j} \lambda^{i'}$. Therefore, minimizing T^i is equivalent to minimizing

$$\lambda_j - \mu_j, \tag{6}$$

the negative of the *residual capacity* of server j.[3] The additively separable expression (6) may be viewed as the cost of choosing server j.

Habitat selection (See Brown, 1998). Several species of competing animals forage in an environment divided into m habitats. Habitat selection is cost-free. The fitness of each individual i is determined by the rate at which it harvests food items from the habitat j it chooses. In the absence of any competitors, the harvest rate would depend only on the habitat productivity R_j and the degree to which i's species is adapted to foraging at j, which can be expressed by the (small) probability w_j^i that i will be able to locate and consume a specified food item in j within a specified period of time. Competition (both intra- and inter-specific) affects this rate in two ways. As more individuals forage within j, the probability that any single food item will be harvested increases, but i's chance of being the finder decreases. The Poisson approximation (Feller, 1968, p. 282) gives $1 - e^{-w_j}$ for the former probability, where $w_j = \sum_{i' \in I_j} w_j^{i'}$ and I_j is the set of all individuals (of all species) that forage within j. It follows that, if there are many competitors, i's approximate harvest rate H^i is given by

$$H^i = w_j^i R_j \frac{1 - e^{-w_j}}{w_j}. \tag{7}$$

At equilibrium, the habitat choice of each individual i maximizes the multiplicatively separable expression (7). Equivalently, j minimizes $1/H^i$, which is the typical time between finding one food item and finding the next one. Note that

$$\frac{1}{H^i} = \frac{1}{w_j^i R_j} l(w_j),$$

[3] The residual capacity is normally positive at equilibrium; otherwise the expected waiting time is infinite.

where the function l is defined by $l(x) = x/(1 - e^{-x})$. Therefore, this increasing function may be viewed as expressing the crowding cost.

The main questions this paper addresses are the existence of pure-strategy Nash equilibria in the class of weighted congestion games with separable preferences, and whether myopic adjustments by individual players necessarily lead to such an equilibrium. For some subclasses of these games, the existing literature provides answers to these questions, and for others it does not. Section 2 below summaries the known positive results and presents a new negative result that completes the large-picture analysis. It also presents an open problem that concerns a special, but important, case. The reason such a case-by-case analysis is warranted is that, as the above examples illustrate, many applications only involve games with separable preferences that have certain additional properties.

1.1 Resource-specific variable costs

Separability, as defined above, means that the variable costs (equivalently, the crowding costs) have the same functional form for all resource–player pairs. This model is seemingly less general than one with a different variable-cost function c_j for each resource j. However, the following proposition shows that this is in fact not the case: the former model subsumes the latter. Thus, allowing resource-specific variable costs would not make the model any more general.

Proposition 1. Every weighted congestion game with resource-specific variable-cost functions $c_1, c_2, \dots, c_m$ can also be presented using a single variable-cost function c.

The proof of the proposition, which is given in the Appendix, is based on the following simple argument. Even if the functions $c_1, c_2, \dots, c_m$ are not identical, they can be "stitched" together into a single function c, which can effectively replace each of them, if the values that the corresponding arguments $w_1, w_2, \dots, w_m$ take lie in nonoverlapping intervals in the real line. Such separation can always be achieved by multiplying each weight w_j^i by the jth power of a sufficiently large constant β, and countering these rescalings by applying the opposite operations to the arguments of the functions $c_1, c_2, \dots, c_m$.

Note that this argument depends critically on the possibility of resource-specific weights. Therefore, it would not apply if only the subclass of games described in Section 2.3 below were considered. On the other hand, as the proof of Proposition 1 shows, the move to a single variable-cost function does not require—or create—player-specific fixed costs.

2 Existence of Equilibrium and Convergence

As detailed below, weighted congestion games with separable preferences do not always have pure-strategy Nash equilibria. However, such equilibria always exist in certain subclasses of these games, in which, moreover, any sufficiently long sequence of myopic unilateral moves by players is bound to lead to an equilibrium. To precisely describe these results, the following definitions are required.

An *improvement path* (Monderer and Shapley, 1996) in a noncooperative game is any finite or infinite sequence of strategy profiles, each differing from the preceding one only in the strategy of a single player i, such that the change of strategy makes i better off. A *best-(reply) improvement path* has the additional property that each change of strategy represents a best reply against the other players' strategies, so that player i could not gain more by choosing another strategy instead. A game has the *finite improvement property* (FIP) or the *finite best-(reply) improvement property* if every improvement path or best-improvement path, respectively, is finite. The former property implies the latter, which in turn implies that the game has a pure-strategy Nash equilibrium ('an equilibrium', for short). In general, the reverse implications do not hold. A finite game has the FIP if and only if it admits a (generalized ordinal) *potential*, which is any function P over strategy profiles that strictly increases along every improvement path.[4] An *exact potential* P is defined by the stronger property that any unilateral change of strategy is beneficial to the player i changing strategy *if and only if* it increases P, and in this case, the change in P exactly equals i's gain. (Note that the notion of exact potential is a cardinal, rather than ordinal, one.)

Monderer and Shapley (1996) showed that the only kind of finite games that admit an exact potential are the congestion games introduced by Rosenthal (1973). These games differ from those in Definition 1 in several significant respects. On the one hand, they are more general in that players may be allowed to choose combinations of resources rather than single ones, and congestion does not necessarily increase the costs. On the other hand, they are less general than the games considered here is that they are *unweighted* in the sense that all weights are 1, and the costs are player-independent (but may be resource-specific). Applying similar limitations to the games in Definition 1 gives the three subclasses of games described below.

2.1 Games with player-independent costs

The costs are player-independent if for every resource j the base costs satisfy

$$a_j^1 = a_j^2 = \cdots = a_j^n,$$

or equivalently, if the fixed costs in (3) satisfy a similar condition. The job balancing example presented above has this form. Games with *resource*-independent costs are essentially a special case. Specifically, if $a_1^i = a_2^i = \cdots = a_m^i$ for every player i, then these constants can all be normalized to 1 without affecting the players' preferences as a result.

Even-Dar et al. (2003) and Fabrikant et al. (2004) proved the following result.

Theorem 1 (Even-Dar et al., Fabrikant et al.). Every weighted congestion game with separable preferences and player-independent costs has a pure-strategy Nash equilibrium, and moreover has the finite improvement property.

The proof of Theorem 1 is based on the following observation, which is actually valid for more general settings than the present one. If only a single player i changes his choice of strategy, from some resource k to another resource j, then the cost after the move for every player who was negatively affected by it (which necessarily means that the player also uses

[4] One example of a potential for a finite game that has the FIP is the function that assigns to each strategy profile the number of improvement paths that terminate at it.

j) is equal to the cost c^i for i. This implies that the multiset of costs $\{c^1, c^2, \dots, c^n\}$ after the move is *lexicographically smaller* than before it. (A multiset of n real numbers A is lexicographically smaller than another such multiset B if there is some real number r such that the number of elements of A that are equal to r is less than the corresponding number for B, and for every $r' > r$ the two numbers are equal. This binary relation is clearly asymmetric and transitive.) It follows that any improvement path must be finite.

2.2 Games with player-independent weights

The weights are player-independent if for every resource j

$$w_j^1 = w_j^2 = \cdots = w_j^n,$$

so that

$$w_j = w_j^1 n_j,$$

where n_j is the number of players who use resource j. A weighted congestion game with separable preferences that has this property can be presented as an unweighted game with a distinct variable-cost function c_j for each resource j (see Section 1.1). That function is defined by

$$c_j(x) = c(w_j^1 x),$$

so that the right-hand side of (3) can be written as

$$c_j(n_j) + b_j^i. \tag{8}$$

Therefore, a straightforward generalization of Rosenthal's (1973) argument (see Facchini et al., 1997; Konishi et al., 1997; Hollard, 2000) gives the following.

Theorem 2 (Facchini et al., Konishi et al.). Every weighted congestion game with separable preferences and player-independent weights has a pure-strategy Nash equilibrium, and moreover has the finite improvement property.

For an explicit form of a potential for games as in Theorem 2, see Mavronicolas et al. (2007, Theorem 1). If preferences are additively separable, this potential is in fact an exact one (Facchini et al., 1997; Konishi et al., 1997). Moreover, this result holds for more general settings than the present one. In particular, it would hold also if the number of users had a positive, or ambiguous, effect on the costs.

As shown below, the assumption in Theorem 2 that the players' weights are identical is crucial for the equilibrium existence result. This is not so for the separability assumption, which is only required for the FIP result. In fact, a pure-strategy Nash equilibrium exists in every unweighted congestion game with player-specific payoff functions (Milchtaich, 1996, Theorem 2). In other words, an equilibrium would exist even if for each i and j (8) were replaced by any (player- as well as resource-specific) nondecreasing function of the number of users n_j. However, the game would not then necessarily have the finite improvement, or even best-improvement, property (Milchtaich, 1996, Fig. 1).

2.3 Games with resource-independent weights

The case of resource-independent weights, in which, for every player i,

$$w_1^i = w_2^i = \cdots = w_m^i,$$

does not fall under any general category for which the existence of equilibrium is guaranteed. The only known positive result for games of this kind concerns the special case of three or fewer players. Mavronicolas et al. (2007, Corollary 3) showed that an equilibrium always exists in this case. On the other hand, they presented an example of a three-player weighted congestion game with separable preferences and resource-independent weights that does not have the finite best-improvement property, i.e., it admits a best-improvement cycle.[5] The question of the existence of equilibrium for games with more than three players was left open.

The following theorem shows that, in fact, with an arbitrary number of players an equilibrium does not always exist.

Theorem 3. There is a weighted congestion game with separable preferences and resource-independent weights, with eight players and five resources and a strictly increasing and strictly concave variable-cost function c, in which a pure-strategy Nash equilibrium does not exist.

Proof. Let c be any strictly increasing and strictly concave real-valued function on the positive ray that returns the values in the following table:

x	3	3.9	4	5	6	7	8	8.9	9	9.9	10	12
$c(x)$	1	9.5	10.3	12	13.2	14.3	15.2	15.95	16	16.23	16.25	16.26

(Since for these values the marginal costs, or more precisely first-order divided differences, are positive and decreasing, a function as above exists.) The players' weights are given by $w_j^1 = 6$, $w_j^2 = w_j^3 = w_j^4 = 3$, $w_j^5 = 2.9$, $w_j^6 = 2$ and $w_j^7 = w_j^8 = 1$ (for all j). The fixed cost of each of the five resources for each player is either prohibitively high, and effectively excludes the player from that resource,[6] or it is given in the following table:

Player	1	2	3	4	5	6	7	8
Fixed	$b_1^1 = 0$	$b_1^2 = 0$	$b_1^3 = 0$	$b_4^4 = 0$	$b_2^5 = 0$	$b_3^6 = 0$	$b_3^7 = 0$	$b_2^8 = 0$
costs	$b_2^1 = 0.04$	$b_3^2 = 3$		$b_5^4 = 10$		$b_4^6 = 1$		$b_4^8 = 8$

Thus, three players are effectively confined to a single resource, and for five players there are two resources that they can potentially use. It is not difficult to check that, in view of these limitations, in any pure-strategy Nash equilibrium:

- player 4 uses resource 4 if player 6 does not use it, and uses resource 5 otherwise,
- player 8 uses resource 4 if player 4 does not use it, and uses resource 2 otherwise,

[5] More precisely, the three-player example in Mavronicolas et al. (2007, Theorem 2) involves resource-specific variable-cost functions, so that it does not strictly belong to the class of games considered here. However, Theorem 3 below renders this point largely moot.

[6] Alternatively, it can simply be assumed that each player is not allowed to use certain resources.

- player 1 uses resource 2 if player 8 does not use it, and uses resource 1 otherwise,
- player 2 uses resource 1 if player 1 does not use it, and uses resource 3 otherwise.

It follows from these propositions that, if player 6 uses resource 4, then player 8 also uses that resource and only player 7 uses resource 3; and if player 6 uses resource 3, then player 2 also uses that resource and only player 4 uses resource 4. In both cases, the resource player 6 uses is not optimal for him: switching from 4 to 3 or vice versa would decrease the cost. This proves that an equilibrium does not exist. ■

Theorem 3 shows that, for the kind of games considered in this subsection, existence of equilibrium is not guaranteed for general (or even concave) variable-cost functions (equivalently, general crowding costs). However, since many applications involve *specific* such functions, it might be interesting to find functional forms of c that *do* guarantee existence. The little that is known about this issue is summarized below.

2.3.1 Linear variable cost

Particularly important among the games with resource-independent weights are those with a linear variable-cost function. Without loss of generality, linearity means that, in the additive formulation of the cost, c is the identity function:

$$c(x) = x. \tag{9}$$

The M/M/1 queuing example presented above has this property. Unlike the general case, for linear costs the existence of equilibrium is guaranteed. This follows as a special case from a result of Mavronicolas et al. (2007, Theorem 6), which is a straightforward extension of an earlier result of Fotakis at al. (2005, Theorem 1). Both results hold for more general settings than the present one.

Theorem 4 (Mavronicolas et al.). Every weighted congestion game with separable preferences and resource-independent weights in which the variable-cost function c is linear has a pure-strategy Nash equilibrium, and moreover has the finite improvement property.

The proof of Theorem 4 is based on the fact that the additive formulation of every game as in the theorem admits a *weighted potential* P. A weighted potential (Monderer and Shapley, 1996) is a cardinal concept quite similar to an exact potential. It is defined by the requirement that any unilateral change of strategy is beneficial to the player i changing strategy if and only if it increases P, and in this case, the change in P is equal to i's gain times his (resource-independent) weight.

2.3.2 Homogeneous crowding cost

The crowding cost is homogeneous if for some $k > 0$

$$l(x) = x^k.$$

Replacing the cost (2) with its kth root turns l into the identify function. Since this transformation does not affect the players' ordinal preferences, there is no loss of generality in assuming that l *is* the identity, i.e., $k = 1$. Thus, unlike the previous case, the identity function appears in the multiplicative rather than additive formulation of the costs (cf. (9)).

Gairing et al. (2006) called games of this kind *weighted congestion games with player-specific capacities*. In the additive formulation of the costs (3) these games are characterized by

$$c(x) = \log x.$$

Georgiou et al. (2006) conjectured that all weighted congestion games with player-specific capacities have pure-strategy Nash equilibria. However, they were only able to establish that three-player games always have such equilibria, and moreover have the finite best-improvement property. Part of the difficulty is that, even with only three players, weighted congestion games with player-specific capacities do not always have the finite improvement property (Gairing et al., 2006, Theorem 3). Thus, the question of the existence of equilibrium in games with more than three players is still open.

3 Network Congestion Games

A noteworthy property of weighted congestion games with separable preferences is that each such game is isomorphic to a weighted *network* congestion game in which the players have *identical* cost functions and differ from one another only in their weights and the paths they are allowed to take in a specified network, with all the paths connecting specified origin and destination vertices o and d. An example of an applicable network is shown in Fig. 1. A player i can take an o–d path r if and only if r's first edge is marked b_j^i, for some allowable resource j for i in the game with separable preferences. In this case, the cost c^i for i of taking r is given by (3) and (1), where I_j is the set of all players whose paths include the second edge in r. This defines an isomorphism (Monderer and Shapley, 1996) between the weighted congestion game with separable preferences and the weighted network congestion game.

This observation points to a link between the questions discussed in this paper and the equilibrium existence problem in network congestion games (Milchtaich, 2006b, 2009). However, the latter is a quite different problem in that the network topology figures prominently in it.

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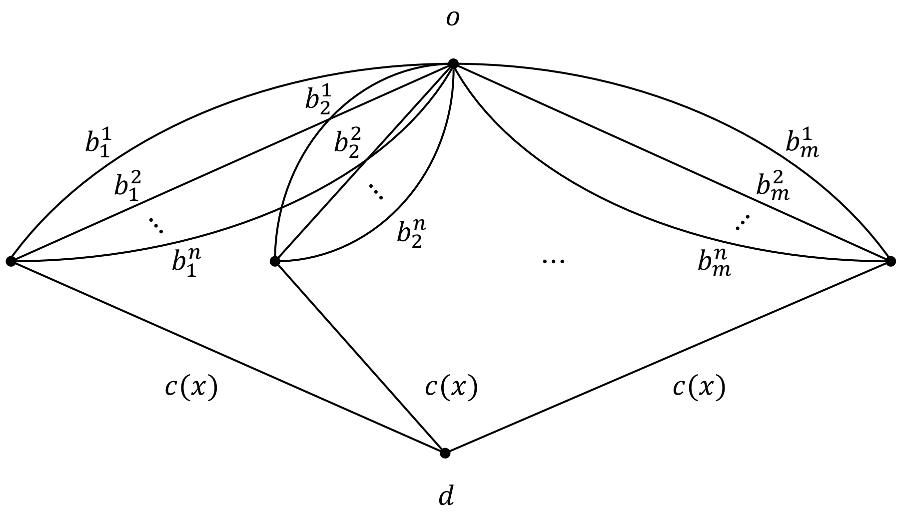

Fig. 1. A weighted network congestion game isomorphic to a (general) weighted congestion game with separable preferences. The cost of each edge is either constant and equal to b_j^i, for some i and j, or it is given by the variable-cost function c. A player i can use an edge with cost b_j^i if and only if j is an allowable resource for i.

Appendix

Proof of Proposition 1. Let a weighted congestion game with resource-specific variable-cost functions $c_1, c_2, \ldots, c_m$ be given. Choose $\underline{w}$ and $\overline{w}$ in such a way that $0 < \underline{w} < w_j^i < \overline{w}$ for all i and j, and define $\beta = n\,\overline{w}/\underline{w}$. For every i and j, define

$$\widetilde{w}_j^i = \beta^j w_j^i. \tag{10}$$

These definitions imply that, for every player i and resource j,

$$\beta^{j+1}\underline{w} > \beta^j \sum_{i'=1}^{n} w_j^{i'} = \sum_{i'=1}^{n} \widetilde{w}_j^{i'} \geq \widetilde{w}_j^i > \beta^j \underline{w}. \tag{11}$$

The function $c: (0, \infty) \to \mathbb{R}$ will be defined as follows. Given $x > 0$, consider the resource with the highest index j such that $x > \beta^j \underline{w}$ (and choose $j = 1$ if $x \leq \beta \underline{w}$). Define

$$c(x) = c_j(\frac{x}{\beta^j}) - b_j, \tag{12}$$

where $b_j = \sum_{k=2}^{j}\big(c_k(\underline{w}) - c_{k-1}(n\overline{w})\big)$ (and $b_1 = 0$). The resource-specific constants $b_1, b_2, \ldots, b_m$ are required to guarantee the monotonicity of c. Finally, for every player i and resource j, define

$$\tilde{b}_j^i = b_j^i + b_j. \tag{13}$$

The weighted crowding game defined by the function c in (12) and the modified weights and constants in (10) and (13) is identical to the original game. To see this, suppose that resource j is chosen by a nonempty subset of players I. In the original game, the cost for each of these players i is given by (3), where $w_j = \sum_{i' \in I} w_j^{i'}$. In the modified game defined above, w_j is replaced by $\widetilde{w}_j = \sum_{i' \in I} \widetilde{w}_j^{i'}$. However, it follows from (11) and (12) that

$$c(\widetilde{w}_j) = c_j(\frac{\widetilde{w}_j}{\beta^j}) - b_j,$$

so that $c(\widetilde{w}_j) + \tilde{b}_j^i = c_j(w_j) + b_j^i$ by (10) and (13). ■

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