Auctions With a Random Number of Identical Bidders

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Abstract

We examine and compare the (normally, mixed) symmetric equilibrium bidding strategies in first-price and all-pay common value multiple item auctions with a random number of bidders, who only seek one of the identical items and have the same budget.

Keywords: Auctions, stochastic number of bidders, random-player games, identical bidders, symmetric equilibrium

JEL Classification: D4

1 Introduction

A number $m \geq 1$ of identical items are sold in an auction with an uncertain number of identical bidders, who only seek a single item, which they value at 1, and have a budget of $0 < w \leq 1$. The bidders do not know the actual number of other bidders, and share the same beliefs about that number, which are expressed by the probabilities $p_0, p_1, p_2 ...$ that the number is $0, 1, 2, ...$ The auction is either a first-price or an all-pay sealed-bid auction, with the m highest bids winning and ties broken randomly. A bidder places a bid b in $[0, w]$. His payoff is $1 - b$ if he obtains an item, $-b$ if he does not obtain it but nevertheless pays his bid, and 0 otherwise.

Bidders can use mixed strategies, which are random bids. A random bid can be expressed by its (nondecreasing and right-continuous) *distribution function* F, such that, for $0 \leq b \leq 1$, the probability that the bid does not exceed b is $F(b)$. An alternative presentation is given by the nondecreasing *bidding strategy* $B : [0, 1] \to [0, 1]$, such that the random bid is $B(X)$, where X is a uniformly distributed random variable in the unit interval, which may be thought of as the bidder's randomly generated "type". With the additional technical assumptions of right continuity and continuity at 1, the bidding strategy is uniquely determined by the distribution function, which is essentially its inverse. More precisely,

$$F(B(x)) \geq x, \qquad 0 \leq x \leq 1,$$

with equality if B is increasing at x, equivalently, if the probability of the bid $B(x)$ is 0. (Strict inequality may hold only if the distribution has an atom at $B(x)$.)

We look for a symmetric Nash equilibrium strategy, that is, a strategy that, when used by all participants, is also a best response for each of them. If $p_k = 0$ for all $k \geq m$ or for all $k < m$, respectively, then bidding 0 is a dominant strategy or everybody bidding w is an equilibrium. To exclude these uninteresting equilibria, we assume that $0 < \sum_{k=0}^{m-1} p_k < 1$. As we show, if the budget w is sufficiently high, then for both first-price and all-pay auctions this sum is equal to the equilibrium expected payoff of each bidder.

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2 Equilibrium analysis

In our model, the number of bidders is exogenous; it does not reflect individual strategic decisions. However, participation is costless, since the minimum bid is 0. Also, unlike other models of auctions with a stochastic number of bidders (e.g., [1],[2],[3],[4]), in which the participants' (ex post) valuations may differ, in our model they are identical. Nevertheless, formal similarity is maintained by presenting the symmetric equilibrium strategy $B(x)$ as nondecreasing in the bidder's (payoff-irrelevant) "type" x. However, a non-constant strategy must yield the same expected payoff c to all prescribed bids, which means that, in the first-price or all-pay auction, respectively, for all $0 \leq x \leq 1$

$$P(x)\,(1 - B(x)) = c \tag{1}$$

or

$$P(x) - B(x) = c, \tag{2}$$

where $P(x)$ is the probability that a bid of $B(x)$ results in winning an item.

If B is (strictly) increasing, a bidder does *not* win if there are at least m higher bids and any number j of lower bids. Therefore,

$$P(x) = 1 - \sum_{j \geq 0} \sum_{k \geq m} \binom{j+k}{j} p_{j+k} x^j \,(1-x)^k \,, \qquad 0 \leq x \leq 1. \tag{3}$$

A bid of $B(0)$ has the same winning probability as a bid of 0, and cannot yield a lower gain. Therefore, necessarily $B(0) = 0$. Hence, both (1) and (2) imply that

$$c = P(0) = \sum_{k=0}^{m-1} p_k. \tag{4}$$

Since $P(1) = 1$, it follows that the support of the mixed strategy is the interval

$$[B(0), B(1)] = [0, 1-c]. \tag{5}$$

If $w < 1 - c$, (5) is inconsistent with the budget constraint, which implies that the bidding strategy cannot in fact be increasing. Indeed, there must be an atom at w, so that the maximum bid has a positive probability q_w. If w is small enough, then $q_w = 1$, so that the equilibrium is pure with everyone bidding w. For larger w, lower bids also occur, and lie in an interval $[0, v]$, for some v strictly smaller than w. As shown below, the first-price and the all-pay auctions differ in the value of v and the probabilities of lower bids.

The smallest maximizer of the bidding strategy is $\overline{x} := 1 - q_w$. If $\overline{x} < 1$, so that ties at the top are possible, (3) underestimates the corresponding winning probability, which is instead obtained by substituting $\overline{x}$ for x in

$$\begin{aligned} \tilde{P}(x) &= 1 - \sum_{j \geq 0} \sum_{k \geq m} \left(1 - \frac{m}{k+1}\right) \binom{j+k}{j} p_{j+k} x^j \,(1-x)^k \\ &= 1 - \sum_{k \geq m} p_k \sum_{j=0}^{k-m} \left(1 - \frac{m}{k-j+1}\right) \binom{k}{j} x^j \,(1-x)^{k-j} \,. \end{aligned} \tag{6}$$

The function $\tilde{P}$ is continuous and strictly increasing. Therefore,

$$\tilde{c} := \min_x \tilde{P}(x) = \tilde{P}(0) = c + \sum_{k=m}^{\infty} \frac{m}{k+1} p_k > c.$$

Suppose that $w < 1 - c$, and hence $B(\overline{x}) = w$. The expected payoff for $\overline{x}$ in the first-price or all-pay auction is $\tilde{P}(\overline{x})\,(1-w)$ or $\tilde{P}(\overline{x}) - w$, respectively. If $\tilde{c}\,(1-w)$ or $\tilde{c} - w$, respectively, is greater than c, then it is the symmetric equilibrium payoff, and the equilibrium strategy is to bid w with probability 1, that is, $\overline{x} = 0$. If the specified inequality does not hold, then the symmetric equilibrium payoff is c, and for the first-price or all-pay auction, respectively, $\overline{x}$ is uniquely specified by

$$\tilde{P}(\overline{x})\,(1-w) = c \tag{7}$$

or

$$\tilde{P}(\overline{x}) - w = c. \tag{8}$$

Lowering the bid even a little bit from w makes the winning probability jump from $\tilde{P}(\overline{x})$ down to $P(\overline{x})$ or lower. To compensate for this discontinuous decrease in the winning probability, the bid must be sufficiently lower than w. Hence, all equilibrium bids lower than w lie in some interval $[0, v]$, with $0 < v < w$. For these bids, equivalently, for $0 \le x < \overline{x}$, (1) or (2) must hold for the first-price or all-pay auction, respectively, which together with (3) gives the equilibrium bidding strategy $B(x)$ as well as the explicit expression

$$v = 1 - \frac{c}{P(\overline{x})} \tag{9}$$

or

$$v = P(\overline{x}) - c. \tag{10}$$

These considerations lead to the following.

Theorem 1 *In both a first-price and an all-pay auction, the symmetric equilibrium strategy is unique and yields the expected payoff $\tilde{c}\,(1-w)$ or $\tilde{c}-w$, respectively, if this expression is greater than c (low budget), and c otherwise (high budget). In the low-budget case, the probability of bidding w is 1, and in the high-budget case, it is $1-\overline{x}$, where $\overline{x}$ is 1 if $w \ge 1-c$ (non-binding budget constraint) and it is implicitly given by (7) or (8) otherwise. All lower bids fall in an interval $[0, v]$, where v is given by (9) for the first-price auction or (10) for the all-pay auction, and the corresponding distribution function is*

$$F_f(b) = P^{-1}\left(\frac{c}{1-b}\right), \qquad 0 \le b \le v \tag{11}$$

or

$$F_a(b) = P^{-1}(b + c), \qquad 0 \le b \le v. \tag{12}$$

The result that, in the high-budget case, the two kinds of auctions yield the same equilibrium expected payoff implies that, for any given winning probability, the bid when both winners and losers pay must be lower. Thus, the equilibrium bid distribution in the first-price auction first-order stochastically dominate that in the all-pay auction. Indeed, by (11) and (12),

$$F_f(b) = F_a(\frac{bc}{1-b}) \tag{13}$$

for all b as in (11), and $b > \frac{bc}{1-b}$ in the interior.

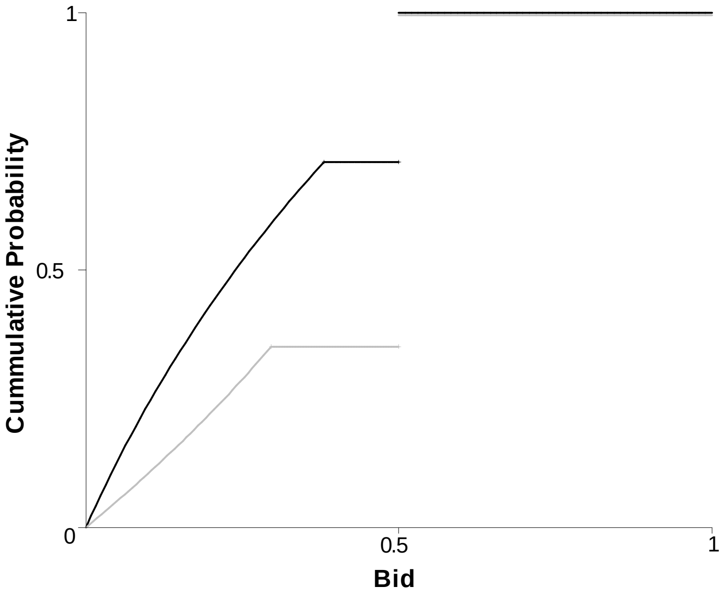

Figure 1: The (cumulative) distribution function of the symmetric equilibrium strategy for the first-price auction (gray line) and the all-pay auction (black line) with a single object and a number of bidders that has a Poisson distribution with expectation $\lambda = 1$. The budget is $w = 0.5$.

3 Special cases

The participants' beliefs about the number of their competitors are a primitive in our model; they are given rather than derived. One way to derive them would be to assume a *common prior*, which is a description of the participants' selection process as seen from the outside. The corresponding participants' beliefs are their *posterior beliefs*, which combine the prior with the information provided by the fact of own participation. Our assumption that bidders have identical beliefs entails a simple connection between the prior and the posterior, namely, $p_{k-1} = \frac{k}{\bar{n}}\pi_k$ ($k = 1, 2, ...$), where π_k is the prior probability that there are k bidders, $\bar{n}$ is the expected number of bidders and p_{k-1} is the posterior probability that there are $k - 1$ *other* bidders ([4],[5]). In other words, compared with the prior, the bidders' beliefs are length biased.

The above relation between the prior and the posterior preserves the two families of distributions considered below; it may only change the parameters of the distribution. For a Poisson distribution, the posterior moreover coincides with the prior, and it is not difficult to see that this is the only kind of distribution for which the coincidence holds. For binomial distributions $B(n, p)$ (and negative binomial distributions $NB(r, p)$), the posteriors are (negative) binomial distributions with n lower (respectively, r higher) by 1 than that of the prior.

3.1 Poisson distribution

The Poisson distribution models the limiting case of a large number of potential bidders, whose actual participation in the auction has a small probability and is independent of the others' participation. With the expected number of actual bidders denoted by λ, $p_k = e^{-\lambda}\frac{\lambda^k}{k!}$, $k =$

$0, 1, 2, \ldots$. Therefore,

$$P(x) = 1 - e^{-\lambda} \sum_{j \geq 0} \frac{\lambda^j x^j}{j!} \sum_{k \geq m} \frac{\lambda^k (1-x)^k}{k!} = e^{-\lambda(1-x)} \sum_{k=0}^{m-1} \frac{(\lambda(1-x))^k}{k!}, \qquad 0 \leq x \leq 1.$$

In the special case of a single object, $m = 1$,

$$P(x) = e^{-\lambda(1-x)},$$

so that $c = e^{-\lambda}$ by (4). Therefore, by Theorem 1, with a non-binding budget constraint ($w \geq 1 - e^{-\lambda}$) the unique symmetric equilibrium distribution in the first-price or all-pay auction, respectively, has the monotonically increasing density

$$f_f(b) = F_f'(b) = \frac{1}{\lambda\,(1-b)}, \qquad 0 \leq b \leq 1 - e^{-\lambda} \tag{14}$$

or the monotonically decreasing density (which is the mirror image of (14))

$$f_a(b) = F_a'(b) = \frac{1}{\lambda\,(b + \, e^{-\lambda})}, \qquad 0 \leq b \leq 1 - e^{-\lambda}. \tag{15}$$

For the complementary case of a binding budget constraint ($w < 1 - e^{-\lambda}$), (6) gives

$$\begin{aligned}\tilde{P}(x) &= 1 - e^{-\lambda} \sum_{j \geq 0} \frac{\lambda^j x^j}{j!} \sum_{k \geq m} \left(1 - \frac{m}{k+1}\right) \frac{\lambda^k (1-x)^k}{k!} \\ &= \frac{m}{y}\left(1 - e^{-y} \frac{y^{m-1}}{(m-1)!}\right) + \left(1 - \frac{m}{y}\right) e^{-y} \sum_{k=0}^{m-2} \frac{y^k}{k!},\end{aligned}$$

where $y = \lambda\,(1-x)$. With a single object, this simplifies to $\tilde{P}(x) = \frac{1-e^{-y}}{y}$. Therefore, by (7) and (8), in the first-price or all-pay auction, respectively, $\overline{x}$ is a solution of

$$\frac{e^{\lambda} - e^{\lambda x}}{\lambda - \lambda x} = \frac{1}{1-w}$$

or

$$\frac{e^{\lambda} - e^{\lambda x}}{\lambda - \lambda x} = 1 + w e^{\lambda}.$$

Each equation has at most one solution $\overline{x}$ in $[0, 1)$, which gives the probability $1 - \overline{x}$ that a participant bids w. (If there is no solution in $[0, 1)$, the probability is 1.) All lower bids are between 0 and $1 - e^{-\lambda \overline{x}}$ in the first-price auction and between 0 and $e^{-\lambda}\left(e^{\lambda \overline{x}} - 1\right)$ in the all-pay auction, in which range the density is as in (14) or (15), respectively. Figure 1 shows an example of the corresponding distribution functions.

3.2 Binomial distribution

For the binomial distribution, $p_k = \binom{n}{k} p^k q^{n-k}$, $k = 0, 1, ..., n$, for some integer $n \geq m$ and $0 < p, q < 1$ with $q = 1 - p$. Therefore,

$$\begin{aligned} P(x) &= 1 - \sum_{k=m}^{n} \sum_{j=0}^{n-k} \binom{j+k}{k} \binom{n}{j+k} p^{j+k} q^{n-j-k} x^j (1-x)^k \\ &= 1 - \sum_{k=m}^{n} \binom{n}{k} (p(1-x))^k \sum_{j=0}^{n-k} \binom{n-k}{j} (px)^j q^{n-j-k} \\ &= \sum_{k=0}^{m-1} \binom{n}{k} (p(1-x))^k (px+q)^{n-k} \\ &= I_{px+q}(n-m+1, m), \end{aligned}$$

where $I_x(a, b)$ is the regularized incomplete beta function:

$$I_x(a,b) = \frac{(a+b-1)!}{(a-1)!(b-1)!} \int_0^x t^{a-1} (1-t)^{b-1} dt.$$

In other words, $P(x)$ is the probability that n independent trials, each with probability of success equal to $(1-x)p$, result in less than m successes.

In the special case of a single object, $m = 1$,

$$P(x) = (px+q)^n$$

and $c = q^n$. Hence, by (1), with a non-binding budget constraint ($w \geq 1 - q^n$) the equilibrium bidding strategy in the first-price auction is

$$B(x) = 1 - \left(1 + \frac{p}{q} x\right)^{-n}, \qquad 0 \leq x \leq 1$$

and the distribution function is

$$F_f(b) = \frac{q}{p} \left((1-b)^{-\frac{1}{n}} - 1 \right), \qquad 0 \leq b \leq 1 - q^n.$$

The distribution function in the corresponding all-pay case can be computed using (13), which gives

$$F_a(b) = \frac{1}{p} \left((b+q^n)^{\frac{1}{n}} - q \right), \qquad 0 \leq b \leq 1 - q^n.$$

To study the case of a binding budget constraint, assume further that $n = m = 1$, so that the number of other bidders is either one or zero with probabilities p and q, respectively. Hence, $c = q$, $\tilde{c} = 1 - \frac{p}{2}$ and

$$\tilde{P}(x) = 1 - \frac{p}{2} (1-x), \qquad 0 \leq x \leq 1.$$

By (7) and (8), with a budget $\frac{p}{1+q} < w < p$ the (positive) probability $\overline{x}$ of bidding less than w in the first-price or all-pay auction, respectively, is

$$\overline{x} = 1 - \frac{2}{p} \left(1 - \frac{q}{1-w} \right)$$

or

$$\overline{x} = \frac{2w}{p} - 1.$$

The largest such bid v is given, respectively, by

$$v = 1 - \frac{1}{\frac{2}{1-w} - \frac{1}{q}}$$

or

$$v = 2w - p.$$

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