Poisson Price Dispersion*

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Abstract

We study a competitive market for a homogeneous good, in which the only uncertainty concerns the number of identical sellers, who are sampled by a finite Poisson process from a continuum of potential participants. It is shown that, in equilibrium, there is price dispersion. Specifically, prices conform to a Poisson process on an interval, which is a proper subset of that between the sellers' cost and the buyers' reservation price. Although prices arbitrarily close to the latter may occur in equilibrium, they are less frequent than prices at the lower end of the pricing interval. *JEL Classification:* C7, D4, D8, L1.

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1 Introduction

The phenomenon of price dispersion was first analyzed theoretically by Stigler [29], who argued that "... it is important to emphasize immediately the fact that dispersion is ubiquitous even for homogeneous goods" (p. 213). Since Stigler's seminal paper, which introduced search as the agent's instrument for coping with "ignorance in the market" (in his terminology), numerous economic models that can account for price dispersion have been suggested.

In the model we present here, supply or demand uncertainties for a homogeneous good create price dispersion. Uncertainty stems from the assumption that

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agents decide on their strategies before they know if they will be actively participating in the market, and the number of competitors they will face if they will be active. It is only after strategies are chosen that uncertainty is resolved by a finite Poisson process, which samples from the continuum of potential participants. If the uncertainty is on the supply side (i.e., sellers, who post prices), two conflicting forces operate. Sellers aim to raise their posted prices to the buyers' reservation price, but this may jeopardize their chances of selling, since only the lowest offers are matched. Similarly, if the uncertainty is on the demand side, each buyer would preferably make the lowest bid possible, however, this may lower the probability that the offer would be sufficient to obtain the object.

The equilibrium exhibits price dispersion, which is a (nonhomogeneous) Poisson process in prices. If the uncertainty is on the supply side, there is a minimum price, strictly higher than the sellers' cost, above which any price not exceeding the reservation price is possible. Furthermore, the price process exhibits decreasing density.

The model reflects certain aspects of narrowly defined segments of the housing and labor markets. For example, in a housing market, a substantial portion of the supply is determined by random, exogenous factors relating to the participants (a seller has to leave town, a divorced couple has to sell their house, etc.). Suppose all sellers have to post their asking prices in a weekend newspaper before they know how many other houses are for sale. They would sell as long as there is a buyer willing to pay the posted price who cannot find a better deal. Similarly, consider the job market for junior faculty in a narrowly defined field in economics. The number of candidates (of comparable quality) is generally well known (since most of them apply to all the departments), but the number of departments actively recruiting in the specific field is uncertain. As a result of the uncertainty, there may be price dispersion even with uniform quality.

The main contributions of this paper to the extensive literature on price dispersion is the introduction of finite point processes as a model for the selection of market participants and the complete characterization of the equilibrium price process. This paper demonstrates how the framework of random-player games, in general, and Poisson games, in particular, may be applied to the analysis of uncertain economic environments.

2 Price Dispersion Literature

Prescott [24] was the first to consider a model of stochastic demand faced by a fixed number of sellers of a homogeneous good. He showed that the equilibrium pricing strategy exhibits price dispersion. Prescott's goal was to show that the fact that some goods are not exchanged in the market does not imply inefficiency.[1] Bryant [3] considered stochastic demand in his study of the (partial) competitive equilibrium with price-setting firms. He showed that the existence of potential entrants determines the equilibrium pricing strategies, which in the case of stochastic demand results in price dispersion. Eden [10] showed that if agents and firms are allowed to trade based on the probability of sale in a spot market, this leads to marginal cost pricing. Dana [9] introduced costly capacity and allowed the firm to choose a set of prices (and quantity limit on each price). He proved that there is pure strategy equilibrium in price distributions, even without pure strategy equilibrium in prices (as in Bryant's [3] model). Furthermore, price dispersion increases as competition becomes more intense. Although the above literature and the current paper share some basic insights, the main difference is that while in the stochastic demand literature the strategic portion of the market is deterministic, in this paper the uncertainty is coupled with the strategic component of the market. That is, the number of sellers posting prices is stochastic, while the demand is deterministic.

Janssen and Rasmusen [13] studied a Bertrand competition with an unknown number of sellers, in which each potential seller is active with a given probability. They showed that in equilibrium there is price dispersion and characterized it. Janssen and Rasmusen [13] studied the pricing strategy of firms at the interim stage: after they are chosen to be "active" but before they know how many other firms are active. Although the aim is similar, the focus of their paper is different. This paper describes an environment in which the ex-ante and interim decisions of firms are identical, and studies the implications on the equilibrium price process.

Peters [20] studied Bertrand competition among firms with capacity constraints.[2] After observing prices, each buyer has to choose a firm (as in the recent directed-search literature). If the demand of the agents approaching a specific firm exceeds its capacity, the firm allocates its capacity among the buyers. In this framework, there is a trade-off between the price and the probability of being allocated the desired demand. Peters showed that there is price dispersion in equilibrium (i.e., a mixed equilibrium in prices). Using similar logic, Montgomery [18] showed that wage dispersion may be the equilibrium outcome in a directed-search model of the labor market, in which heterogeneous firms post vacancies and wage offers that are known to the workers. A high offer may attract many workers, reducing the applicants' probability of getting the job. Coles and Eeckhout [6] showed that the restriction of the directed search approach—whereby sellers are committed to the posted prices—is an equilibrium of a game that allows each seller to specify a mechanism for allocating the good (if the posted price attracts more than a single

[1] A more elaborate model is presented in Peters and Winter [22].

[2] The exogenous capacity constraint assumption was relaxed in Peters [21].

buyer). However, a continuum of other equilibria also exists.

Another strand of models derive price dispersion in equilibrium when advertisements are takes into account. Firms may advertise at a central location (e.g. Shilony [28], Varian [30]), or target specific consumers (e.g. Butters [5]). Some consumers know the price distribution (through the advertisements), while others do not know it or have "brand loyalty". Price dispersion in these models allows firms to discriminate between informed and uninformed consumers. Baye and Morgan [1] introduced a middleman, who collects price data from sellers and sells the right to access this information to consumers. The search literature accounting for price dispersion (e.g. Burdett and Judd [4], Rob [27]) assumes basic heterogeneity in the market, either in the ex-ante availability of information (as in [4]) or the search cost (as in [27]).

3 The Poisson Process

This section briefly reviews finite Poisson processes on the real line. For a more extensive review, see Milchtaich [17].

A *finite point process* X on the real line $\mathbb{R}$ is a random selection of a finite multiset in $\mathbb{R}$. A multiset differs from an ordinary set in that it may contain multiple copies of the same point. If the points selected are almost surely distinct, the process is said to be *simple*. For each Borel set B in $\mathbb{R}$, the (random) number of points selected in B, each point counted with its multiplicity, is denoted by $X(B)$. The joint probability distributions of the numbers of points in all finite collections of disjoint intervals in $\mathbb{R}$ completely specify the process. A *finite Poisson process* is defined by the assumption that, for each Borel set B, $X(B)$ has a Poisson distribution. This implies that the process can be simple only if it has no *fixed atoms*, i.e., for every $x \in \mathbb{R}$, $X(\{x\}) = 0$ almost surely. It also implies that the process has *independent increments*: for every finite family of disjoint Borel sets $\{B_i\}_{i=1}^k$, the random variables $\{X(B_i)\}_{i=1}^k$ are independent. The importance of the Poisson process comes from the result that if a finite point process without fixed atoms has the latter property (also known as *complete independence*), then it is *necessarily* a simple finite Poisson process.

Theorem 1 (Prékopa [23]) *A finite point process without fixed atoms is a finite Poisson process if and only if it is simple and has independent increments.*

For every finite Poisson process X, there is a unique finite Borel measure Λ, called the *parameter measure* of the process, such that for every finite family of disjoint Borel sets $\{B_i\}_{i=1}^k$ and corresponding list of nonnegative integers $\{n_i\}_{i=1}^k$:

$$\Pr(X(B_i) = n_i,\ i = 1,2,\ldots,k) = e^{-\sum_{i=1}^k \Lambda(B_i)} \prod_{i=1}^k \frac{\Lambda(B_i)^{n_i}}{n_i!}. \tag{1}$$

Λ is also the *mean measure* of the process, i.e., it gives the expected number of points selected in each Borel set B, each point counted with its multiplicity. Equation (1) shows that this measure completely specifies the finite Poisson process (and vice versa). The parameter measure itself is completely specified by the *generalized distribution function* $F_\Lambda(x) := \Lambda((-\infty, x])$, defined on the real line. This function is nondecreasing and continuous from the right, and is continuous if and only if Λ has no atoms, which is the case if and only if the finite Poisson process is simple. The derivative $f(x) := dF_\Lambda(x)/dx$ (which exists for almost every x) is called the *density* of a simple Poisson process.

An alternative presentation of finite Poisson processes is sometimes useful. For any finite *partition* of $\mathbb{R}$ into Borel sets $\{B_i\}_{i=1}^k$, the sum $\lambda := \sum_{i=1}^k \Lambda(B_i) = \Lambda(\mathbb{R})$ gives the expected total number of points selected. For such a partition, (1) can be written as follows:

$$\Pr(X(B_i) = n_i,\ i = 1,2,\ldots,k) = \left(e^{-\lambda}\frac{\lambda^n}{n!}\right)\left(\frac{n!}{\prod_{i=1}^k n_i!}\prod_{i=1}^k\left(\frac{\Lambda(B_i)}{\lambda}\right)^{n_i}\right), \tag{2}$$

where $n = \sum_{i=1}^k n_i$. This shows that every finite Poisson process X can be implemented by first determining the total number of points n according to the Poisson distribution with parameter λ, and then sampling each of the n points independently of the others according to the probability measure $(1/\lambda)\Lambda$. These two stages correspond to the first and second expressions on the right-hand side. of (2), respectively.

This presentation of a finite Poisson process as a *mixed sample process* (where 'mixed' refers to the randomness of the sample size n) may be used to compute the *posterior* on the number and positions of the other points, given that a particular point was selected. Since different points are selected independently, information about positions cannot be extracted from this. The posterior on the number of points can be computed by noting that the process selects $n+1$ points with probability $e^{-\lambda}\lambda^{n+1}/(n+1)!$, and for each of them the number of other points is n. Therefore, the posterior probability that there are n other points is proportional to $(n+1)e^{-\lambda}\lambda^{n+1}/(n+1)! = \lambda e^{-\lambda}\lambda^n/n!$. Since this is proportional to the prior probability that the total number of points is n, the two probabilities must in fact be equal. This shows that the finite point process giving the posterior (on the number and positions of the *other* points) is equal to the prior (i.e., the original process, which specifies the number and positions of *all* the points). Indeed, this *environmental equivalence* property uniquely characterizes finite Poisson processes (Myerson [19, Theorem 2]; Milchtaich [17, Theorem 1]).

Theorem 2 *A finite point process with finite mean measure has the environmental equivalence property if and only if it is a finite Poisson process.*

4 The Model

In a simple market for homogeneous widgets, the demand is deterministic and consists of b (≥ 1) identical buyers. Each buyer has an inelastic demand for one widget and a positive reservation price, which, without loss of generality, is assumed to be unity. Therefore, the buyers' utility is $1-p$ if they purchase one widget at price p and zero otherwise.

The supply side consists of a continuum of identical risk-neutral potential sellers, indexed on the real line or some interval in it. Each seller can potentially sell one unit, produced at zero cost.[3] The potential sellers' asking prices are given by a Borel measurable *asking price function p*. For each seller s who is actually chosen to participate in the market, the asking price is $0 \leq p(s) \leq 1$.[4] The selection of the actual sellers is modeled as a simple finite Poisson process S on $\mathbb{R}$, with the parameter measure M. For each Borel set B, the number of actual sellers with indices in B is the random variable $S(B)$. The expected total number of sellers is:

$$\lambda := M(\mathbb{R}) \; (> 0). \tag{3}$$

The b actual sellers who quote the lowest prices (or, if there are less than b sellers, *all* the sellers) sell their widgets at the quoted prices. If more than one seller quotes the b lowest price, the matched sellers are chosen with equal probabilities.

5 The Price Process

The finite Poisson process S that determines the set of actual sellers, and the asking price function p, together determine the actual multiset of prices. In particular, for each Borel set P in the real line, the (random) number of sellers who quote a price in P is given by:

$$X(P) := S(p^{-1}(P)). \tag{4}$$

[3] This is a normalization assumption: the only real requirement is that the cost of production is less than the buyers' reservation price.

[4] The assumption that potential sellers choose selling prices before they know if they will actually participate in the market is not a real restriction. It only requires that, when the sellers choose prices, they do not know the number or the identities of the other actual sellers.

Thus the random multiset of prices is also a finite point process on the real line. As the following proposition shows, this *price process* X is, in fact, a finite Poisson process. The easiest way to see this is to consider the presentation described above of a finite Poisson process as a mixed sample process.

Proposition 1 (see Milchtaich [17, Section 2.3]) *The price process X is a finite Poisson process with parameter measure $\Lambda(\cdot) := M\left(p^{-1}(\cdot)\right)$. Thus it satisfies the environmental equivalence property and has independent increments.*

A seller's environment is the set of all other actual sellers and their asking prices. According to the environmental equivalence property, the environment of each seller s coincides with the prior. In other words, the mere presence in the market and the price quoted do not provide s any information on the number of other sellers or their asking prices. Moreover, the sellers' posterior about these coincides with the prior on the *total* number of sellers and their asking prices, which is the price process X itself.

Using environmental equivalence, the expected profit $\Pi(p)$ for a seller quoting price p can be computed. If $0 < p \leq 1$ and there are n other sellers quoting the same price and m sellers quoting prices less than p, then the seller's probability of selling (at price p) is $(b-m)/(n+1)$ if this expression is between 0 and 1, zero if the expression is negative, and one if it is greater than 1. Hence, by (1),

$$\Pi(p) = p \sum_{m,n=0}^{\infty} \min\left\{\max\left\{\frac{b-m}{n+1}, 0\right\}, 1\right\} e^{-\Lambda([0,p])} \frac{\Lambda([0,p))^m}{m!} \frac{\Lambda(\{p\})^n}{n!}. \quad (5)$$

Equilibrium is defined by the condition that only the prices giving the maximum expected profit can occur. This condition is easiest to express in terms of the *support* of the parameter measure Λ, which is the collection of all prices p such that, for all $\varepsilon > 0$, $\Lambda([p-\varepsilon, p+\varepsilon]) > 0$. However, equilibrium may also be considered a property of the asking price function $p(s)$. If $p(s)$ is a Nash equilibrium in the game in which the players are the potential sellers, then the parameter measure of the resulting price process is an equilibrium in the following sense.

Definition 1 *The parameter measure Λ is an equilibrium if, for every price p in the support of Λ and every other price q,*

$$\Pi(p) \geq \Pi(q). \quad (6)$$

This condition clearly implies that the expected profit is the same for all prices in the support of the equilibrium. This common value is the *equilibrium profit*.

6 Characterization of the Equilibrium

This section shows that the equilibrium can be completely characterized. The first result concerns the price dispersion.

Proposition 2 *An equilibrium has no atoms, and its support has the form* $[\alpha, 1]$*, for some* $0 < \alpha < 1$.

Proof. The support of any equilibrium Λ is a closed subset of $[0,1]$. Let α be the smallest element in this set. The proof consists of the following three claims.

Claim 1 $\alpha \geq e^{-\lambda} > 0$.

The expected profit of a seller with asking price 1 is at least $e^{-\lambda}$. This is because, by definition of the Poisson process, there is probability $e^{-\lambda}$ that there are no other sellers. A seller asking a price less than $e^{-\lambda}$ has a lower expected profit, and thus, by definition of equilibrium, such a price is not in the support of Λ.

Claim 2 Λ *has no atoms, and* $\Pi(p) = p \Pr(X([0,p]) < b)$ *for all* $0 \leq p \leq 1$.

Suppose that Λ has an atom at p, that is, $\Lambda(\{p\}) > 0$. An atom lies in the support of Λ. Therefore, the previous step gives $0 < p \leq 1$. For any $0 < q < p$ with $\Lambda(\{q\}) = 0$, the expected number of sellers also asking price q is zero, and thus such sellers almost surely do not exist. Therefore,

$$\Pi(q) = q \Pr(X([0,q]) < b) \geq q \Pr(X([0,p)) < b). \tag{7}$$

Since q can be chosen arbitrarily close to p, (6) and (7) give

$$\Pi(p) \geq p \Pr(X([0,p)) < b).$$

However, this inequality contradicts (5), since that equation implies that

$$\begin{aligned}
\Pi(p) &< p \sum_{m,n=0}^{\infty} \min\{\max\{b-m, 0\}, 1\} e^{-\Lambda([0,p])} \frac{\Lambda([0,p))^m}{m!} \frac{\Lambda(\{p\})^n}{n!} \\
&= p \sum_{m=0}^{\infty} \min\{\max\{b-m, 0\}, 1\} e^{-\Lambda([0,p))} \frac{\Lambda([0,p))^m}{m!} \\
&= p \Pr(X([0,p)) < b).
\end{aligned}$$

This proves that atoms do not exist, and thus the equality in (7) holds for *all* q.

Claim 3 *The support of* Λ *is* $[\alpha, 1]$*, and* $\alpha < 1$.

Since α is the smallest element in the support of Λ, and is not an atom, it cannot also be the biggest element there. Hence, $\alpha < 1$. Suppose some point $\alpha < q \leq 1$ is not in the support of Λ. Let p be the highest price in $[\alpha, q]$ that is in the support. Since none of the points in $(p, q]$ is in the support, $\Lambda((p,q]) = 0$, and therefore there is probability 1 that no seller quotes a price in $(p, q]$. The previous step gives:

$$\begin{aligned}\Pi(p) &= p\Pr(X([0,p]) < b) = p\Pr(X([0,q]) < b) \\ &< q\Pr(X([0,q]) < b) = \Pi(q).\end{aligned}$$

However, this contradicts (6). ■

As Proposition 2 shows, an equilibrium necessarily involves price dispersion: all the prices between some fraction α of the buyers' reservation price and the reservation price itself may occur. In addition, an equilibrium has no atoms, and therefore a seller almost surely does not compete with others quoting the same price. This implies, in particular, that a seller quoting price α almost surely sells. Hence, the expected profit for such a seller is given by

$$\Pi(\alpha) = \alpha. \tag{8}$$

This shows that the profit in equilibrium is equal to the minimum asking price. For a seller asking a higher price p, the probability of selling is equal to the probability that the number of other sellers asking prices that are not higher than p is less than b. Hence, by (1),

$$\Pi(p) = p\,\varphi_b(F_\Lambda(p)), \tag{9}$$

where F_Λ is the generalized distribution function of Λ and

$$\varphi_b(t) := e^{-t}\sum_{n=0}^{b-1}\frac{t^n}{n!} \tag{10}$$

is the probability that a Poisson random variable with parameter t is less than b, which can also be written (Feller [11, Section VI.10]) as

$$\varphi_b(t) = \frac{1}{(b-1)!}\int_t^\infty e^{-x}x^{b-1}dx. \tag{11}$$

In particular, (9) gives

$$\Pi(1) = \varphi_b(\lambda). \tag{12}$$

By Proposition 2, (8) and (12) are equal. Therefore, the equilibrium profit α is explicitly given by

$$\alpha = \varphi_b(\lambda). \tag{13}$$

Using this result, the equilibrium can now be specified.

Theorem 3 *A unique equilibrium Λ exists, which is given by the following generalized distribution function:*

$$F_\Lambda(p) = \varphi_b^{-1}\left(\frac{\alpha}{p}\right) \tag{14}$$

for $\alpha \le p \le 1$, and $F_\Lambda(p) = 0$ or $F_\Lambda(p) = \lambda$ for $p < \alpha$ or $p > 1$, respectively (with α given by (13)).

Proof. First, the uniqueness of the equilibrium is established. As shown above, any equilibrium Λ is supported in $[\alpha, 1]$, and hence satisfies $F_\Lambda(p) = 0$ for $p < \alpha$ and $F_\Lambda(p) = \lambda$ for $p > 1$. Consider any $\alpha \le p \le 1$. Since, by Proposition 2, (9) and (12) are equal, (13) gives

$$\varphi_b(F_\Lambda(p)) = \frac{\alpha}{p}. \tag{15}$$

Since, for all $t > 0$,

$$\frac{d\varphi_b}{dt} = -e^{-t}\frac{t^{b-1}}{(b-1)!} < 0, \tag{16}$$

the function $\varphi_b : [0,\infty) \to (0,1]$ is one-to-one and onto, and hence invertible. Therefore, (14) follows from (15).

It remains to prove that an equilibrium exists. Define the selling price function by

$$p(s) = \frac{\alpha}{\varphi_b(F_M(s))}.$$

By (16) and (13), $p(s)$ is a nondecreasing continuous function on the real line, and tends to α as s tends to $-\infty$ and to 1 as it tends to ∞. The price process X satisfies (4) and therefore has the parameter measure $\Lambda(\cdot) := M\left(p^{-1}(\cdot)\right)$. For every $\alpha < q < 1$, let s be the greatest solution of $p(s) = q$. Then, $p^{-1}((-\infty, q]) = (-\infty, s]$, and therefore:

$$\begin{aligned} F_\Lambda(q) &= \Lambda((-\infty, q]) = M\left(p^{-1}((-\infty, q])\right) \\ &= M((-\infty, s]) = F_M(s) \\ &= \varphi_b^{-1}\left(\frac{\alpha}{p(s)}\right) = \varphi_b^{-1}\left(\frac{\alpha}{q}\right). \end{aligned} \tag{17}$$

By (16) and (13), this shows that $F_\Lambda(q)$ is continuous, strictly increasing for $\alpha < q < 1$, and tends to 0 or λ as q tends to α or 1, respectively. It follows that (17) also holds for $q = \alpha$ and $q = 1$, that the support of Λ is $[\alpha, 1]$, and that Λ has no atoms. The latter property implies that for every $0 \le p \le 1$ the profit is given by (9). By (17), this implies that, for every $\alpha \le p \le 1$, $\Pi(p) = p\ \varphi_b(F_\Lambda(p)) = \alpha$.

For $0 \leq p < \alpha$, $F_\Lambda(p) = 0$, and therefore, by (9), $\Pi(p) = p\,\varphi_b(0) = p < \alpha$. This proves that Λ is an equilibrium. ■

The next result specifies the density of the price process. In particular, it shows that, in equilibrium, prices closer to the minimum price α are more likely to occur.

Proposition 3 *In the support of the equilibrium Λ, the density $f = F'_\Lambda$ satisfies*

$$f(p) = \frac{1}{p}\left(1 + \frac{b-1}{F_\Lambda(p)} + \frac{(b-1)(b-2)}{(F_\Lambda(p))^2} + \cdots + \frac{(b-1)!}{(F_\Lambda(p))^{b-1}}\right). \tag{18}$$

Hence, it decreases monotonically there.

Proof. Taking the derivatives of both sides of (15) and using (16) gives

$$\begin{aligned}\left(-e^{-F_\Lambda(p)}\frac{(F_\Lambda(p))^{b-1}}{(b-1)!}\right) f(p) &= -\frac{\alpha}{p^2} = -\frac{1}{p}\varphi_b(F_\Lambda(p)) \\ &= -\frac{1}{p}e^{-F_\Lambda(p)}\sum_{n=0}^{b-1}\frac{(F_\Lambda(p))^n}{n!},\end{aligned}$$

which simplifies to (18). ■

7 Comparative Statics

Equations (13) and (14) show that the equilibrium depends only on the number of buyers b and the expected number of sellers λ. The finite Poisson process S selecting the actual sellers is irrelevant. The following proposition considers the effect of these two parameters on the equilibrium profit.

Proposition 4 *If λ increases or b decreases, the equilibrium profit α decreases. If λ and b are equal (i.e., the market is balanced in expectation), then $\alpha < \frac{1}{2}$, and as they both increase concomitantly, α approaches $\frac{1}{2}$.*

Proof. The first part follows immediately from (10) and (11). The second part follows from a result of Ramanujan [25, 26], who showed that, for $\lambda \geq 1$,

$$\varphi_\lambda(\lambda) = \sum_{n=0}^{\lambda-1} e^{-\lambda}\frac{\lambda^n}{n!} = \frac{1}{2} - \theta(\lambda)\, e^{-\lambda}\frac{\lambda^\lambda}{\lambda!},$$

where $\frac{1}{3} \leq \theta(\lambda) \leq \frac{1}{2}$ (see also Choi [7]). Since $\lim_{\lambda\to\infty} e^{-\lambda}\frac{\lambda^\lambda}{\lambda!} = 0$, this implies that $\lim_{\lambda\to\infty}\varphi_\lambda(\lambda) = \frac{1}{2}$. ■

The equilibrium profit α may be rather sensitive to changes in the expected number of sellers. In other words, the elasticity η of α with respect to λ may be rather high, even if the market is balanced in expectation. Indeed, by (13) and (16),

$$\begin{aligned}\eta|_{\lambda=b} &= \left.\frac{\lambda}{\alpha}\frac{\partial\alpha}{\partial\lambda}\right|_{\lambda=b} = -\frac{\lambda}{\alpha}e^{-\lambda}\frac{\lambda^{\lambda-1}}{(\lambda-1)!} \\ &= -\left(\frac{1}{\alpha}e^{-\lambda}\frac{\lambda^{\lambda+\frac{1}{2}}}{\lambda!}\right)\sqrt{\lambda}.\end{aligned}$$

By Proposition 4 and Stirling's formula, the expression in parenthesis tends to $\sqrt{\frac{2}{\pi}}$ as λ tends to infinity. For example, if $\lambda = b = 100$, this approximation gives an elasticity of about -8. Exact computation shows that increasing λ to 101 decreases the equilibrium profit α by 8.2%, from 0.4867 to 0.447.

The next result extends Proposition 4 by describing the effect of increasing supply or decreasing demand on the whole equilibrium price distribution, rather than the minimum price only.

Proposition 5 *Let p be any fixed price between the minimum price in equilibrium α and the buyers' reservation price 1. If λ increases or b decreases, the expected number of sellers posting prices less than p increases.*

Proof. In the equilibrium Λ, the expected number of sellers posting prices less than p is $F_\Lambda(p)$. By (15) and (13),

$$\varphi_b(F_\Lambda(p)) = \frac{\varphi_b(\lambda)}{p}.$$

By (11), this gives:

$$\int_{F_\Lambda(p)}^{\infty} e^{-x}x^{b-1}dx = \frac{1}{p}\int_{\lambda}^{\infty} e^{-x}x^{b-1}dx.$$

Therefore, if λ increases, so does $F_\Lambda(p)$. Suppose that λ does not change. The above equation, which can also be written as

$$\int_{F_\Lambda(p)}^{\lambda} e^{-x}x^{b-1}dx = (\frac{1}{p}-1)\int_{\lambda}^{\infty} e^{-x}x^{b-1}dx, \tag{19}$$

interpolates the dependency of $F_\Lambda(p)$ on b for all real $b \geq 1$. Taking the partial derivatives with respect to b of both sides of (19) gives:

$$\int_{F_\Lambda(p)}^{\lambda} e^{-x}x^{b-1}\ln x\,dx - e^{-F_\Lambda(p)}\left(F_\Lambda(p)\right)^{b-1}\frac{\partial}{\partial b}F_\Lambda(p) \tag{20}$$

$$= \left(\frac{1}{p}-1\right)\int_{\lambda}^{\infty} e^{-x}x^{b-1}\ln x\,dx.$$

The integral on the right-hand side of (20) is (strictly) greater than $\ln\lambda$ times the integral on the right-hand side of (19). The integral on the left-hand side is *less* than $\ln\lambda$ times the corresponding one in (19). Therefore, (19) and (20) imply that, for fixed p,

$$\frac{\partial}{\partial b}F_\Lambda(p) < 0.$$

Thus, if b decreases, $F_\Lambda(p)$ increases. ■

7.1 Numerical Examples

In the simple case of a single buyer ($b = 1$), the expected profit (5) is $\alpha = e^{-\lambda}$. By (14), this implies that the equilibrium Λ is given by $F_\Lambda(p) = \lambda + \ln p$, and the density by

$$f(p) = \begin{cases} 1/p & e^{-\lambda} \leq p \leq 1 \\ 0 & \text{otherwise} \end{cases}. \tag{21}$$

Note that where the density is positive it is independent of the expected number of sellers λ.

With more than one buyer, a graphical presentation of the equilibrium is more illuminating than an analytical one. Figure 1 shows the effect of varying the number of buyers b, with the expected number of sellers kept fixed at $\lambda = 10$. If the market is balanced in expectation ($b = 10$), the minimum price in equilibrium (which equals the sellers' expected profit) is $\alpha = 0.458$ (which is less than 0.5, in accordance with Proposition 4). In expectation, six sellers quote prices between α and 0.5, two others quote prices between 0.5 and 0.65, and the expected number of sellers asking for more than 0.8 is less than one. Thus, the price distribution is heavily skewed towards the minimum price (see Proposition 3). However, all higher prices not exceeding the buyers' reservation price of 1 also occur (see Proposition 2). Reducing the number of buyers to eight decreases the minimum price to $\alpha = 0.22$ (see Proposition 4). Moreover, the entire price distribution shifts to the left (see Proposition 5), with six sellers in expectation now quoting prices less than 0.3 and two others between 0.3 and 0.5. Increasing the number of buyers to twelve has the opposite effect on the prices.

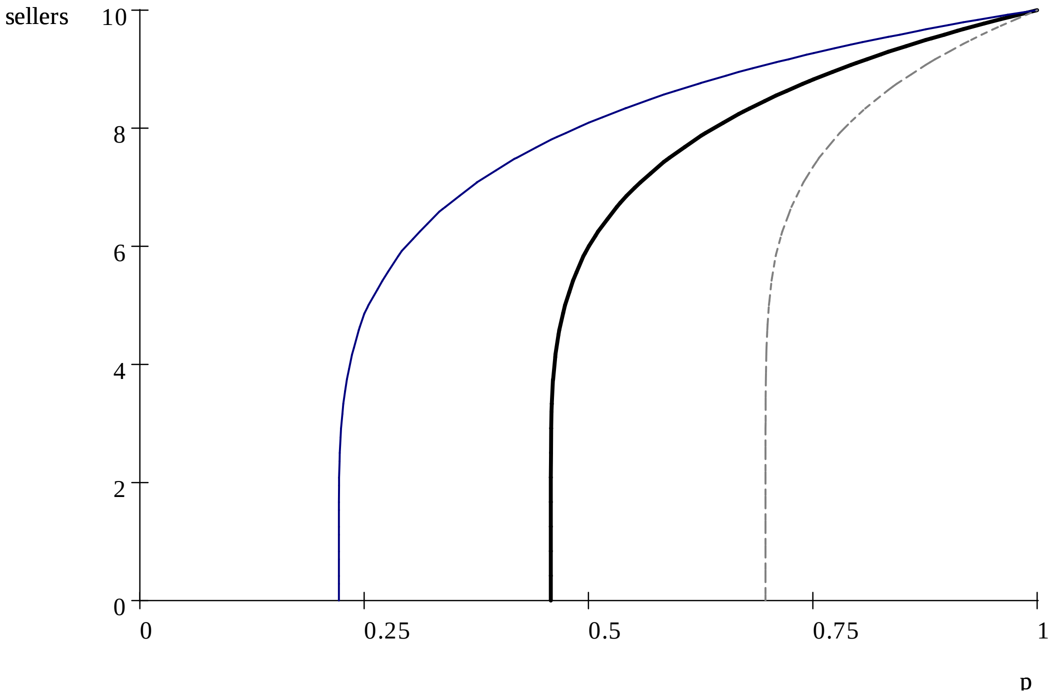

Figure 1: The effects of changes in the number of buyers b on the equilibrium price distribution. The expected number of sellers is fixed at $\lambda = 10$. The generalized distribution function of the equilibrium is shown for $b = 10$ (heavy curve), $b = 8$ (light curve), and $b = 12$ (dashed curve). Each function gives the expected number of sellers quoting prices less than p in equilibrium.

8 Conclusion

This paper demonstrates how random-player games, and Poisson games in particular, can be used to model uncertain economic environments. In agreement with the literature, it shows that price dispersion of a homogeneous product may exist in a competitive market, if the environment, specifically the number of participants, is uncertain. The main contribution of the present model is in providing an alternative framework to study price dispersion. This framework is simple, and the model has sharp predictions, which can be tested empirically. Thus, the distribution of prices and the equilibrium profits for various parameter values can be completely characterized. This makes it possible to analyze the sensitivity of the price process, and the minimum price in particular, to changes in the market structure.

The present framework can easily be adapted to model auctions, in which price dispersion stems not from heterogeneity in information but from uncertainty regarding the number of bidders. Unlike much of the existing literature on uncertain number of bidders (Matthews [15], McAfee and McMillan [16], Harstad, Kagal and Levin [12], Levin and Ozdenoren [14]), which assumes an independent private-value-single-object model, this model would assume a pure common-value multi-object model without informational asymmetries, i.e., all the parties involved know the value of the objects being auctioned. The only uncertainty involves the number of bidders. The model can also be adapted to allow for private values and other differences among bidders. This only requires a richer type space from which bidders are selected. Standard questions relating to auction design and the advantages of alternative designs can be studied in such a way.

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