POTENTIAL IN POPULATION GAMES

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A novel notion of potential in population games is presented. A population game is defined, very broadly, as any bivariate function $g(x, y)$ on a convex set in a linear topological space. This function may specify the payoff to an individual population member from choosing *strategy* x (in a symmetric population game) or the mean payoff to individuals from playing according to *strategy profile* x (in an asymmetric game), with the choices in the population as a whole expressed by the *population strategy* y. These notions of population game and potential include a number of earlier notions as special cases. Potential is closely linked with (a general notion of) equilibrium. It increases along every *improvement curve*: the population-game analog of an improvement path in an N-player game.

1 Introduction

The term population games has been in use from the early days of evolutionary game theory (see Maynard Smith 1982 and references therein). However, it does not always carry the same meaning. Most often, the term refers to what may be called symmetric population games. Such games model a large population of identical agents, whose payoff $g(x, y)$ is determined by the action or strategy x they choose and a single second strategy, the population strategy y, which in one way or the other reflects the collective or aggregate action of the population as a whole. A population game may also be asymmetric. Different population members may have different allowable actions or receive different payoffs from choosing the same actions, or it may matter who are the other agents choosing each action. In such cases, x and y are *profiles*, not single actions or strategies.

As this paper shows, it is useful in the context of population games to separate between formalism and interpretation. A single, simple model is capable of accommodating a large swath of interpretations, and pertain to both symmetric and asymmetric population games. A population game is defined as any bivariate function $g(x, y)$ whose two arguments are elements of a convex set X in a linear topological space. In a symmetric population game, X may be, for example, the unit simplex in an n-dimensional Euclidian space, whose elements represent all mixed strategies with n pure strategies or actions. In an asymmetric context, the elements of X may be functions, for example, assignments of (pure or mixed) strategies to individual agents (i.e., strategy profiles). In this case, $g(x, y)$ may represent the mean or aggregate payoff to individuals who would play according to x when the actual choices of the population members are given by y. Section 2 presents a number of such alternative interpretations of the model, points to the examples in this paper where these interpretations apply, and identifies the particular interpretations referred to as "population game" elsewhere in the literature (which is very voluminous indeed; only a small sample of papers is covered here).

The heart of the paper is Section 3, which concerns potential. It first recalls the meaning of this concept in (asymmetric) N-player games and in symmetric such games and presents a couple of results that are used in the sequel. Then, a novel definition of potential in population games is presented, which is meaningful in the broad setting introduced in the preceding section.

An advantage of the proposed definition of potential is that it does not require straying from the set X of strategies or strategy profiles, as it only involves the variation of the potential Φ along line segments in X. By contrast, Sandholm (2001, 2015) defines potential in terms of its partial derivatives, which generally requires extending the domain of Φ beyond X. For such extended Φ, Section 3.4 presents a generalization of Sandholm's formulation which is equivalent to the definition of potential here. In particular, a sufficient condition for Φ to be a potential for a population game g is that g is the differential of Φ.

In N-player games, an obvious sufficient condition for a strategy profile to be an equilibrium is that it maximizes the potential. Section 4 shows that in a population game, even a local maximum point of a potential is an equilibrium. And if the potential is concave, then being a (global) maximum point is (also) a necessary condition for equilibrium. Sufficient conditions for the existence and uniqueness of equilibrium immediately follow from these results.

In finite N-player games, the existence of a potential implies the so-called finite improvement property (Monderer and Shapley 1996). Starting at any strategy profile, an equilibrium is necessarily reached if the players change their strategies one-by-one, in whatever order, in such a way that the payoff of each player changing his strategy increases as a result (an *improvement path*). Even if the game is not finite (that is, the players' strategy sets are not all so), the improvement is still finite in the sense that the sum of the moving players' payoff increments is globally bounded, as it is equal to the difference between the values of the potential at the initial and final strategy profiles. Section 5 shows that a similar equality holds for population games. However, improvement paths are replaced here by *improvement curves*. Such curves are traced by the population strategy y as individual population members change their choices of strategies or actions to increase their payoffs. It is assumed that the changes involve only a small fraction of the population within any short time interval, so that the change of y is continuous in time. A corresponding difference from the N-player case is that the sum of the individual payoff increments is replaced here (essentially) by an integral.

The last section of the paper presents applications.

In Section 6.1, it is shown that any N-player game with suitable strategy spaces can be presented as a population game. Whereas the normal presentation of a game specifies the payoff to each player for each strategy profile y, the alternative (and functionally equivalent) population-game presentation specifies the *sum* of the payoffs that individual players would get by *unilaterally* deviating to play according to a second strategy profile x. For an N-player game with multilinear payoff functions, it is shown that a potential exists if and only if the corresponding population game has a potential. Moreover, the two potentials are the same. A corollary of this equality is a novel explicit formula for the potential in finite games.

Certain symmetric N-player games also have a corresponding population game. And if these symmetric games have a potential, then so do the population games. These facts are established in Section 6.2, which then applies them to the setting of random matching in symmetric N-player games.

A much-studied special case of the symmetric setting is that of symmetric $n \times n$ games. These games can be viewed either as symmetric two-player games or as population games, with each interpretation carrying its own form of potential. As an illustration, Section 6.3 compares these two potentials in the 2×2 case, in which a potential always exists.

A (very large) superset of $n \times n$ games is population games there X is the unit simplex in a Euclidean space $\mathbb{R}^n$ and g is linear in the first argument. (Such linearity holds, in particular, whenever g expresses an *expected* payoff.) For this class of games, Section 6.4 compares the notion of potential proposed in this paper with that proposed by Sandholm (2015) and (2001) for the symmetric and asymmetric cases respectively. It is shown that the latter potentials are essentially a subset of the former. For example, a symmetric 2×2 game has a potential according to Sandholm's definition only if the two players' payoff are always equal. As indicated, this limitation does not apply to the definition of potential in this paper or to that of potential in symmetric games.

Sandholm's treatment of improvement curves assumes a rigid law of motion that dictates how the population strategy (or population state) changes over time. In this paper, by contrast, improvement curves may bend in different directions, depending on who are the first population members to change their choices. This stands in direct analogy with improvement paths in N-player games, several of which may pass through a given strategy profile, with each path reflecting a different order of moves. The total improvement (of individual payoffs) along an improvement path is given by a line integral. If the population game has a potential, then, as indicated, this integral is equal to the potential difference between the endpoints of the curve.

An important and familiar class of population games is nonatomic congestion games. Section 6.5 shows how some of the earlier literature about these games fits into the present framework. It analyzes two concrete, but rather general, models, one involving a symmetric congestion game and the other an asymmetric game, and applies to them some of the above results.

2 Population games

A *population game*, as defined in this paper, is any (bivariate) function $g: X^2 \longrightarrow \mathbb{R}$ where X is a convex set in a (Hausdorff real) linear topological space (for example, the unit simplex in a Euclidean space $\mathbb{R}^n$). A fair number of interpretations are possible. Here is a non-exhaustive list:

1. *Symmetric population games.* X is interpreted as the space of *strategies*, and $g(x, y)$ as the payoff to an individual using strategy x when the *population strategy* is y. The population strategy is an encapsulated description of the choice of strategies in the entire population. For example:
    a. Strategy y is the population's *mean*, or average, strategy with respect to some nonatomic *population measure* μ, which attaches zero mass to every individual in an infinite population.[1] See Examples 2 and 6.

[1] An infinite population may represent the limiting case of an increasingly large population, with the effect of each player's action on each of the other players correspondingly decreasing. Alternatively, it may represent all possible *characteristics* of players, or potential players, when the number of *actual players* is finite.

   b. Strategy y describes the *distribution* of strategies in the population. In this case, X necessarily consists of *mixed strategies*, that is, probability measures on some underlying space of actions or (pure[2]) strategies, and $g(x, y)$ is linear in the first argument and expresses the expected payoff of an individual whose choice of action is random with distribution x. See Examples 3 and 4.

2. *Asymmetric population games.* X is interpreted as the space of *strategy profiles*. These may represent the individual choices of actions in greater or lesser detail.
   a. Each element of X may be an ordinary strategy profile in an N-player game, with $g(x, y)$ expressing the aggregate payoff of single players who unilaterally deviate to play according to strategy profile x when all the other players play according to y. See Example 1.
   b. A strategy profile y may be a mapping from an infinite set of players I to an action set A or to the space of all distributions over A. For a population strategy y, which specifies the players' actual choices, $g(x, y)$ gives the mean payoff for single players who unilaterally deviate to play according to strategy profile x. The mean is with respect to a specified population measure μ. See Example 7.
   c. A strategy profile y may instead refer to a particular partition of the set of players into a finite number of classes of identical players, with each class possibly having a different set of actions. In this case, y only specifies the distribution of actions within each class. The total mass of each class is given by a population measure μ. See Examples 5 and 7.

Earlier definitions of population games are largely compatible with one or more interpretations of the above definition.

Maynard Smith (1982) loosely describes population games as games in which the payoffs depend on what other members of the population are doing. His actual concern, however, is with random pairwise contests and with situations in which players are "playing the field". The latter possibility means that the payoff of an individual adopting a particular strategy only depends on some average property of the population as a whole, or some section of it.

The definition of population game in Bomze and Pötscher (1989) is interpretation 1.a above. However, these authors also consider 1.b.

In Hammerstein and Selten's (1994) definition of (symmetric) population game, X is a compact convex set in a Euclidean space. The population is either monomorphic, meaning that everyone uses the same element of X, or bimorphic, with one element of X used with frequency p and another with frequency $1 - p$.

Sandholm (2015; see Example 4 below) defines (symmetric) population games as the special case of interpretation 1.a where the set of pure strategies is finite (so that X can be viewed as the unit simplex in a Euclidean space) and $g(x, y)$ is continuous in the second argument, the population strategy (or population state) y. Similarly, Sandholm's (2001; see Example 5 below) definition of asymmetric population games is effectively the special case of interpretation 2.c where the action sets are finite and $g(x, y)$ is linear in the first argument and is continuous in the second argument.

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[2] "Pure" and "mixed" are relative terms. In particular, a pure strategy may itself be a probability vector.

2.1 Equilibrium

Depending on the interpretation of g, the number $g(y, y)$ may represent the payoff, mean payoff or aggregate payoff in a population with population strategy y. Any of these interpretations suggests the following natural definition.

Definition 1 An *equilibrium* in a population game g is any element $y \in X$ such that

$$g(y, y) \geq g(x, y), \qquad x \in X. \tag{1}$$

Here too there are several possible interpretations.

1. In a *symmetric* population game, an equilibrium y is a symmetric equilibrium *strategy*. The equilibrium condition (1) may mean any of the following:
   a. In a monomorphic population where everyone plays strategy y, single individuals cannot increase their payoff by choosing any alternative strategy x. See Example 3.
   b. For a population strategy y that describes the population's mean strategy or distribution of strategies, and a payoff function g that is linear in the first argument, inequality (1) expresses the condition that almost everyone in the population is using a strategy that is a best response to y. In other words, the possibly polymorphic population is in an equilibrium state. See Examples 2, 4 and 6.
2. In an *asymmetric* population game, an equilibrium y is a *strategy profile.* Condition (1) may (indirectly) assert the optimality for individual players of playing according to y.
   a. With a finite number of players, the condition may be just an alternative formulation of the Nash equilibrium condition. See Example 1.
   b. With an infinite population, it may similarly mean that almost everyone in the population would not benefit from choosing an alternative strategy. See Examples 5 and 7.

3 Potential

Potential in population games is closely related to the concept of (exact) potential in N-player games. It is therefore useful to start with discussing this concept, first in the context of asymmetric N-player games and then in that of symmetric games.

3.1 In N-player games

In an (asymmetric) N-player game, each player i has a strategy space X_i and a payoff function $h_i: X \longrightarrow \mathbb{R}$, where $X = X_1 \times X_2 \times \cdots \times X_N$ is the space of all strategy profiles. The game can be written as a single function, $h = (h_1, h_2, \ldots, h_N): X \longrightarrow \mathbb{R}^N$.

For a game $h: X \longrightarrow \mathbb{R}^N$, a function $P: X \longrightarrow \mathbb{R}$ is an (*exact*) *potential* (Monderer and Shapley 1996) if, whenever a single player i changes his strategy, the resulting change in i's payoff is equal to the change in P. Formally, for all players i,

$$h_i(x_i, y_{-i}) - h_i(y) = P(x_i, y_{-i}) - P(y), \;\; x_i \in X_i, y \in X, \tag{2}$$

where the notation (x_i, y_{-i}) refers to the strategy profile where player i plays x_i and everyone else plays according to $y = (y_1, y_2, \ldots, y_N)$. Fixing x_i and rearranging (2) to read

$$h_i(y) = P(y) + \big(h_i(x_i, y_{-i}) - P(x_i, y_{-i})\big), \qquad y \in X$$

shows that, in a potential game, each player's payoff is equal to the sum of P and some function of the other players' strategies. (Obviously, the converse is also true: any function P having the last property is a potential.) An immediate corollary is that the potential is unique up to an additive constant.

A useful observation (Monderer and Shapley 1996, Lemma 2.10) is that potentials of finite games are preserved under mixed extensions. That is, in a game where the players' strategy spaces X_i are finite sets (henceforth, *pure* strategies), replacing each of them with the set $\Delta(X_i)$ of probability measures on X_i (*mixed* strategies) and each payoff function h_i with its multilinear extension give a game that has a potential if and only if the original game h has one. Indeed, it is easy to see that, if condition (2) holds, then a similar condition holds for mixed strategies of player i and mixed strategy profiles, with h_i and P replaced with their multilinear extensions. The multilinear extension of P is therefore a potential for the mixed extension of h. In the other direction, the restriction of a potential for the mixed extension of h to pure strategy profiles is obviously a potential for h.

A characterization of potential in finite games follows from this observation. In the next proposition, h_i denotes both the payoff function of player i and its multilinear extension.

Proposition 1 For a finite N-player game $h: X \longrightarrow \mathbb{R}^N$, a function $P: X \longrightarrow \mathbb{R}$ is a potential if and only if

$$P(x) - P(y) = \int_0^1 \sum_{i=1}^{N} \big(h_i(x_i, px_{-i} + (1-p)y_{-i}) - h_i(y_i, px_{-i} + (1-p)y_{-i})\big)\, dp\,, \quad x, y \in X. \tag{3}$$

Proof. As remarked, P is a potential for h if and only if its multilinear extension, which may also be denoted by P, is a potential for the mixed extension of h. In this case, the sum in (3) is equal to

$$\sum_{i=1}^{N} \big(P(x_i, px_{-i} + (1-p)y_{-i}) - P(y_i, px_{-i} + (1-p)y_{-i})\big) = \frac{d}{dp} P(px + (1-p)). \tag{4}$$

The necessity of condition (3) follows from this equality by integration. Sufficiency follows from the fact that the equality in (3) reduces to that in (2) in the special case where y and x are equal in all but the ith component (and so $(x_i, y_{-i}) = x$). ■

Note that Eq. (3) provides an explicit form for the potential. For fixed, arbitrary y and any choice of value for $P(y)$, the value of $P(x)$ is given by that formula for all $x \in X$.

Proposition 1 extends to N-player games $h: X \longrightarrow \mathbb{R}^N$ where for every player i the strategy space X_i is a convex set in a linear topological space and the payoff function h_i is multilinear. This is because it follows from (2) that a potential P for such a game must also be multilinear, and therefore satisfy (4).

3.2 In symmetric N-player games

Symmetric N-player games differ from the asymmetric games considered in the previous subsection in that the players share a single strategy space X and a single payoff function $g: X^N \longrightarrow \mathbb{R}$ that is invariant to permutations of its second through Nth arguments. If one player uses strategy x and the other players use $y, \dots, z$, in any order, the first player's payoff is $g(x, y, \dots, z)$.

Potential in symmetric games has essentially the same meaning as in asymmetric games. The only difference is that, here, the potential is necessarily a symmetric function, meaning that it is invariant to permutations of its N arguments. Thus, for a symmetric game $g: X^N \longrightarrow \mathbb{R}$, a symmetric function $F: X^N \longrightarrow \mathbb{R}$ is a potential if

$$g(x, z, \dots) - g(y, z, \dots) = F(x, z, \dots) - F(y, z, \dots), \qquad x, y, z, \dots \in X, \tag{5}$$

where the ellipsis (…) stands for any $N - 2$ elements of X.

It is easy to see that, similarly to the asymmetric case, a symmetric function F is a potential for a symmetric game g if and only if the difference $g - F$ is a function that does not depend on the first argument. By the symmetry of the potential, this implies that it is unique up to an additive constant. In addition, if g itself is a symmetric function (in other words, if the game is *doubly symmetric*), then g is its own potential. This case represents a common-interest game, where the players' payoffs are always equal.

Proposition 2 A symmetric N-player game $g: X^N \longrightarrow \mathbb{R}$ has a potential if and only if

$$g(x, y, \dots) + g(y, z, \dots) + g(z, x, \dots) = g(y, x, \dots) + g(z, y, \dots) + g(x, z, \dots), \; x, y, z, \dots \in X. \tag{6}$$

For $N = 2$, the potential $F: X^2 \longrightarrow \mathbb{R}$ is then given by

$$F(x, y) = g(x, y) + g(y, z) - g(z, y), \tag{7}$$

where $z \in X$ is any fixed strategy.

Proof. Subtract the equality (5) from the sum of two similar equalities, one in which x and z are interchanged and one in which y and z are interchanged. Because of the symmetry of the potential F, the terms in which it appears cancel out, and the result is the equality in (6). This proves the necessity of condition (6).

To prove sufficiency, define a function $F: X^N \longrightarrow \mathbb{R}$ by assigning to some fixed, arbitrary point $(y^1, y^2, \dots, y^N) \in X^N$ the value 0 and assigning to every other point $(x^1, x^2, \dots, x^N)$ the sum of the players' changes of payoffs as they move one by one to that point from the initial point, with the order of moves determined by the players' indices, starting with N and ending with 1. Thus,

$$\begin{aligned} F(x^1, x^2, \dots, x^N) = {} & g(x^N, y^1, \dots, y^{N-1}) - g(y^N, y^1, \dots, y^{N-1}) + g(x^{N-1}, x^N, y^1, \dots, y^{N-2}) \\ & - g(x^{N-1}, x^N, y^1, \dots, y^{N-2}) + \cdots + g(x^1, x^2, \dots, x^N) - g(y^1, x^2, \dots, x^N). \end{aligned} \tag{8}$$

Condition (5) holds, because the first strategy x^1 appears on the right-hand side of (8) only in the penultimate term. Therefore, F is a potential if and only if it is a symmetric function, equivalently, invariant to the transposition of any two consecutive arguments x^j and x^{j+1}. Because of the invariance of g to permutations of its second through Nth arguments, interchanging x^j and x^{j+1} does not affect any term on the right-hand side of (8) where they both appear in those places. Therefore, the invariance boils down to the equality

$$\begin{aligned} & g(x^{j+1}, x^{j+2}, x^{j+3}, \dots, y^j) + g(x^j, x^{j+1}, x^{j+2}, \dots, y^{j-1}) - g(y^j, x^{j+1}, x^{j+2}, \dots, y^{j-1}) \\ & \quad = g(x^j, x^{j+2}, x^{j+3}, \dots, y^j) + g(x^{j+1}, x^j, x^{j+2}, \dots, y^{j-1}) - g(y^j, x^j, x^{j+2}, \dots, y^{j-1}). \end{aligned}$$

Replacing on both sides of the last equality the first term $g(\cdot, x^{j+2}, x^{j+3}, \dots, y^j)$ with $g(\cdot, y^j, x^{j+2}, \dots, y^{j-1})$ (which is equal to the first because it is obtained by permutation of the second through Nth arguments) makes the equality an instance of that in (6) (with $x = x^{j+1}$, $y = y^j$ and $z = x^j$). This proves the sufficiency of condition (6).

The formula (7) is obtained from (8) as a special case by setting $z = y^1$ and dropping a constant. ■

The remark in the previous subsection concerning mixed extensions also applies to symmetric games. A symmetric N-player game that is the mixed extension of a finite symmetric game has a potential if and only if the equality in (6) holds for all *pure* strategies $x, y, z, \ldots$. For $N = 2$, the potential F is then given by (7).

3.3 In population games

For population games, which generally represent interactions involving many identical players whose individual actions have negligible effects on the other players, the definition of potential may be naturally adapted by replacing the increment of the potential with a derivative.

Definition 2 For a population game $g: X^2 \to \mathbb{R}$, a function $\Phi: X \longrightarrow \mathbb{R}$ is a *potential* if

$$\frac{d}{dp}\Phi(px + (1-p)y) = g(x, px + (1-p)y) - g(y, px + (1-p)y), \qquad 0 \le p \le 1, x, y \in X. \quad (9)$$

Note that the requirement here is twofold: the derivative on the left-hand side of (9) exists (as a one-sided, right or left derivative, for $p = 0$ or 1 respectively), and the equality holds.

It is easy to see that the potential is unique up to an additive constant. That is, the difference between any two functions satisfying the condition in the definition is a constant function on X.

When the potential exists, it is normally possible to present it explicitly by transforming the differential condition in Definition 2 into an integral condition. The following theorem concerns population games where $g: X^2 \to \mathbb{R}$ is a continuous function. Linearity of g in the first argument means

$$g(tx + (1-t)y, z) = tg(x, z) + (1-t)g(y, z), \qquad 0 \le t \le 1, x, y, z \in X. \qquad (10)$$

Theorem 1 For a continuous population game $g: X^2 \to \mathbb{R}$, a function $\Phi: X \longrightarrow \mathbb{R}$ is a potential if and only if

$$\Phi(x) - \Phi(y) = \int_0^1 \big(g(x, px + (1-p)y) - g(y, px + (1-p)y)\big)\, dp\,, \qquad x, y \in X \quad (11)$$

and g is linear in the first argument.

Theorem 1 is proved – and extended – by the next two propositions. Say that a population game $g: X^2 \to \mathbb{R}$ satisfies the *continuity condition* if the function $p \mapsto g(x, px + (1-p)y) - g(y, px + (1-p)y)$ is continuous in the interval $[0,1]$ for all $x, y \in X$. A sufficient condition for this is that g is continuous in the second argument.

Proposition 3 Consider a population game $g: X^2 \to \mathbb{R}$ that satisfies the continuity condition. A necessary condition for a function $\Phi: X \longrightarrow \mathbb{R}$ to be a potential for g is (11). If g is linear in the first argument, then this condition is also sufficient.

Proof. Eq. (11) follows from (9) by integration. Conversely, if (10) and (11) hold, then for every $x, y \in X$ and $0 \le t \le 1$

$$\Phi(tx + (1-t)y) - \Phi(y)$$

$$= \int_0^1 t\left(g(x, ptx + p(1-t)y + (1-p)y) - g(y, ptx + p(1-t)y + (1-p)y)\right) dp$$
$$= \int_0^t g(x, px + (1-p)y) - g(y, px + (1-p)y)\, dp,$$

where the second equality involves a change of the integration variable. Eq. (9) now follows from the first fundamental theorem of calculus. ■

Proposition 4 A necessary condition for a continuous population game $g: X^2 \to \mathbb{R}$ to have a potential is that g is linear in the first argument.

Proof. Suppose that g has a potential Φ. Fix $x, y, z \in X$ and $0 \le t \le 1$, and write $x_t = tx + (1-t)y$. For $0 < \epsilon < 1$, consider the strategy w defined by

$$\begin{aligned} w &= \epsilon x_t + (1-\epsilon)z \\ &= \epsilon t x + (1-\epsilon t)\bar{z}, \end{aligned}$$

where $\bar{z} = \epsilon(1-t)/(1-\epsilon t)\, y + (1-\epsilon)/(1-\epsilon t)\, z$. As w lies on the line segment connecting x_t and z as well as on that connecting x and $\bar{z}$, and $\bar{z}$ lies on the line segment connecting y and z, by the mean value theorem and Definition 2 there exist $0 < \eta < \epsilon$, $0 < \eta' < \epsilon t$ and $0 < \eta'' < \epsilon(1-t)/(1-\epsilon t)$ such that

$$\begin{aligned} \Phi(w) - \Phi(z) &= \epsilon \frac{d}{dp}\bigg|_{p=\eta} \Phi(px_t + (1-p)z) \\ &= \epsilon\left(g(x_t, \eta x_t + (1-\eta)z) - g(z, \eta x_t + (1-\eta)z)\right), \end{aligned}$$

$$\begin{aligned} \Phi(w) - \Phi(\bar{z}) &= \epsilon t \frac{d}{dp}\bigg|_{p=\eta'} \Phi(px + (1-p)\bar{z}) \\ &= \epsilon t\left(g(x, \eta' x + (1-\eta')\bar{z}) - g(\bar{z}, \eta' x + (1-\eta')\bar{z})\right), \end{aligned}$$

$$\begin{aligned} \Phi(\bar{z}) - \Phi(z) &= \epsilon \frac{1-t}{1-\epsilon t} \frac{d}{dp}\bigg|_{p=\eta''} \Phi(py + (1-p)z) \\ &= \epsilon \frac{1-t}{1-\epsilon t}\left(g(y, \eta'' y + (1-\eta'')z) - g(z, \eta'' z + (1-\eta'')z)\right). \end{aligned}$$

The expression on the left-hand side of the first equation is equal to the sum of those in the second and third equations. The same must therefore hold for the right-hand sides, and so

$$\begin{aligned} &g(x_t, \eta x_t + (1-\eta)z) - g(z, \eta x_t + (1-\eta)z) \\ &\quad = t\left(g(x, \eta' x + (1-\eta')\bar{z}) - g(\bar{z}, \eta' x + (1-\eta')\bar{z})\right) \\ &\quad + \frac{1-t}{1-\epsilon t}\left(g(y, \eta'' y + (1-\eta'')z) - g(z, \eta'' z + (1-\eta'')z)\right). \end{aligned}$$

Taking the limit $\epsilon \to 0$ (hence, $\eta, \eta', \eta'' \to 0$ and $\bar{z} \to z$) gives the equality in (10). ■

Condition (11) provides an explicit formula for the potential. For fixed, arbitrary $y \in X$ and any choice of value for $\Phi(y)$, it gives the value of $\Phi(x)$ for all $x \in X$.

The formula for the potential takes a particularly special form if g is linear in both arguments.

Proposition 5 A bilinear population game $g: X^2 \to \mathbb{R}$ has a potential if and only if

$$g(x, y) + g(y, z) + g(z, x) = g(y, x) + g(z, y) + g(x, z), \qquad x, y, z \in X. \tag{12}$$

The potential $\Phi: X \to \mathbb{R}$ is then given by

$$\Phi(x) = \frac{1}{2}\big(g(x,x) + g(x,y) - g(y,x)\big), \tag{13}$$

where y is any fixed element of X.

Proof. By Proposition 3, a necessary and sufficient condition for a function $\Phi: X \longrightarrow \mathbb{R}$ to be a potential is (11), which by the linearity of g in the second argument can be written as

$$\Phi(x) - \Phi(z) = \frac{1}{2}\big(g(x,x) + g(x,z) - g(z,x) - g(z,z)\big), \qquad x, z \in X. \tag{14}$$

Replacing z with any fixed y, the necessity of condition (14) implies that a potential must coincide up to an additive constant with the function defined by (13). Plugging that function into the equality in (14) and simplifying give condition (12). ∎

An immediate corollary of Propositions 2 and 5 concerns games that can be viewed either as symmetric two-player games or as population games. An important class of such games is considered in Section 6.3.

Corollary 1 A bilinear population game $g: X^2 \to \mathbb{R}$ has a potential if and only if it has a potential when viewed as a symmetric two-player game. In this case, a potential $\Phi: X \to \mathbb{R}$ for the population game and a potential $F: X^2 \to \mathbb{R}$ for the symmetric game are connected by

$$\Phi(x) = \frac{1}{2}F(x,x), \qquad x \in X. \tag{15}$$

3.4 Differential formulation

The term potential is borrowed from physics, where it refers to a scalar field whose gradient gives the force field. Force is analogous here to incentive, which is expressed by the payoff difference. The analogy can be taken one step further by connecting the payoff function g with the differential of the potential. This requires the potential Φ to be defined on a larger set than X.

For a convex set X in a linear topological space, consider the cone

$$\hat{X} = \{\, tx \mid x \in X, t > 0 \,\}.$$

For example, if X is the set of all mixed strategies, that is, all probability measures on some set of pure strategies, then $\hat{X}$ is the set of all non-zero positive finite measures. For a function $\Phi: \hat{X} \longrightarrow \mathbb{R}$, consider the directional derivative in the direction $\hat{x}$ at the point $\hat{y}$,

$$d\Phi(\hat{x}, \hat{y}) \coloneqq \frac{d}{dt}\bigg|_{t=0^+} \Phi(t\hat{x} + \hat{y}). \tag{16}$$

If this (right) derivative exists for all $\hat{x}, \hat{y} \in \hat{X}$, then the function $d\Phi: \hat{X}^2 \longrightarrow \mathbb{R}$ defined by (16) is the *differential* of Φ.

Theorem 2 Let $g: X^2 \to \mathbb{R}$ be a population game, and let $\Phi: \hat{X} \longrightarrow \mathbb{R}$ be a continuous function with a differential $d\Phi: \hat{X}^2 \longrightarrow \mathbb{R}$ that is continuous in the second argument. The restriction of Φ to X is a potential for g if and only if there is some function $\psi: X \longrightarrow \mathbb{R}$ such that

$$g(x,y) = d\Phi(x,y) + \psi(y), \qquad x, y \in X. \tag{17}$$

The meaning of condition (17) is that g is "almost" given by the differential of Φ; the difference between the two functions depends only on the population strategy y. Even in the very simple case of symmetric 2×2 games (Example 3 below), this difference ψ is generally not a constant function.

The proof of the theorem is based on the following lemma.

Lemma 1 For a convex set X in a linear topological space, a continuous function $\Phi: \hat{X} \longrightarrow \mathbb{R}$ with a differential $d\Phi: \hat{X}^2 \longrightarrow \mathbb{R}$ that is continuous in the second argument satisfies

$$\frac{d}{dp}\Phi(px + (1-p)y) = d\Phi(x, px + (1-p)y) - d\Phi(y, px + (1-p)y),\ 0 \leq p \leq 1, x, y \in X. \quad (18)$$

Proof (an outline). Using elementary arguments, the following can be established.

FACT. A continuous real-valued function defined on an open real interval is continuously differentiable if and only if it has a continuous right derivative.

Now, replacing $\hat{y}$ in (16) with $p\hat{x} + \hat{y}$ gives

$$d\Phi(\hat{x}, p\hat{x} + \hat{y}) = \left.\frac{d}{dt}\right|_{t=p^+} \Phi(t\hat{x} + \hat{y}), \qquad p \geq 0, \hat{x}, \hat{y} \in \hat{X}. \quad (19)$$

By the above Fact, the continuity of Φ and the continuity of $d\Phi$ in the second argument, for $0 < p < 1$ the right derivative in (19) is actually a two-sided derivative and it depends continuously on $\hat{y}$. Therefore, the right-hand side in (18) is equal to the expression

$$\left.\frac{d}{dt}\right|_{t=p} \Phi(tx + (1-p)y) - \left.\frac{d}{dt}\right|_{t=1-p} \Phi(px + ty),$$

which by the chain rule is equal to the derivative on the left-hand side. Hence, equality (18) holds for $0 < p < 1$. The validity of the equality also at the endpoints now follows from the continuity properties of Φ and $d\Phi$. ∎

Proof of Theorem 2. By (18), condition (17) implies (9). Conversely, if (9) holds, then together with (18) it gives (by setting $p = 0$) that (17) holds for the function $\psi(y) \coloneqq g(y, y) - d\Phi(y, y)$. ∎

Theorem 2 may be viewed as an alternative definition of potential in population games. While this definition is not as general as Definition 2, it may be more familiar. In particular, the definition of potential in Sandholm (2015) is a special case. See Section 6.4.

The theorem may also be looked at from the opposite perspective. As it shows, any suitably extendable function Φ on a convex set X in a linear topological space is a potential for some population game. That game g is the restriction of $d\Phi$ to X^2 (which corresponds to a choice of $\psi = 0$ in (17)).

4 Potential and equilibrium

An important property of potential is its intimate connection with the equilibria in the game. In an N-player game, every maximum point of a potential is obviously an equilibrium. For games where the potential satisfies a concavity and smoothness condition, the converse also holds: every equilibrium maximizes the potential (Neyman 1997).

In a population game $g: X^2 \to \mathbb{R}$, an equilibrium is an element $y \in X$ that satisfies condition (1). Depending on whether the game is symmetric or asymmetric (see the discussion in Section 2.1), y may represent either an equilibrium *strategy* or a *strategy profile*. For a game that has a potential, the following theorem connects the potential with the game's equilibria.

Theorem 3 In a population game with a potential Φ, every strategy (profile) y that is a local maximum point of Φ is an equilibrium (strategy). If the potential Φ is concave, then the converse also holds, indeed, every equilibrium (strategy) is a *global* maximum point of Φ.

Proof. Setting $p = 0$ in (9) gives that a potential Φ for a population game $g: X^2 \to \mathbb{R}$ satisfies

$$\left.\frac{d}{dp}\right|_{p=0^+} \Phi(px + (1-p)y) = g(x, y) - g(y, y), \qquad x, y \in X. \tag{20}$$

An element $y \in X$ is an equilibrium if and only if the right-hand side of the equality is nonpositive for all x, equivalently, if and only if this is so for the left-hand side. The last condition is *implied* by y being a local maximum point of Φ, and if Φ is concave, the condition *implies* that y is a global maximum point. ■

It follows from Theorem 3 that a sufficient condition for a population game with a potential Φ to have at least one equilibrium (strategy) is that X is compact and Φ is continuous. A sufficient condition for a population game to have *at most* one equilibrium (strategy) is that it has a *strictly* concave potential. This is because a strictly concave function on the convex set X can have no more than one maximum point.

5 Finite improvement property

In an N-player game h, an *improvement path* of length L (≥ 1) is any list of strategy profile $x^0, x^1, \dots, x^L$ such that, for $l = 1,2, \dots, L$, x^l differs from x^{l-1} only in the strategy of a single player i_l, for whom the payoff increment $h_{i_l}(x^l) - h_{i_l}(x^{l-1})$ is positive. Clearly, a potential P increases along every improvement path and, moreover, the potential difference between the endpoints, $P(x^L) - P(x^0)$, is equal to the sum of the players' payoff increments. If the game is finite, then the monotone increase of the potential implies that h has the *finite improvement property* (Monderer and Shapley 1996): the length of any improvement path is less than the number of strategy profiles in the game. The same conclusion holds for finite symmetric N-player games that have a potential.

Population games cannot be expected to have a property similar to the FIP even if they have a potential, because they are not finite games. However, it is shown below that a version of the result that the sum of the payoff increments is equal to the payoff difference between the endpoints holds here too.

Consider a population game $g: X^2 \longrightarrow \mathbb{R}$ such that the topology on the space in which X lies is induced by a norm $\|\cdot\|$. Suppose that g has a potential $\Phi: X \longrightarrow \mathbb{R}$ that can be extended to an open neighborhood of X in such a way that the Fréchet derivative $D\Phi$ (see below) exists and its restriction to X is uniformly continuous. A *rectifiable curve* C in X is any continuous function from $[0,1]$ to X, $t \mapsto y(t)$, such that

$$\sup_P \sum_{l=1}^{L} \|y(t_l) - y(t_{l-1})\| < \infty, \tag{21}$$

where the supremum is over all partitions P of the unit interval, that is, all finite collections of points $0 = t_0 < t_1 < \cdots < t_L = 1$, $L \geq 1$. (The supremum value represents the length of $\mathcal{C}$.) The *norm* of a partition P is defined as $\lambda(P) \coloneqq \max_{1\leq l\leq L}(t_l - t_{l-1})$.

Theorem 4 For a population game g and a potential Φ as above, and any rectifiable curve $\mathcal{C}$,

$$\lim_{\lambda(P)\to 0} \sum_{l=1}^{L} \big(g(y(t_l), y(t_{l-1})) - g(y(t_{l-1}), y(t_{l-1}))\big) = \Phi(y(1)) - \Phi(y(0)). \tag{22}$$

In words, for a sufficiently fine partition of $[0,1]$, the sum on the left-hand side of (22), which concerns the corresponding points on the curve, is arbitrarily close to the difference between the potential values at the endpoints.

Proof of Theorem 4. For any partition P, Eq. (20) gives that

$$\begin{aligned}\Phi(y(1)) - \Phi(y(0)) - \sum_{l=1}^{L}\big(g(y(t_l), y(t_{l-1})) - g(y(t_{l-1}), y(t_{l-1}))\big) \\ = \sum_{l=1}^{L}\left(\Phi(y(t_l)) - \Phi(y(t_{l-1})) - \frac{d}{dp}\bigg|_{p=0} \Phi(py(t_l) + (1-p)y(t_{l-1}))\right).\end{aligned} \tag{23}$$

It follows from the definition of the Fréchet derivative[3] that

$$\frac{d}{dp}\Phi(px + (1-p)y) = D\Phi(px + (1-p)y)(x - y), \qquad 0 \leq p \leq 1, x, y \in X. \tag{24}$$

Therefore, by the mean value theorem, there is some $0 < p < 1$ such that the expression on the right-hand side of (23) is equal to

$$\sum_{l=1}^{L}\big(D\Phi(py(t_l) + (1-p)y(t_{l-1})) - D\Phi(y(t_{l-1}))\big)(y(t_l) - y(t_{l-1}))$$

and its absolute value is therefore bounded by

$$\sup_{\substack{x,y\in X \\ \|x-y\|\leq \max_{1\leq l\leq L}\|y(t_l)-y(t_{l-1})\|}} \|D\Phi(x) - D\Phi(y)\| \times \sum_{l=1}^{L}\|y(t_l) - y(t_{l-1})\|.$$

By the (automatically, uniform) continuity of $y(\cdot)$ on $[0,1]$ and the uniform continuity of $D\Phi(\cdot)$ on X, the supremum in the last expression tends to 0 as $\lambda(P) \to 0$. It follows, in view of (21), that both sides of (23) also tend to 0. ∎

As the next proposition shows, the equality in Theorem 1 is a special case of that in Theorem 4. For other cases in which the latter equality takes a simple form, see Section 6.4 and Example 6.

[3] The Fréchet derivative at a point y is a bounded linear functional $D\Phi(y)$ (whose norm is denoted $\|D\Phi(y)\|$) such that $(1/\|h\|)\big(\Phi(y+h) - \Phi(y) - D\Phi(y)(h)\big) \to 0$ whenever $\|h\| \to 0$. In particular, the directional derivative in any direction x exists and is given by $d\Phi(x, y) = D\Phi(y)(x)$.

Proposition 6 If $\mathcal{C}$ in Theorem 4 is a line segment, that is, for some $x, y \in X$

$$y(t) = tx + (1 - t)y, \qquad 0 \le t \le 1, \tag{25}$$

then the limit in (22) is equal to the integral in (11).

Proof. As the equalities in (9) and (24) have the same left-hand side, the expressions on the right-hand side are equal. The equality implies that g satisfies the continuity condition (which follows from the assumed continuity of $D\Phi$ in X) and is linear in the first argument (which following from setting $p = 0$). The linearity and assumption (25) give that the sum in (22) is equal to

$$\sum_{l=1}^{L}(t_l - t_{l-1})\big(g(x, t_{l-1}x + (1 - t_{l-1})y) - g(y, t_{l-1}x + (1 - t_{l-1})y)\big).$$

This is a Riemann sum of the integral in (11). ■

The significance of Theorem 4 comes from the case where $\mathcal{C}$ describes a gradual, continuous change of the population strategy that results when, at every moment, a small fraction of the population changes its choice of strategies or actions. As discussed in Section 2, the strategies of the individual population members determine the population strategy y. Different individual choices would yield a possibly different population strategy y'. The expression $g(y', y)$ gives the mean payoff for individual population members who unilaterally deviate to their latter strategy when everyone else chooses their former strategy. As a special case, $g(y, y)$ is the mean payoff when no one deviates. The mean change of payoff from these unilateral changes of strategy is therefore given by

$$g(y', y) - g(y, y). \tag{26}$$

By the assumed continuity of $t \mapsto y(t)$, the population strategy $y(t)$ changes only little during any short time interval $[t, t']$. The mean change of payoff for individual population members that results from their own change of strategy (if any) during this time interval is therefore closely approximated by (26), with $y = y(t)$ and $y' = y(t')$. Therefore, the sum of these payoff changes throughout the entire period has a mean that is approached by the sum on the left-hand side of (22) as the partition of $[0,1]$ becomes finer. Theorem 4 shows that this mean is given by the potential difference between the endpoints of $\mathcal{C}$.

Individual population members may be expected to change their strategy only if they gain from doing so. In this case, the curve $\mathcal{C}$ that the population strategy traces in X is an *improvement curve*. Theorem 4 shows that the gains from the unilateral deviations sum up to the difference between the values that the potential Φ takes at the two endpoints – just as they do in the N-player case. The total gains are therefore bounded by twice the supremum of $|\Phi|$.

6 Applications

6.1 N-player games

For an N-player game $h: X \longrightarrow \mathbb{R}^N$ (see Section 3.1), the function $g: X^2 \longrightarrow \mathbb{R}$ defined by

$$g(x,y) = \sum_{i=1}^{N} h_i(x_i, y_{-i}) \tag{27}$$

gives the sum of the payoffs that individual players would get by unilaterally switching to play according to strategy profile x when all the other players play according to y. It is easy to see that a strategy profile $y \in X$ satisfies (1) if and only if it is a (Nash) equilibrium in h.

Suppose that for every player i in h the strategy space X_i is a convex set in a linear topological space, and so the same holds for the product space $X = X_1 \times X_2 \times \cdots \times X_N$. Then, g may be viewed as a population-game representation of h. The observation in the previous paragraph means that g and h share the same equilibria. The next theorem shows that if each of the payoff functions in h is linear in each of its arguments, then the two games also share the same potential. That is, if either game has a potential, then it is also the other game's potential.

Proposition 7 Let $h: X \longrightarrow \mathbb{R}^N$ be an N-player game where for every player i the strategy space X_i is a convex set in a linear topological space and the payoff function h_i is multilinear, and let $g: X^2 \longrightarrow \mathbb{R}$ be the corresponding population game. A real-valued function on X is a potential for h if and only if it is a potential for g.

Proof. The multilinearity of the payoff functions in h implies that g is also multilinear, and therefore satisfies the continuity condition. By Proposition 3, a function $\Phi: X \longrightarrow \mathbb{R}$ is a potential for g if and only if it satisfies condition (11). As remarked as the end of Section 3.1, a function $P: X \longrightarrow \mathbb{R}$ is a potential for h is and only if it satisfies condition (3). However, in view of (27), these two conditions are one and the same. ∎

Example 1 *Finite games*. Proposition 1 characterizes and gives an explicit form for the potential in a finite N-player game, using the game's mixed extension. As the proof of Proposition 7 shows, this form is actually a population-game potential in disguise. It is the potential of the population game g defined by (27).

6.2 Symmetric N-player games

For a symmetric N-player game $g_N: X^N \longrightarrow \mathbb{R}$ (see Section 3.2; a subscript indicating the number of players is added here for clarity), the function $g: X^2 \longrightarrow \mathbb{R}$ defined by

$$g(x,y) = g_N(x, y, \dots, y) \tag{28}$$

gives the payoff of a player playing x when all the other players play y. A strategy $y \in X$ satisfies (1) if and only if it is (symmetric Nash) equilibrium strategy in g_N.

If the strategy space X in g_N is a convex set in a linear topological space, then g is a population game. By the above observation, the equilibria in the population game are precisely the equilibrium strategies in the symmetric game. The next proposition connects the two games' potentials.

Proposition 8 Let $g_N: X^N \longrightarrow \mathbb{R}$ be a symmetric N-player game where the strategy space X is a convex set in a linear topological space, and let $g: X^2 \longrightarrow \mathbb{R}$ be the corresponding population game. If g_N is linear in the first argument (equivalently, if g is so) and has a potential $F: X^N \longrightarrow \mathbb{R}$, then the function $\Phi: X \longrightarrow \mathbb{R}$ defined by

$$\Phi(x) = \frac{1}{N} F(x, x, \dots, x)$$

is a potential for g.

Proof. As indicated in Section 3.2, the difference $g_N - F$ does not depend on the first argument. Therefore, F too is linear in that argument. As F is a symmetric function, this means that it is actually multilinear. The multilinearity and symmetry of the potential F give that, with $x_p = px + (1-p)y$,

$$g(x, x_p) - g(y, x_p) = g_N(x, x_p, \dots, x_p) - g_N(y, x_p, \dots, x_p)$$
$$= F(x, x_p, \dots, x_p) - F(y, x_p, \dots, x_p) = \frac{1}{N}\frac{d}{dp} F(x_p, x_p, \dots, x_p) = \frac{d}{dp}\Phi(x_p).$$

This means that Φ is a potential for g. ■

Example 2 *Random matching in a symmetric N-player game with a multilinear payoff function*. Random matching (Bomze and Weibull 1995, Broom et al. 1997, and many other papers) refers to the random selection of N individuals who are matched to play a symmetric game $g_N: X^N \longrightarrow \mathbb{R}$. The strategy space X is a convex set in a linear topological space and the function g_N is multilinear. The players are picked up independently and according to the same distribution (i.i.d.) from an (effectively) infinite population of potential players, whose individual probability of being selected is (practically) zero.

Because of the multilinearly of g_N, a player's expected payoff depends only on the player's own strategy x and on the population's *mean strategy* y. Specifically, the expected payoff is expressed by the population game $g: X^2 \longrightarrow \mathbb{R}$ defined in (28). As indicated, $y \in X$ is an equilibrium in this game if and only if it is an equilibrium strategy in the symmetric game g_N. If g_N has a potential F, then by Proposition 8 the potential "along the diagonal" is, up to the multiplicative constant N, a potential for g. It follows, by Theorem 3, that a sufficient (and if the function $x \mapsto F(x, x, \dots, x)$ is concave, also necessary) condition for $y \in X$ to be an equilibrium in the population game g is that for all strategies x in some neighborhood of y

$$F(x, x, \dots, x) \le F(y, y, \dots, y).$$

The meaning of being an equilibrium is that, if the population's mean strategy is y, then (almost) everyone's strategy is optimal: no alternative strategy would yield a higher expected payoff in the game g_N against randomly selected opponents.

6.3 Symmetric $n \times n$ games

A symmetric $n \times n$ game is a symmetric two-player game $g: X^2 \to \mathbb{R}$ where the strategy space X is the unit simplex in $\mathbb{R}^n$ and g can be expressed by a square, $n \times n$ (payoff) matrix A. With strategies written as column vectors and T denoting transposition,

$$g(x, y) = x^{\mathrm{T}} A y, \qquad x, y \in X. \tag{29}$$

The interpretation is that both players share a common set of n actions, and a (mixed) strategy $x = (x_1, x_2, \dots, x_n) \in X$ specifies the probability x_i with which a player chooses the ith action, for $i = 1, 2, \dots, n$. A strategy x is *pure* if all the probabilities but one are zero.

A symmetric two-player game can be viewed also as a (symmetric) population game (Maynard Smith 1982). Instead of being matched with a specific opponent, a player is randomly matched with other members of a large population. The population strategy y is the mean strategy in the population, which is also the distribution of actions there. In fact, it makes no difference whether individuals in the population are playing pure or mixed strategies. The duality of the game – both symmetric and population game – is a special case of Example 2, as for $N = 2$ there is no formal difference between the functions on the right- and left-hand sides of (28). Only the interpretations differ.

Whether g is a potential game does not depend on the interpretation. This is shown by Corollary 1, which in addition shows that a potential $F: X^2 \to \mathbb{R}$ for g as a symmetric game and a potential $\Phi: X \to \mathbb{R}$ for it as a population game are connected by (15).

Example 3 *Symmetric* 2×2 *games.* Every symmetric 2×2 game g, with payoff matrix $A = (a_{ij})_{i,j=1}^{2}$, is a potential game, with the potential

$$F(x, y) = (a_{11} - a_{21})x_1 y_1 + (a_{22} - a_{12})x_2 y_2. \tag{30}$$

(The arguments of F are mixed strategies, $x = (x_1, x_2) = (x_1, 1 - x_1)$ and $y = (y_1, y_2) = (y_1, 1 - y_1)$.) This is because F is evidently a symmetric function and its difference from g is a function that depends only on the second argument y, as it is easy to check that $g(x, y) - F(x, y)$ equals $\psi(y) ≔ a_{21}y_1 + a_{12}y_2$.

The same conclusion can also be deduced from g being the mixed extension of a finite symmetric game with only two strategies. That game trivially satisfies condition (6), because at least two of any (pure strategies) x, y, z are identical. As remarked at the end of Section 3.2, this fact means that g also has a potential, which is given by formula (7). Choosing $z = (0,1)$ in (7) and subtracting the constant a_{12} give (30).

As a population game, g has the potential

$$\Phi(x) = \frac{1}{2}(a_{11} - a_{21})x_1^2 + \frac{1}{2}(a_{22} - a_{12})x_2^2.$$

This fact follows immediately from Corollary 1 or from Proposition 8, as Φ satisfies (15). It can also be verified by checking that the condition in Definition 2, Theorem 1 or Theorem 2 holds. For the latter theorem, note that

$$d\Phi(x, y) = F(x, y), \qquad x, y \in X, \tag{31}$$

and therefore (17) holds with the ψ defined above.

The quadratic function Φ is strictly concave if and only if $a_{11} + a_{22} < a_{12} + a_{21}$. In this case, by Theorem 3 and the remarks following it, the unique strategy $y = (y_1, y_2) = (y_1, 1 - y_2)$ maximizing Φ is the unique (Nash) equilibrium in g.

If the above inequality does not hold, equivalently, if Φ is convex, then it is locally maximized at one or both end points. By the same theorem, this means that the population game g has at least one pure-strategy equilibrium. Such an equilibrium describes a monomorphic population in which everyone is using the same action. An equilibrium y that is not pure may exist too. Such an equilibrium may represent either a monomorphic population, in which everyone uses the mixed strategy y, or a polymorphic population, in which y is the mean strategy.

6.4 Population games on the unit simplex

Many applications involve populations games $g: X^2 \longrightarrow \mathbb{R}$ such that X is the unit simplex in a Euclidean space $\mathbb{R}^n$ and g is linear in the first argument. (Symmetric $n \times n$ games are the special case where g is linear also in the second argument.) Writing $x = (x_1, x_2, \dots, x_n) \in X$ as $\sum_{j=1}^{n} x_j e_j$, where $\{e_1, e_2, \dots, e_n\}$ is the standard basis in $\mathbb{R}^n$, the linearity means that

$$g(x, y) = x \cdot f(y), \qquad x, y \in X, \tag{32}$$

with the dot denoting scalar product and $f = (f_1, f_2, \dots, f_n): X \longrightarrow \mathbb{R}^n$ defined by

$$f_j(y) = g(e_j, y), \qquad j = 1,2, \dots, n. \tag{33}$$

The next proposition characterizes the potential of g, if it exists. It concerns a candidate function on the unit simplex that has a smooth extension to the whole nonnegative orthant.

Proposition 9 Let $g: X^2 \to \mathbb{R}$ be a population game where X is the unit simplex in $\mathbb{R}^n$ and g is linear in the first argument (and so has the presentation (32)), and let $\Phi: \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ be a continuously differentiable function. The restriction of Φ to X is a potential for g if and only if there is some function $\psi: X \longrightarrow \mathbb{R}$ such that

$$f_j(y) = \frac{\partial \Phi}{\partial y_j}(y) + \psi(y), \qquad y \in X, j = 1,2, \dots, n. \tag{34}$$

The proposition readily extends to population games where X is the product of several simplices. The extension is outlined at the end of this subsection.

Proof of Proposition 9. The necessary and sufficient condition in Theorem 2 for (the restriction of) Φ to be a potential can be written as

$$g(x, y) = x \cdot \nabla\Phi(y) + \psi(y), \qquad x, y \in X,$$

with $\nabla\Phi$ denoting the gradient of Φ. It follows from (32) that this condition is equivalent to (34). ∎

An explicit formula for the potential can be derived from Proposition 9. Assuming that the vector-valued function f is continuous, consider its line integral along a piecewise smooth curve $\mathcal{C} \subseteq X$ that starts at a point y and ends at a point x. By (34),

$$\int_{\mathcal{C}} f(z) \cdot dz = \int_{\mathcal{C}} (\nabla\Phi(z) + \psi(z)e) \cdot dz = \Phi(x) - \Phi(y), \tag{35}$$

where $e = (1,1, \dots, 1)$. (As $e \cdot z = 1$ for all $z \in X$, $e \cdot dz = 0$.) This equation shows that the line integral of f is path independent and also that, for any fixed starting point $y \in X$, the potential can be presented as

$$\Phi(x) = \int_{\mathcal{C}} f(z) \cdot dz + \text{constant}, \qquad x \in X.$$

Eq. (35) is nothing but Eq. (22), adapted to the present context. This follows from (32), which gives that the sum in (22) is equal to

$$\sum_{l=1}^{L} f(y(t_{l-1})) \cdot \big(y(t_l) - y(t_{l-1})\big),$$

and so the limit in (22) is, by definition, the line integral presented in (35).

The significance of the last finding is that, at shown in Section 5, it means that, if C is an improvement curve representing the evolution of the population strategy as individual population members unilaterally change their choice of strategies and increase their payoff as a result, then the aggregate of these individual gains is given by the line integral in (35). By either that equation or (22), the aggregate gain is also equal to the potential difference between the endpoints of C.

The last conclusion has much similarity with Lemma 4.1 in Sandholm (2001), which concerns evolutionary dynamics, that is, a vector field $V: X \longrightarrow \mathbb{R}^n$ defining an equation of motion $\dot{x} = V(x)$ (with the dot indicating time derivative). However, a fundamental difference between the present setting and Sandholm's is that, here, there is no assumption about *who* are the individuals changing their strategy choices at each moment (among those who can gain from doing so), and therefore an improvement curve may turn in different directions. This is a direct analog of an improvement path in an N-player game, which may go in different directions depending on the order of moves. Evolutionary dynamics, by contrast, assumes that a population strategy y may change in only one direction, that specified by $V(y)$. The conformance with the individuals' incentives is achieved by what Sandholm calls the *positive correlation* assumption:

$$V(x) \cdot f(x) > 0 \text{ whenever } V(x) \neq 0.$$

The assumption implies that a potential Φ is a Lyapunov function for the evolutionary dynamics. That is, it increases along the solution trajectories of $\dot{x} = V(x)$. These trajectories are the specific improvement curves picked up by V.

It should be noted that Sandholm's notion of potential is somewhat more restricted than in this paper. A more detailed examination of his models in light of the present setting follows.

Example 4 (Sandholm 2015) The unit simplex X in $\mathbb{R}^n$ represents all distributions over a set of n possible actions. Each user in a continuum of total mass 1 has to choose an action. The proportion y_j of users choosing each action j defines the population strategy (or population state) $y = (y_1, y_2, \dots, y_n) \in X$. The payoff from choosing action j is given by $f_j(y)$, where $f_j: X \longrightarrow \mathbb{R}$ is a continuous function. A mixed strategy $x \in X$ therefore yields the payoff $g(x, y)$ given in (32). The population strategy y is said to be a Nash equilibrium if every action used by a non-zero fraction of the users is payoff-maximizing. This condition is easily seen to be equivalent to (1).

Suppose that the function $f = (f_1, f_2, \dots, f_n): X \longrightarrow \mathbb{R}^n$ has a continuous extension (which is also denoted by f) to the entire nonnegative orthant $\mathbb{R}^n_+$. Sandholm (2015) calls a continuously differentiable function $\Phi: \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ a potential function if $f = \nabla\Phi$, or more explicitly

$$f_j(y) = \frac{\partial \Phi}{\partial y_j}(y), \qquad y \in \mathbb{R}^n_+, j = 1,2, \dots, n. \tag{36}$$

This definition corresponds to the special case of (34) where $\psi = 0$. It is similar to the meaning of potential in mathematical physics, where f represents a vector field, a force field for example. The existence of a potential function means that f is a *conservative* vector field. If f is continuously differentiable, then a necessary and sufficient condition for this is

$$\frac{\partial f_j}{\partial y_k} = \frac{\partial f_k}{\partial y_j}, \qquad j, k = 1,2, \dots, n. \tag{37}$$

By Proposition 9, condition (36) is sufficient for Φ to be a potential for the symmetric population game $g: X^2 \to \mathbb{R}$ in the sense of Definition 2. However, it is not a necessary condition, and is in fact unnecessarily restrictive as it only picks up potentials Φ for which ψ in (34) is identically zero. A symmetric population game may well have a potential even if f is not a conservative vector field.[4]

Consider, for example, a symmetric $n \times n$ game $g: X^2 \to \mathbb{R}$, viewed as a population game. With the notation in Section 6.3,

$$f(y) = Ay, \qquad y \in X.$$

Condition (37) therefore means that the payoff matrix A is symmetric, equivalently, g is a symmetric function. As indicated in Section 3.2, this represents the very special case of a *doubly* symmetric game: a common-interest game. A potential may exist also in $n \times n$ games where A is not symmetric. Indeed, for $n = 2$ (Example 3), it exists for *all* A.

Example 5 (Sandholm 2001) In an asymmetric version of Example 4, a unit-mass continuum of users is divided into a finite number r of classes of identical users. Class k $(= 1,2, \dots, r)$ has total mass $m^k > 0$. Each of its members needs to choose one of a finite number of actions, numbered from 1 to n^k. The total number of actions is therefore $n = \sum_{k=1}^{r} n^k$. A population strategy y specifies the mass of users choosing each action. It is thus an n-tuple where the first n^1 coordinates $y_{11}, y_{12}, \dots, y_{1n^1}$ refer to class 1 (and therefore constitute a vector in the unit simplex in $\mathbb{R}^{n^1}$ multiplied by m^1), the next n^2 coordinates refer to class 2, and so on. The set $X \subseteq \mathbb{R}_+^n$ of all possible population strategies is therefore the product of r simplices, each of which is a scaled unit simplex in a Euclidean space.

The payoff for members of class k $(= 1,2, \dots, r)$ from choosing action j $(= 1,2, \dots, n^k)$ is $f_{kj}(y)$, where $f_{kj}: X \longrightarrow \mathbb{R}$ is a continuous function. The population strategy y is said to be a Nash equilibrium if, in each class, every action used by a non-zero fraction of users is payoff-maximizing. This condition is easily seen to be equivalent to y being an equilibrium (in the sense of Definition 1) in the asymmetric population game $g: X^2 \longrightarrow \mathbb{R}$ defined by (32). The interpretation of $g(x, y)$ is that it gives the mean payoff to individual users (of all classes) who unilaterally deviate to play the mixed strategy specified by x for their class (with frequencies interpreted as probabilities) when the actual population strategy is y.

Suppose that the function $f = (f_{11}, \dots, f_{1n^1}, f_{21}, \dots, f_{2n^2}, \dots, f_{rn^r}): X \longrightarrow \mathbb{R}^n$ has a continuous extension, which is also denoted by f, to the entire nonnegative orthant $\mathbb{R}_+^n$. Sandholm (2001) calls a continuously differentiable function $\Phi: \mathbb{R}_+^n \longrightarrow \mathbb{R}$ a potential function if $f = \nabla \Phi$, or more explicitly

$$f_{kj}(y) = \frac{\partial \Phi}{\partial y_{kj}}(y), \qquad y \in \mathbb{R}_+^n,\ k = 1,2, \dots, r,\ j = 1,2, \dots, n^k. \tag{38}$$

Thus, as in Example 4, the existence of a potential function means that f is a conservative vector field. However, again, while (38) is sufficient for Φ to be a potential also in the sense of Definition 2, it is not necessary. The population game g defined above may have a potential even if f is not a conservative vector field. This fact follows from a straightforward

[4] As shown above, a necessary condition for the existence of a potential is that the line integral of f in X is path independent. The condition that f is a conservative vector field is equivalent to the stronger condition of path independence in the entire positive orthant $\mathbb{R}_+^n$.

extension of Proposition 9. The extended result states that the restriction of a continuously differentiable function $\Phi\colon \mathbb{R}_+^n \longrightarrow \mathbb{R}$ to a set $X \subseteq \mathbb{R}_+^n$ as above is a potential for a population game $g\colon X^2 \longrightarrow \mathbb{R}$ of the form (32) if and only if it satisfies a condition similar to (34) in which the index j is replaced with the double index kj and $\psi(y)$ is replaced with a class-specific $\psi_k(y)$. Condition (38) only picks up the case where all the ψ_k's are identically zero.

6.5 Nonatomic congestion games

An important class of population games is nonatomic congestion games, which model the negative externalities of resource use with a continuum of users. Such games can be symmetric (e.g., Milchtaich 2012, 2021), with all the users identical, or asymmetric (e.g., Milchtaich 2000, 2004), with different users potentially facing different choices or receiving different payoffs from making identical choices. The following two examples present one, specific model of each kind.

Example 6 *Symmetric nonatomic congestion game.* An infinite population I of identical users share a finite number n of common resources (for example, road segments). Each user $i \in I$ has to choose a subset of resources (for example, a route, comprising several road segments), which can be expressed as a binary vector $\sigma(i) = (\sigma_1(i), \sigma_2(i), \dots, \sigma_n(i))$, with $\sigma_j(i) = 1$ or 0 indicating that resource j is included or is not included, respectively, in i's choice. The vector must belong to a specified (finite) collection $S \subseteq \{0,1\}^n$, which describes the allowable subsets of resources (for example, all routes from town A to town B). A *pure strategy profile* is a mapping $\sigma\colon I \longrightarrow S$ such that for each resource j the set of all i with $\sigma_j(i) = 1$ is measurable in the sense that it belongs to a specified σ-algebra $\mathcal{I}$ of subsets of I. The *load* on each resource j is then defined as the measure of its set of users with respect to a specified nonatomic probability measure on $(I, \mathcal{I})$, the *population measure* μ. The load y_j can therefore be written as

$$y_j = \int \sigma_j(i)\, d\mu(i).$$

The load vector $y = (y_1, y_2, \dots, y_n) = \int \sigma(i)\, d\mu(i)$ is also the population's mean strategy. This vector, the *population strategy*, lies in the convex hull of S, a compact subset of $[0,1]^n$ denoted X.

The cost of using each resource j (for example, the time is takes to traverse a road) depends on the load y_j and is given by $c_j(y_j)$, where $c_j\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a continuous and strictly increasing *cost function*. The total cost for user i is the sum of the costs of the resources included in i's choice, $\sum_{j=1}^n \sigma_j(i) c_j(y_j)$. The user's payoff is the negative of the cost. A natural, linear extension of the payoff to mixed strategies[5] $x = (x_1, x_2, \dots, x_n) \in X$ is given by

$$g(x, y) := -\sum_{j=1}^{n} x_j c_j(y_j) = -x \cdot c(y),$$

where $c(y) = \big(c_1(y_1), c_2(y_2), \dots, c_n(y_n)\big)$. This formula defines a population game $g\colon X^2 \longrightarrow \mathbb{R}$.

[5] Note that such mixtures, which are weighted averages of the elements of S, are generally not probability vectors.

The function $\Phi: X \longrightarrow \mathbb{R}$ defined by

$$\Phi(x) = -\sum_{j=1}^{n} \int_0^{x_j} c_j(t)\, dt$$

is a potential for g. This fact follows from Theorem 2, as the expression defining Φ is actually meaningful for all $x \in \mathbb{R}_+^n$ and it is not difficult to check that (17) holds, with $\psi = 0$. Since the cost functions are continuous and strictly increasing, the function Φ is continuous and strictly concave. Therefore, by Theorem 3 and the remarks following it, the population game g has exactly one equilibrium y, which is the unique maximum point of Φ in X.

The interpretation of the equilibrium condition is as follows. Since the population measure μ is nonatomic, there exists a pure strategy profile σ having the equilibrium y as its load vector. By the equilibrium condition (1), expressed in terms of costs,

$$y \cdot c(y) = \min_{x \in X} \big(x \cdot c(y)\big) \leq \sigma(i) \cdot c(y)$$

for all $i \in I$. However, the expression on the left-hand side is the integral with respect to μ of that on the right-hand side, which implies that the (weak) inequality must hold as equality for (μ-) almost all users i. Thus, in σ, almost everyone uses one of the least costly resources.

If a population strategy y is not an equilibrium, then for any pure strategy profile σ corresponding to y there is a profile σ' such that for all $i \in I$

$$\sigma'(i) \cdot c(y) \leq \sigma(i) \cdot c(y)$$

and the inequality is strict for all i in some subset of I of positive μ-measure. The mean gain for users from unilaterally shifting from playing according to σ to playing according to σ' is given by

$$-(y' - y) \cdot c(y),$$

where $y' = \int \sigma'(i)\, d\mu(i)$. The same expression gives the first-order approximation of the mean gain if the strategy changes are not unilateral but rather all users, in whatever order, actually change from playing according to σ to playing according to σ'. (While these strategy changes take place, the argument of c changes; it does not remain y. However, the effect of these changes of the population strategy on the players' mean payoff is of a second order.) Therefore, if users keep on changing their strategies in this manner, with only a small fraction of them doing so during any short time interval, the mean total gain for users from their own changes of strategy is given by the line integral

$$-\int_{\mathcal{C}} c(z) \cdot dz,$$

where $\mathcal{C}$ is the improvement curve traced in X by the changing population strategy. (This assumes that the integral is well defined, which is the case if $\mathcal{C}$ is rectifiable.) By Theorem 4 (see also Section 6.4), the mean total gain is equal to the difference between the values of the potential Φ at the two endpoints of $\mathcal{C}$.

Example 7 *Asymmetric nonatomic congestion game.* A finite or infinite set I of *representative players* is endowed with a population measure μ, which is a probability measure defined on a σ-algebra $\mathcal{I}$ of subsets of I. Each $i \in I$ may be either a single user, as in Example 6, or represent a continuum of identical users, with the latter possibility necessarily holding if $\mu(\{i\}) > 0$. There are n resources, of which exactly one has to be chosen by each user. A *mixed strategy profile* is a measurable mapping $x: I \longrightarrow \Delta$, where Δ is the unit simplex in $\mathbb{R}^n$. For each $i \in I$, the mixed strategy $x(i) = (x_1(i), x_2(i), \dots, x_n(i))$ is a probability vector, which may represent two things. If i is a single user, then $x(i)$ expresses the probability with which the user chooses each of the n resources. If i represents a continuum of users, then $x(i)$ gives the frequencies of the different resources in these users' choices.

For a mixed strategy profile y, the cost of resource j is $c_j(\mu(y_j))$, where $c_j: \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a continuous and strictly increasing cost function and $\mu(y_j)$ is shorthand for the integral $\int_I y_j(i)\, d\mu(i)$, which is the load on j. Representative player i gets from using j also a constant benefit (or cost, if negative) of $f_j(i)$, where $f_j: I \longrightarrow \mathbb{R}$ is an integrable function, that is, an element of $L^1(I, \mathcal{I}, \mu)$. The payoff for i from using resource j is therefore $f_j(i) - c_j(\mu(y_j))$, and the payoff from using a mixed strategy $x(i) = (x_1(i), x_2(i), \dots, x_n(i))$ is

$$\sum_{j=1}^{n} x_j(i) \left(f_j(i) - c_j(\mu(y_j)) \right).$$

The formula

$$g(x, y) \coloneqq \int_I \sum_{j=1}^{n} x_j(i) \left(f_j(i) - c_j(\mu(y_j)) \right) d\mu(i) \ = \sum_{j=1}^{n} \left(\mu(x_j f_j) - \mu(x_j)\, c_j(\mu(y_j)) \right)$$

gives the mean payoff for individual users who unilaterally deviate and choose according to x when everyone else chooses according to y. (Note that these are not the representative players who deviate but the individual users they represent.) This formula defines a population game $g: X^2 \longrightarrow \mathbb{R}$, where X is the space of all mixed strategy profiles. X is obviously a convex set, and it follows easily from Alaoglu's theorem that it is also a compact set if viewed as a subset of $(L^\infty(I, \mathcal{I}, \mu))^n$ with L^∞ endowed with the weak* topology (as the dual space of L^1).

The function $\Phi: X \longrightarrow \mathbb{R}$ defined by

$$\Phi(x) = \sum_{j=1}^{n} \left(\mu(x_j f_j) - \int_0^{\mu(x_j)} c_j(t)\, dt \right)$$

is a potential for g (Milchtaich 2004). As in Example 6, this fact follows easily from Theorem 2, using the linearity of integration with respect to μ. Since the cost functions are continuous and increasing, the potential Φ is easily seen to be continuous and concave. Therefore, by Theorem 3 and the remark following it, the set of equilibria in g is the (nonempty) set of global maximum points of Φ.

The meaning of the equilibrium condition (1) for a mixed strategy profile y is that for (μ-) almost all representative players i

$$\sum_{j=1}^{n} y_j(i) \left(f_j(i) - c_j(\mu(y_j)) \right) = \max_j \left(f_j(i) - c_j(\mu(y_j)) \right).$$

The equality means that the mixed strategy $y(i)$ assigns positive probability only to payoff-maximizing resources.

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