Generic Uniqueness of Equilibrium in Large Crowding Games

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A crowding game is a noncooperative game in which the payoff of each player depends only on the player's action and the size of the set of players choosing that particular action: The larger the set, the smaller the payoff. Finite, n-player crowding games often have multiple equilibria. However, a large crowding game generically has just one equilibrium, and the equilibrium payoffs in such a game are always unique. Moreover, the sets of equilibria of the m-replicas of a finite crowding game generically converge to a singleton as m tends to infinity. This singleton consists of the unique equilibrium of the "limit" large crowding game. This equilibrium generically has the following graph-theoretic property: The bipartite graph, in which each player in the original, finite crowding game is joined with all best-response actions for (copies of) that player, does not contain cycles.

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1. Introduction. An n-player crowding game (Milchtaich, 1996a, 1998; Konishi *et al.*, 1997) is a finite noncooperative game in which the payoff of each player is affected by the actions of the other players only through the total number of players choosing the same action as that player. This effect is negative: Crowding, or congestion, decreases payoffs. However, the quantitative relation between crowding and payoffs may vary and may depend both on the particular player and on the particular action. A player may prefer action a to action b when few other players choose these actions and may have the reverse preferences when many people choose them. Another player may prefer action b to action a in the first case and action a to action b in the second case. This is one of the differences between n-player crowding games and the somewhat similar congestion games (Rosenthal, 1973). In a congestion game, all players are equally affected by crowding. On the other hand, the effects of crowding on payoffs are not necessarily negative. In addition, each player does not choose a single action but a combination of actions. A common feature of n-player crowding games and congestion games is that, in both classes of games, each player's contribution to crowding is the same. In other words, all players have the same "weight."

An n-player crowding game has at least one (Nash) equilibrium in pure strategies. (This, incidentally, would not be true if the players did not all have the same weight; see Milchtaich, 1996a, Section 8.) Often, however, there is more than one equilibrium. To take an extreme example, if the number of players is equal to the number of actions and the negative effects of crowding on payoffs are sufficiently great, then every one-to-one assignment of actions to players is an equilibrium. A player may be willing to interchange his action with that of another player, but the negative effect of his own weight on the payoff makes a *unilateral* deviation unprofitable for him. Hence the multiplicity of equilibria. This example raises the question: To what extent does multiplicity of equilibria persist as the number of players is increased and each player's weight is correspondingly decreased?

One way of formulating the idea that the "same" crowding game can be played by an increasing number of players is to consider m-replicas of the game in which each of the n original players is replaced by m "scaled-down" copies. The payoffs of these m players are affected by crowding in exactly the same way as that of the original player, but their weight is m times smaller than his. To compare meaningfully (pure-strategy) equilibria across such replicas, two equilibria of the same m-replica should be identified whenever the number of copies of an original player choosing a particular action in one equilibrium is the same as in the other equilibrium. The number of copies of an original player choosing each action can be expressed formally as a mixed strategy for that player. (This, however, is not the same as a *randomized* strategy for the original player. In the present context, a mixed strategy is not interpreted as a lottery over actions.) Do the sets of mixed-strategy equilibria converge as m tends to infinity? Section 6 shows that these sets indeed converge, and their limit is a set consisting of a single point, whenever the "limit game" has a unique equilibrium. Such a limit game is an example of a large crowding game.

Large crowding games, which are formally defined in Section 2, are closely related to (essentially, a special case of) the nonatomic games studied by Schmeidler (1973). In both classes of games, all players share the same, finite set of actions. In Schmeidler's (1973) model, the set of players is the unit interval $[0, 1]$, and the population measure, which gives the "size" of each set of players, is Lebesgue measure. In the present context, however, this setup is too restrictive. The "limit" of the sequence of m-replicas of an n-player crowding game is a large crowding game with n "representative" players, corresponding to the n players in the finite game. The (purely atomic) population measure is defined on this set of representative players, and assigns equal mass to them. In general, *any* probability measure (with or without atoms) defined on any (finite or infinite) set of representative players can be used as the population measure in a large crowding game. Each representative player stands for one or (possibly many) more independent, identical players, whose weight is zero. The measure of a set of representative players is interpreted as the total weight of the players whose representative player is in that set. A representative player's mixed strategy is interpreted as the distribution of actions of the players he represents. Thus, the total weight of the players choosing a particular action, and hence also the payoff that each player would get by switching to that action, are completely determined by the profile of mixed strategies of the representative players. If that profile has the property that the mixed strategy of almost every representative player involves only best-response actions for the players he represents, then it is an equilibrium of the large crowding game.

Section 3 shows that every large crowding game has at least one equilibrium (in mixed strategies), and that the total weight of the players choosing each action is the same at all equilibria of the game. Consequently, the set of best-response actions and the payoff of each player, at equilibrium, are unique. Section 4 shows that, generically, the equilibrium itself is unique.

The term "generic" is used in this paper in the topological sense (see Mas-Colell, 1985, Section 8.2). In a complete metric space, a property is generic (from the Baire category point of view) if it holds in some dense G_δ in that space. According to the Baire category theorem, in a complete metric space a set is a dense G_δ if and only if it is the intersection of countably many dense open sets. The generic uniqueness of equilibrium in large crowding games defined over a fixed set of representative players, population measure, and set of actions is proved in Section 4 by showing that, with respect to a natural topology on these games—which is induced by a complete metric—the mean number of best-response actions, at equilibrium, is an upper semicontinuous function which has a discontinuity at each large crowding game with multiple equilibria. The set of points of continuity of an upper semicontinuous function defined on a complete metric space is always a dense G_δ.

These results show that the sets of mixed-strategy equilibria of the m-replicas of an n-player crowding game nearly always converge to a singleton as m tends to infinity. This singleton consists of the unique equilibrium of the limit large crowding game. As shown in Section 4, the equilibrium generically has the following graph-theoretic property: The bipartite graph, in which each of the n players in the original game is joined with all best-response actions for (copies of) that player, does not contain cycles. Thus, for example, for every pair of players, at most one action that is a best response for one of the players is also a best response for the other player (otherwise the graph would contain a quadrilateral). This result has a number of potential applications in biology (see Milchtaich, 1996b, and Section 4).

There is a more general sense than converging m-replicas in which a sequence of finite crowding games can be convergent. This is convergence "in distribution": Individual players in one game in the sequence may not be identifiable with players in another game, but the distributions of the *types of players* in these games converge. A player's type specifies, for each action, the functional relation between the payoff of the player when he is choosing that action and the total weight of the players choosing it. The type distribution of a large crowding game indicates "how many" players of each type there are. This is analogous to the description of an exchange economy in terms of the distribution of agents' characteristics (Hart *et al.*, 1974). Mas-Colell (1984) introduced the concept of a (Cournot-Nash) equilibrium distribution. An equilibrium distribution is a probability measure on the product space of types of players and actions. Its marginal on the space of types of players coincides with a given type distribution. Its marginal on the set of actions determines the payoff associated with each action, for each type of player. An equilibrium distribution is supported in the set of all ordered pairs consisting of a type of player and a best-response action for that type. As shown in Section 5, the set of all equilibrium distributions with a fixed marginal on the space of types of players coincides with the set of all type–

action distributions of a fixed large crowding game with that type distribution and equilibria of that game. This implies that at least one, and generically only one, equilibrium distribution corresponds to every probability measure on the space of types of players. As shown in Section 6, when a sequence of finite crowding games has the property that the corresponding sequence of type distributions converges to a type distribution for which there is a unique equilibrium distribution, and the number of players tends to infinity, the sets of type–action distributions converge to a limit set consisting of the unique equilibrium distribution of the limit type distribution.

Each of the convergence results described above refers to a particular sequence of finite crowding games: m-replicas of a single n-player crowding game in the first case, and a sequence converging "in distribution" in the second case. In both cases, the number of players tends to infinity. One may conjecture that, in some large class of finite crowding games, similar results hold *uniformly*. Thus, in some precise sense, the following may be true: If n is large, then most n-player crowding games have a small set of equilibria. "Small" may refer either to the number of equilibria of the game or to the diameter of its set of equilibria (the maximum distance between any two equilibria of the game).

2. The model. A large crowding game is defined over a game structure that consists of a (finite or infinite) set I of *representative players*, a *population measure* μ, which is a probability measure on a σ-algebra of subsets of I, and a finite set $\mathcal{A} = \{1, 2, \ldots, \alpha\}$ of *actions*. The set of all probability measures on $\mathcal{A}$ is denoted $\Delta(\mathcal{A})$. An element of $\Delta(\mathcal{A})$ is called a *mixed strategy* (see the introduction). Each mixed strategy corresponds to a unique probability vector $p = (p_1, p_2, \ldots, p_\alpha)$, such that $p_j \geq 0$ for all j and $\sum_{j\in\mathcal{A}} p_j = 1$. A measurable function $\sigma : I \to \Delta(\mathcal{A})$ is called a (mixed-)*strategy profile*. Such a function assigns a mixed strategy $\sigma(i) = (\sigma_1(i), \sigma_2(i), \ldots, \sigma_\alpha(i))$ to every representative player i. If $\sigma_j(i) \in \{0, 1\}$ for (μ)-almost every i and every j, then σ is called a *pure-strategy profile*. Two strategy profiles σ and σ' that are equal almost everywhere will be identified. The *action distribution* of a strategy profile σ is the integral $\int \sigma(i)d\mu(i)$. The action distribution is an element of $\Delta(\mathcal{A})$. Its jth coordinate, $\int \sigma_j(i)d\mu(i)$ (henceforth, $\int \sigma_j$), is interpreted as the total weight of the players choosing action j.

A *large crowding game* g is a mapping that assigns a continuous and strictly decreasing *payoff function* $g_{ij} : [0, 1] \to \mathbb{R}$ to every representative player i and action j, such that, for each j and $0 \leq x \leq 1$, the real-valued function $i \mapsto g_{ij}(x)$ is measurable. Two large crowding games g and g' such that $g_{ij} = g'_{ij}$ for almost every i and every j will be identified. A strategy profile σ is an *equilibrium* of a large crowding game g if, for almost every representative player i and every action j,

$$\sigma_j(i){>}0 \text{ implies } g_{ij}(\textstyle\int \sigma_j) = \max_{k\in\mathcal{A}} g_{ik}(\textstyle\int \sigma_k). \tag{1}$$

In this case, $\max_{k\in\mathcal{A}} g_{ik}(\int \sigma_k)$ is an *equilibrium payoff* for the representative player i, and the action distribution of σ is an *equilibrium action distribution* of g. If σ is a pure-strategy profile, then σ is an equilibrium in *pure strategies*. An equilibrium σ such that, for almost every i and every j, $g_{ij}(\int \sigma_j) = \max_{k\in\mathcal{A}} g_{ik}(\int \sigma_k)$ implies $\sigma_j(i){>}0$ (as well as the converse) is said to be *quasi-strict* (or quasi-strong, in the terminology of Harsanyi, 1973). A *strict* (or strong, in the terminology of Harsanyi, 1973) equilibrium is a quasi-strict equilibrium in pure strategies.

The set of all large crowding games defined over a fixed game structure is denoted $\mathcal{G}$. A sequence $\{g^{(n)}\}$ in $\mathcal{G}$ converges *almost everywhere* to a large crowding game g if, for almost every representative player i, $\lim_n g^{(n)}_{ij}(x) = g_{ij}(x)$ for every strategy j and every $0 \leq x \leq 1$. It follows from the assumed continuity and monotonicity of the payoff functions that, in this case, for almost every representative player i, the sequence $\{g^{(n)}_{ij}(x)\}$ converges to $g_{ij}(x)$ *uniformly* in j and x. The *distance* $d(g, g')$ between two elements g and g' of $\mathcal{G}$ is defined as

$$\int_I \min\left\{1, \max_{\substack{j\in\mathcal{A}\\ 0\leq x\leq 1}} \left|g_{ij}(x) - g'_{ij}(x)\right|\right\} d\mu(i).$$

Clearly, $d(g, g') = 0$ if and only if $g_{ij} = g'_{ij}$ for almost every i and every j, in which case g and g' are identical. As shown in Section 5, the metric space $(\mathcal{G}, d)$ is topologically complete. That is, there exists a complete metric d^* on $\mathcal{G}$ such that the d and d^* topologies are the same. Indeed, a sequence in $\mathcal{G}$ converges to a limit g with respect to either metric topology if and only if every subsequence has a sub-subsequence that converges almost everywhere to g. This property obviously uniquely determines the (common) metric topology. Convergence almost everywhere of sequences in $\mathcal{G}$ implies convergence in the metric topology, and is equivalent to it when the set of representative players is countable.

3. Preliminary results. The following equilibrium existence results are essentially special cases of Theorems 1 and 2 of Schmeidler (1973).

THEOREM 3.1. *Every large crowding game has at least one equilibrium. If the population measure is nonatomic, then, moreover, every large crowding game has an equilibrium in pure strategies.*

The proof of Theorem 3.1 given below is based on Mas-Colell (1984, Theorem 2) and Rath (1992, Theorem 1). For $i \in I$, $g \in \mathcal{G}$, and $p \in \Delta(\mathcal{A})$, the number of best-response actions for representative player i in the game g when the action distribution is $p = (p_1, p_2, \ldots, p_\alpha)$ is denoted $\#BR(i,g,p)$. This number is equal to the cardinality of the set $BR(i,g,p) = \{q \in \Delta(\mathcal{A}) \mid \text{ for some } j,\ q_j = 1 \text{ and } g_{ij}(p_j) = \max_{k\in\mathcal{A}} g_{ik}(p_k)\}$ (of best-response pure strategies). The convex hull of this set, co $BR(i,g,p)$, is the set of all best-response mixed strategies for the representative player i. The collection of all elements of $\Delta(\mathcal{A})$ of the form $\int \sigma d\mu$, where σ is a strategy profile such that $\sigma(i) \in BR(i,g,p)$ for almost every i, is denoted $\int BR(i,g,p)d\mu(i)$. The meaning of $\int \text{co } BR(i,g,p)d\mu(i)$ is similar. The first set, which is obviously a subset of the second, is never empty. For example, for any linear order on the α extremal points of $\Delta(\mathcal{A})$ (that is, on pure strategies), the action distribution of the strategy profile σ defined by $\sigma(i) = \min BR(i,g,p)$ belongs to $\int BR(i,g,p)d\mu(i)$. (The measurability of σ follows from the measurability condition in the definition of a large crowding game.)

LEMMA 3.2. *Let $\{p^{(n)}\}_{n\geq 0} \subseteq \Delta(\mathcal{A})$, $\{q^{(n)}\}_{n\geq 0} \subseteq \Delta(\mathcal{A})$, and $\{g^{(n)}\}_{n\geq 0} \subseteq \mathcal{G}$ be such that $p^{(n)} \to p^{(0)}$, $q^{(n)} \to q^{(0)}$, and $g^{(n)} \to g^{(0)}$. Then,*

(a) *If $q^{(n)} \in \int BR(i, g^{(n)}, p^{(n)})d\mu(i)$ for all $n \geq 1$, then this relation holds for $n = 0$ as well;*

(b) *If $q^{(n)} \in \int \text{co } BR(i, g^{(n)}, p^{(n)})d\mu(i)$ for all $n \geq 1$, then this relation holds for $n = 0$ as well; and*

(c) $\limsup_n \int \#BR(i, g^{(n)}, p^{(n)})d\mu(i) \leq \int \#BR(i, g^{(0)}, p^{(0)})d\mu(i)$.

PROOF. Every subsequence of $\{g^{(n)}\}$ has a sub-subsequence that converges almost everywhere to $g^{(0)}$. Hence, it may be assumed without loss of generality that $\{g^{(n)}\}$ itself converges almost everywhere to $g^{(0)}$. Then, for almost every i, (i) if $q^{(n)} \in BR(i, g^{(n)}, p^{(n)})$ for infinitely many ns, then this relation holds for $n = 0$ as well; (ii) if $q^{(n)} \in \text{co } BR(i, g^{(n)}, p^{(n)})$ for infinitely many ns, then this relation holds for $n = 0$ as well; and (iii) $\limsup_n \#BR(i, g^{(n)}, p^{(n)}) \leq \#BR(i, g^{(0)}, p^{(0)})$. Conclusions (a) and (b) follow from (i) and (ii), respectively, by Fatou's lemma in several dimensions (Schmeidler, 1970, Corollary 2). Conclusion (c) follows from (iii) by Fatou's lemma. □

PROOF OF THEOREM 3.1. By (a) and (b) of Lemma 3.2, for every large crowding game g the correspondences (i.e., multi-valued functions) $p \mapsto \int BR(i,g,p)d\mu(i)$ and $p \mapsto \int \text{co } BR(i,g,p)d\mu(i)$, defined on $\Delta(\mathcal{A})$, have closed graphs. The latter correspondence is evidently convex-valued and therefore, by Kakutani's fixed-point theorem, has a fixed point. If μ is nonatomic, then the former correspondence is also convex-valued (Hildenbrand, 1974, p. 62), and therefore has a fixed point. It follows that there exists a strategy profile σ such that $\sigma(i) \in \text{co } BR(i, g, \int \sigma d\mu)$ for almost every i, and if μ is nonatomic, then there exists a strategy profile σ' such that $\sigma'(i) \in BR(i, g, \int \sigma' d\mu)$ for almost every i. The first strategy profile is an equilibrium of g. The second strategy profile is an equilibrium in pure strategies. □

As a first step towards proving generic uniqueness of equilibrium in large crowding games, the next proposition establishes the uniqueness of the equilibrium action distribution and the equilibrium payoffs.

PROPOSITION 3.3. *If $\hat{\sigma}$ and $\tilde{\sigma}$ are two equilibria of the same large crowding game, then $\int \hat{\sigma}(i)d\mu(i) = \int \tilde{\sigma}(i)d\mu(i)$. Consequently, the equilibrium payoffs in $\hat{\sigma}$ and $\tilde{\sigma}$ are the same.*

PROOF. Since $\sum_{j\in\mathcal{A}} \hat{\sigma}_j = \sum_{j\in\mathcal{A}} \tilde{\sigma}_j = 1$ identically, for every subset $\mathcal{A}'$ of $\mathcal{A}$ and every measurable subset I' of I

$$\begin{aligned} \sum_{j\in\mathcal{A}'} \left(\int \hat{\sigma}_j - \int \tilde{\sigma}_j \right) &= \int_I \sum_{j\in\mathcal{A}'} (\hat{\sigma}_j - \tilde{\sigma}_j) d\mu - \int_{I'} \sum_{j\in\mathcal{A}} (\hat{\sigma}_j - \tilde{\sigma}_j) d\mu \\ &\leq \int_{I\setminus I'} \sum_{j\in\mathcal{A}'} \hat{\sigma}_j d\mu + \int_{I'} \sum_{j\in\mathcal{A}\setminus\mathcal{A}'} \tilde{\sigma}_j d\mu. \end{aligned} \tag{2}$$

If $\hat{\sigma}$ and $\tilde{\sigma}$ are equilibria of a large crowding game g, $\mathcal{A}' = \{j \in \mathcal{A} \mid \int \hat{\sigma}_j > \int \tilde{\sigma}_j\}$, and $I' = \{i \in I \mid \max_{k\in\mathcal{A}} g_{ik}(\int \hat{\sigma}_k) < \max_{k\in\mathcal{A}} g_{ik}(\int \tilde{\sigma}_k)\}$, then the right-hand side of (2) is equal to zero. Since if $i \in I\backslash I'$ and $j \in \mathcal{A}'$, then $\max_{k\in\mathcal{A}} g_{ik}(\int \hat{\sigma}_k) \geq \max_{k\in\mathcal{A}} g_{ik}(\int \tilde{\sigma}_k) \geq g_{ij}(\int \tilde{\sigma}_j) > g_{ij}(\int \hat{\sigma}_j)$ by the monotonicity of g_{ij}. Therefore, by (1), $\hat{\sigma}_j(i) = 0$ for almost every $i \in I\backslash I'$ and every $j \in \mathcal{A}'$. Similarly, if $i \in I'$ and $j \in \mathcal{A}\backslash\mathcal{A}'$, then $\max_{k\in\mathcal{A}} g_{ik}(\int \tilde{\sigma}_k) > \max_{k\in\mathcal{A}} g_{ik}(\int \hat{\sigma}_k) \geq g_{ij}(\int \hat{\sigma}_j) \geq g_{ij}(\int \tilde{\sigma}_j)$. Therefore, $\tilde{\sigma}_j(i) = 0$ for almost every $i \in I'$ and every $j \in \mathcal{A}\backslash\mathcal{A}'$. This proves that $\mathcal{A}'$ is empty, or $\int \hat{\sigma}_j \leq \int \tilde{\sigma}_j$ for all j. It follows, by symmetry, that $\int \hat{\sigma} = \int \tilde{\sigma}$. □

The unique equilibrium action distribution of a large crowding game g is denoted $e(g) = (e_1(g), e_2(g), \ldots, e_\alpha(g))$. The mean number $\int \#BR(i, g, e(g))d\mu(i)$ of best-response actions for the players in g, at equilibrium, is denoted $f(g)$.

Lemma 3.4. *The function $e : \mathcal{G} \to \Delta(\mathcal{A})$, which sends each large crowding game to its unique equilibrium action distribution, is continuous. The function $f : \mathcal{G} \to \mathbb{R}$, which sends each large crowding game to the mean number of best-response actions for the players in that game, at equilibrium, is upper semicontinuous.*

Proof. For every large crowding game g, an element p of $\Delta(\mathcal{A})$ is equal to $e(g)$ if and only if it satisfies $p \in \int \text{co } BR(i, g, p)d\mu(i)$. The continuity of e hence follows from (b) of Lemma 3.2 by setting $q^{(n)} = p^{(n)} = e(g^{(n)})$ for all $n \geq 1$. Setting $p^{(n)} = e(g^{(n)})$ for all $n \geq 0$ in (c) of Lemma 3.2 now shows that f is upper semicontinuous. □

4. Generic uniqueness of equilibrium To prove that a large crowding game generically has a unique equilibrium, it suffices to show that there exists an upper semicontinuous real-valued function on $\mathcal{G}$ with a discontinuity at every large crowding game that has more than one equilibrium. The set of points of continuity of an upper semicontinuous real-valued function f on a complete metric space is a dense G_δ. Indeed, this set is equal to $\bigcap_{r\in\mathbb{Q}}[A_r \cup (\bar{A}_r)^c]$, where $\mathbb{Q}$ is the set of rational numbers, A_r is the G_δ set $f^{-1}((-\infty, r])$, and $(\bar{A}_r)^c$ is the complement of the closure of A_r (see also Kuratowski, 1966, § 34, VII). (The converse, incidentally, is also true: In every topological space, every dense G_δ is the set of points of continuity of some upper semicontinuous function. Specifically, if $G_1, G_2, \ldots$ are open dense sets, and χ_{G_n} denotes the characteristic function of G_n, then $\bigcap_{n\geq 1} G_n$ is precisely the set of points of continuity of the upper semicontinuous function $-\sum_{n\geq 1} 2^{-n}\chi_{G_n}$.)

Let $I_0, I_1, I_2, \ldots$ be a fixed finite or infinite sequence of disjoint measurable subsets of I, such that $\bigcup_{m\geq 0} I_m = I$, the set I_0 contains no atoms of the population measure μ, and I_m is an atom for all $m \geq 1$. The population measure is said to be *purely atomic* if $\mu(I_0) = 0$. For a given strategy profile σ, and for $m \geq 1$, there is a representative player i_m such that $\sigma(i) = \sigma(i_m)$ for almost every $i \in I_m$. Consider the (undirected, possibly infinite) bipartite graph in which the set of vertices is $I_1, I_2, \ldots$ together with all elements of $\mathcal{A}$, and an atom I_m and an action j are joined by an edge if and only if $\sigma_j(i_m) > 0$. If this bipartite graph contains no cycles (that is, if the graph is a forest: Each of its connected components is a tree), and if $\sigma_j(i) \in \{0, 1\}$ for almost every $i \in I_0$ and every j, then the strategy profile σ will be said to be *acyclic*.

It follows from Proposition 3.3 that the set E of equilibria of a large crowding game g is always convex (in the obvious sense). The following lemma shows that if the population measure is purely atomic, then the set ext E of extremal points of E is precisely the set of acyclic equilibria of g. The lemma also shows that (even if the population measure in not purely atomic) an acyclic equilibrium always exists, and gives a sufficient condition for such an equilibrium to be the unique equilibrium of a large crowding game g.

Lemma 4.1. *Every large crowding game has at least one acyclic equilibrium. A large crowding game that has an equilibrium that is both acyclic and quasi-strict has a unique equilibrium. If the population measure is purely atomic, and E is the set of equilibria of a large crowding game g, then for every $\sigma \in E$ the following conditions are equivalent:*

(i) *$\sigma \in$ ext E;*

(ii) *σ is acyclic;*

(iii) *for every $\sigma' \in E$, if the bipartite graph of σ' is a subgraph of the bipartite graph of σ, then $\sigma' = \sigma$.*

PROOF. Suppose, first, that the population measure is purely atomic. The space of all strategy profiles is then compact and metrizable with respect to the topology of almost everywhere convergence of strategy profiles. It follows from Proposition 3.3 that the set E of equilibria of a large crowding game g is closed in this topology. Therefore, by the Krein-Milman theorem, it coincides with the closed convex hull of the set of its extremal points. In particular, ext $E \neq \emptyset$. Therefore, to prove that g has an acyclic equilibrium, it suffices to show that (i) implies (ii). To prove that an equilibrium that is both acyclic and quasi-strict is the unique equilibrium of g, it suffices to show that (ii) implies (iii). For, if an equilibrium σ is quasi-strict, then the bipartite graph of every other equilibrium is a subgraph of the bipartite graph of σ.

((i) $\Rightarrow$ (ii)) Suppose that the bipartite graph of σ contains a cycle, Γ. The edges of the bipartite graph of σ can be directed in such a way that Γ becomes a cycle in the *directed* graph. The number of directed edges (or arcs) in Γ that are incident to a vertex is equal to the number of directed edges that are incident from it. For $m \geq 1$, it may be assumed without loss of generality that $\sigma(i) = \sigma(i')$ for every $i, i' \in I_m$. Therefore, for every ϵ in a neighborhood of zero, the strategy profile $\sigma^{(\epsilon)}$ defined by

$$\sigma_j^{(\epsilon)}(i) = \begin{cases} \sigma_j(i) + \epsilon/\mu(I_m) & \text{if } i \text{ belongs to an atom } I_m \text{ such that there is a directed edge in } \Gamma \\ & \text{incident from } I_m \text{ and to } j \\ \sigma_j(i) - \epsilon/\mu(I_m) & \text{if } i \text{ belongs to an atom } I_m \text{ such that there is a directed edge in } \Gamma \\ & \text{incident from } j \text{ and to } I_m \\ \sigma_j(i) & \text{otherwise} \end{cases}$$

has the same action distribution and the same bipartite graph as σ, and is hence an element of E. It follows that σ is not an extremal point of E.

((ii) $\Rightarrow$ (iii)) Suppose that the bipartite graph of $\sigma' \in E$ is a subgraph of the bipartite graph of σ but $\sigma' \neq \sigma$. For $m \geq 1$, there is a representative player i_m such that $\sigma(i) = \sigma(i_m)$ and $\sigma'(i) = \sigma'(i_m)$ for almost every $i \in I_m$. Consider the subgraph of the bipartite graph of σ obtained by deleting all the edges joining an atom I_m and an action j such that $\sigma'_j(i_m) = \sigma_j(i_m)$. It follows from the assumptions that the set of remaining edges is nonempty. Direct each of these edges from I_m to j if $\sigma'_j(i_m) < \sigma_j(i_m)$ and from j to I_m if $\sigma'_j(i_m) > \sigma_j(i_m)$. For $m \geq 1$, $\sum_{j \in \mathcal{A}} \sigma'_j(i_m) = \sum_{j \in \mathcal{A}} \sigma_j(i_m)$ $(= 1)$. For $j \in \mathcal{A}$, $\sum_{m \geq 1} \sigma'_j(i_m)\mu(I_m) = \sum_{m \geq 1} \sigma_j(i_m)\mu(I_m)$ $(= e_j(g)$, since $\mu(I_0) = 0$ by assumption). Therefore, if there is a directed edge incident to a vertex, then there is also a directed edge incident from it. The directed subgraph must therefore contain a cycle, and hence the (undirected) bipartite graph of σ contains a cycle, too.

((iii) $\Rightarrow$ (i)) If $\sigma \notin$ ext E, then σ is a convex combination of two other elements of E. The bipartite graph of each of these elements is then a subgraph of the bipartite graph of σ.

Suppose now that the population measure is not purely atomic, and let σ be an equilibrium of a large crowding game g. Consider the game structure in which the set of representative players is I_0, the (nonatomic) population measure μ^N is defined by $\mu^N(B) = \mu(B)/\mu(I_0)$ (B a measurable subset of I_0), and the set of actions is $\mathcal{A}$. Let g^N be the large crowding game defined over this game structure by the payoff functions $g^N_{ij}(x) = g_{ij}(\mu(I_0)x + \int_{I \setminus I_0} \sigma_j d\mu)$. If $\mu(I \setminus I_0) > 0$, then consider also the game structure in which the set of representative players is $I \setminus I_0$, the (purely atomic) population measure μ^A is defined by $\mu^A(B) = \mu(B)/\mu(I \setminus I_0)$ (B a measurable subset of $I \setminus I_0$), and the set of actions is $\mathcal{A}$. Let g^A be the large crowding game defined over this game structure by the payoff functions $g^A_{ij}(x) = g_{ij}(\mu(I \setminus I_0)x + \int_{I_0} \sigma_j d\mu)$. It is not difficult to see that the restriction of σ to I_0 is an equilibrium of g^N and that the restriction of σ to $I \setminus I_0$ is an equilibrium of g^A. The equilibrium action distributions of these games are therefore $(1/\mu(I_0)) \int_{I_0} \sigma d\mu$ and $(1/\mu(I \setminus I_0)) \int_{I \setminus I_0} \sigma d\mu$, respectively. Conversely, if σ^N is an equilibrium of g^N and σ^A is an equilibrium of g^A, then the strategy profile whose restriction to I_0 is equal to σ^N and whose restriction to $I \setminus I_0$ is equal to σ^A is an equilibrium of g. This equilibrium is acyclic if and only if σ^A is acyclic and σ^N is an equilibrium in pure strategies. It therefore follows from the first part of the proof and Theorem 3.1 that g has an acyclic equilibrium. If σ itself is acyclic and quasi-strict, then $\sigma|_{I \setminus I_0}$ is acyclic and quasi-strict and $\sigma|_{I_0}$ is strict. In this case, the restriction of any equilibrium σ' of g to I_0 must be equal to $\sigma|_{I_0}$, and therefore $\sigma'|_{I \setminus I_0}$, like $\sigma|_{I \setminus I_0}$, is an equilibrium of g^A. It then follows from the first part of the proof that $\sigma'|_{I \setminus I_0} = \sigma|_{I \setminus I_0}$. Hence, $\sigma' = \sigma$. $\square$

Lemma 4.2. *The (upper semicontinuous) function $f : \mathcal{G} \to \mathbb{R}$, which sends each large crowding game to the mean number of best-response actions for the players in that game, at equilibrium, is continuous at a large crowding game g if and only if g has an acyclic and quasi-strict (and hence a unique) equilibrium.*

Proof. In light of Lemma 4.1, it suffices to show that f is continuous at a large crowding game g if and only if all equilibria of g are quasi-strict. Let σ be an equilibrium of g. If σ is not quasi-strict, then there are a subset I' of I of positive measure and an action j such that, for every $i \in I'$, $g_{ij}(\int \sigma_j) = \max_{k \neq j} g_{ik}(\int \sigma_k)$ but $\sigma_j(i) = 0$. For $n = 1, 2, \ldots$, let $g^{(n)} \in \mathcal{G}$ be defined as follows: $g_{ij}^{(n)} = g_{ij} - 1/n$ if $i \in I'$, and $g_{ik}^{(n)} = g_{ik}$ if $i \notin I'$ or $k \neq j$. Clearly, σ is an equilibrium of $g^{(n)}$, for all $n \geq 1$. For every $i \in I$, $BR(i, g^{(n)}, \int \sigma d\mu) \subseteq BR(i, g, \int \sigma d\mu)$, and if $i \in I'$, then the inclusion is strict. Therefore, $f(g^{(n)}) \leq f(g) - \mu(I')$. Since $g^{(n)} \to g$, the function f has a discontinuity at g.

Conversely, if f has a discontinuity at g, then it follows from the upper semicontinuity of f that there is a sequence $g^{(1)}, g^{(2)}, \ldots$ of large crowding games, converging almost everywhere to g, such that $\lim_n f(g^{(n)}) < f(g)$. For $n \geq 1$, let $\sigma^{(n)}$ be an equilibrium of $g^{(n)}$. For $m \geq 1$, let i_m be a representative player such that, for almost every $i \in I_m$ and every n, $\sigma^{(n)}(i) = \sigma^{(n)}(i_m)$ and $BR(i, g^{(n)}, e(g^{(n)})) = BR(i_m, g^{(n)}, e(g^{(n)}))$. Suppose that $\#BR(i, g, e(g)) = 1$ for almost every $i \in I_0$. (If not, then every acyclic equilibrium of g is *not* quasi-strict, and the proof is complete.) For almost every $i \in I_0$, $\lim_n \#BR(i, g^{(n)}, e(g^{(n)}))$ and $\lim_n \sigma^{(n)}(i)$ exist, and are equal to 1 and to the unique element of $BR(i, g, e(g))$, respectively. By passing to a subsequence, if necessary, it may be assumed that $\lim_n \#BR(i, g^{(n)}, e(g^{(n)}))$ and $\lim_n \sigma^{(n)}(i)$ also exist for almost every $i \notin I_0$. The strategy profile $\lim_n \sigma^{(n)}$ is an equilibrium of g (because if $\sigma^{(n)}(i) \in \text{co } BR(i, g^{(n)}, \int \sigma^{(n)} d\mu)$ for all n, then this relation holds in the limit as well). Since $\lim_n f(g^{(n)}) < f(g)$ and $\lim_n \#BR(i, g^{(n)}, e(g^{(n)})) = \#BR(i, g, e(g))$ $(= 1)$ for almost every $i \in I_0$, there must be at least one $m \geq 1$ such that $\lim_n \#BR(i_m, g^{(n)}, e(g^{(n)})) < \#BR(i_m, g, e(g))$. For every n, the number of actions j such that $\sigma_j^{(n)}(i_m) > 0$ does not exceed $\#BR(i_m, g^{(n)}, e(g^{(n)}))$. Therefore, the number of actions j such that $\lim_n \sigma_j^{(n)}(i_m) > 0$ does not exceed $\lim_n \#BR(i_m, g^{(n)}, e(g^{(n)}))$, and is hence less than $\#BR(i_m, g, e(g))$. It follows that the equilibrium $\lim_n \sigma^{(n)}$ is not quasi-strict. □

The generic uniqueness of equilibrium in large crowding games is an immediate corollary of Lemma 4.2.

Theorem 4.3. *The set of all large crowding games with an acyclic and quasi-strict, and hence a unique, equilibrium is a dense G_δ in $\mathcal{G}$.*

If the population measure is nonatomic, then "acyclic and quasi-strict" is obviously equivalent to "strict." At the other extreme, if the set I of representative players is finite, then the set of all large crowding games with an acyclic and quasi-strict, and hence a unique, equilibrium is, in fact, a dense open set. This follows from the fact that, in this case, the range of the function f is finite, and the set of its points of continuity is therefore open.

If the number of representative players is finite, and the population measure assigns positive mass to all of them, then an equilibrium σ of a large crowding games g is acyclic and quasi-strict if and only if

$$\sum_{i \in I} [\#BR(i, g, e(g)) - 1] = \alpha - c, \tag{3}$$

where c is the number of connected components in the bipartite graph of σ. This follows from the well-known fact that the number of edges in a finite tree is equal to the number of vertices minus one. (If σ is not acyclic or quasi-strict, then the left-hand side of (3) is greater than the right-hand side.) It follows from Theorem 4.3 and (3) that, generically, $\sum_{i \in I} [\#BR(i, g, e(g)) - 1] \leq \alpha - 1$.

A number of potential applications of this result for community ecology are given in Milchtaich (1996b). In that paper, each element of I represents a distinct animal species. The number of individuals of each species is very large. The elements of $\mathcal{A}$ are habitats in which individual animals can search for food or other resources. The quantity or quality of food that an individual of species i is able to find in habitat j, which is given by the payoff function g_{ij}, declines with the total number of individual competitors in that habitat. The *degree of generalism* of animal species i is the quotient $Gen(i) = [\#BR(i, g, e(g)) - 1] \,/\, (\alpha - 1)$. An animal species whose degree of generalism is 0 is a *specialist.* For a specialist, there is, at equilibrium, only one optimal habitat in which individuals of that species look for food. At the other extreme, an animal species whose degree of generalism is 1 is a *full generalist.* For a full generalist, all habitats are equally suitable at equilibrium. It follows from Theorem 4.3 and (3) that the average generalism $Gen = (1/\#I)\sum_{i\in I} Gen(i)$ does not generically exceed $1/\#I$, the reciprocal of the number of species. Hence, if the number of species is large, then the average generalism is likely to be low. In particular, the number of animal species that are *not* specialists is generically less than the number α of habitats.

If the number of representative players is finite, and the population measure assigns positive mass to all of them, then a necessary and sufficient condition for a large crowding game to have an acyclic and quasi-strict (and hence a unique) equilibrium is that (3) holds for some (and hence all) equilibria of that game. An alternative proof of Theorem 4.3, in this special case, could therefore proceed by showing that, generically, every equilibrium σ of a large crowding game g satisfies that equation. This can be demonstrated *heuristically* by comparing the number of equations with the number of unknowns. For each representative player i, there are $\#BR(i, g, e(g)) - 1$ equations, expressing the fact that all actions that, at equilibrium, are optimal for the players represented by i yield the same payoff. The total weight of the players choosing each action is an "unknown." These unknowns are not independent, however. In each connected component of the bipartite graph of σ they must sum up to the measure of the set of representative players that belong to that connected component. Hence, for each connected component, the number of independent unknowns is equal to the number of actions that belong to that connected component minus one. Heuristically, the total number of equations, which is given by the left-hand side of (3), cannot exceed the total number of independent unknowns, which is given by the right-hand side. On the other hand, as mentioned above, the right-hand side never exceeds the left-hand side, either. Hence, equality should hold.

5. Types of players and equilibrium distributions. The space $\mathcal{T}$ of *types of players*, which is a special version of the space of players' characteristics introduced by Mas-Colell (1984), consists of all α-tuples of continuous and strictly decreasing real-valued functions over the unit interval. The *distance* $\rho(\tau, \tau')$ between two types of players, $\tau = (\tau_1, \tau_2, \ldots, \tau_\alpha)$ and $\tau' = (\tau'_1, \tau'_2, \ldots, \tau'_\alpha)$, is defined as $\max_{j\in\mathcal{A}, 0\le x\le 1} \left|\tau_j(x) - \tau'_j(x)\right|$. Since the space $C[0,1]$ of all continuous real-valued functions over the unit interval is separable and complete (with respect to the supremum norm), and since the strictly decreasing continuous functions constitute a G_δ in $C[0,1]$ (which is the intersection of all open sets of the form $\{f \in C[0,1] \mid f(x) > f(y)\}$, x and y rational numbers such that $0 \le x < y \le 1$), the metric space $(\mathcal{T}, \rho)$ is separable and topologically complete (Kuratowski, 1966, § 33, VI). A complete metric ρ^* on $\mathcal{T}$ that induces the same topology as ρ is given by

$$\rho^*(\tau, \tau') = \rho(\tau, \tau') + \sum_{n=1}^{\infty} 2^{-n} \min\left\{1, \max_{j\in\mathcal{A}} \left|\frac{1}{\tau_j(x_n) - \tau_j(y_n)} - \frac{1}{\tau'_j(x_n) - \tau'_j(y_n)}\right|\right\},$$

where $\{(x_n, y_n)\}_{n\ge 1}$ is the collection of all pairs of rational numbers such that $0 \le x_n < y_n \le 1$.

The set $\mathcal{G}$ of all large crowding games that are defined over a fixed game structure can be identified with the set of all measurable functions from I to $\mathcal{T}$, where the measurable structure on the latter set is the collection $\mathcal{B}(\mathcal{T})$ of all Borel sets. Specifically, a game $g \in \mathcal{G}$ will be identified with the function $i \mapsto (g_{i1}, g_{i2}, \ldots, g_{i\alpha})$. The measurability of this function is equivalent to the measurability condition in the definition of a large crowding game. This follows from the fact that a function $h : I \to C[0,1]$ is measurable if and only if, for every $0 \le x \le 1$, the real-valued function $i \mapsto h(i)(x)$ is measurable (see Billingsley, 1968, p. 57). Standard arguments show that the metric d^* on $\mathcal{G}$ defined by

$$d^*(g, g') = \int_I \min\{1, \rho^*((g_{i1}, g_{i2}, \ldots, g_{i\alpha}), (g'_{i1}, g'_{i2}, \ldots, g'_{i\alpha}))\}\, d\mu(i)$$

is complete. Almost everywhere convergence of sequences in $\mathcal{G}$ implies convergence in the d^* topology, and is equivalent to it if I is countable. Conversely, if $g^{(n)} \to g$ in the d^* topology, then, for every $\epsilon > 0$, there is an integer N such that, for every $n > N$, the measure of the set of all representative players i such that $\rho^*((g_{i1}^{(n)}, g_{i2}^{(n)}, \ldots, g_{i\alpha}^{(n)}), (g_{i1}, g_{i2}, \ldots, g_{i\alpha})) > \epsilon$ is less than ϵ. It follows that there is an increasing sequence $\{n_m\}_{m\ge 1}$ of positive integers such that, for almost every i, $\rho^*((g_{i1}^{(n_m)}, g_{i2}^{(n_m)}, \ldots, g_{i\alpha}^{(n_m)}), (g_{i1}, g_{i2}, \ldots, g_{i\alpha})) \to 0$, and hence $g_{ij}^{(n_m)}(x) \to g_{ij}(x)$ uniformly in j and x. Therefore, $g^{(n)} \to g$ in the d^* topology if and only if every subsequence of $\{g^{(n)}\}$ has itself a subsequence that converges almost everywhere to g. The d^* topology is thus completely determined by convergence almost everywhere of sequences. Exactly the same arguments (with ρ replacing ρ^* throughout) show that $g^{(n)} \to g$ in the d topology (which is defined in Section 2) if and only if every subsequence of $\{g^{(n)}\}$ has a sub-subsequence that converges almost everywhere to g. Therefore, the metrics d^* and d induce the same topology on $\mathcal{G}$.

The *type distribution* of a large crowding game g is the measure $\mu \circ g^{-1}$. This measure is an element of $\Delta(\mathcal{T})$, the space of all (Borel) probability measures on $\mathcal{T}$. For every Borel set $B \subseteq \mathcal{T}$, $(\mu \circ g^{-1})(B) = \mu(\{i \in I \mid (g_{i1}, g_{i2}, \ldots, g_{i\alpha}) \in B\})$. Both $\Delta(\mathcal{T})$ and $\Delta(\mathcal{T} \times \mathcal{A})$, the space of all (Borel) probability measures on the product space $\mathcal{T} \times \mathcal{A}$, are metrizable, separable, and topologically complete with respect to the topology of weak convergence of measures (Parthasarathy, 1967, Chapter II, Theorems 6.2 and 6.5). The *type–action distribution* of a large crowding game g and a strategy profile σ is the element η of $\Delta(\mathcal{T} \times \mathcal{A})$ defined by

$$\eta(B \times \{j\}) = \int_{g^{-1}(B)} \sigma_j(i) d\mu(i) \qquad (B \in \mathcal{B}(\mathcal{T}), j \in \mathcal{A}).$$

The marginal of η on $\mathcal{T}$, denoted $\eta_{\mathcal{T}}$, is equal to the type distribution of g. The marginal of η on $\mathcal{A}$, denoted $\eta_{\mathcal{A}}$, is equal to the action distribution of σ.

A probability measure $\eta \in \Delta(\mathcal{T} \times \mathcal{A})$ is an *equilibrium distribution* if it is supported in the set $\{(\tau, j) \in \mathcal{T} \times \mathcal{A} \mid \tau_j(\eta_{\mathcal{A}}(\{j\})) = \max_{k \in \mathcal{A}} \tau_k(\eta_{\mathcal{A}}(\{k\}))\}$. If η is supported in the set $\{(\tau, j) \in \mathcal{T} \times \mathcal{A} \mid \tau_j(\eta_{\mathcal{A}}(\{j\})) > \max_{k \neq j} \tau_k(\eta_{\mathcal{A}}(\{k\}))\}$, then it will be called a *strict* equilibrium distribution. If $\nu \in \Delta(\mathcal{T})$ is the marginal on $\mathcal{T}$ of an equilibrium distribution η, then η will be said to be an equilibrium distribution *of* ν and the marginal of η on $\mathcal{A}$ will be said to be an equilibrium action distribution of ν. The connection between the set of equilibria of a large crowding game and the set of equilibrium distributions of its type distribution is given by the next lemma. This connection is studied at greater length and generality by Green (1984) and Rath (1995). It holds not only for large crowding games, but also for more general games in which players' payoffs depend only on their own actions and on the aggregate actions of others (cf. Theorem 2 of Green, 1984). However, even if the population measure is nonatomic, the lemma would not be true if "equilibrium" were replaced by "equilibrium in pure strategies"; cf. the main result (Theorem 8) of Rath (1995).

LEMMA 5.1. *Suppose that the marginal of $\eta \in \Delta(\mathcal{T} \times \mathcal{A})$ on $\mathcal{T}$ is equal to the type distribution of a large crowding game g. Then η is an equilibrium distribution if and only if there is an equilibrium σ of g such that η is the type–action distribution of g and σ. In this case, η is a strict equilibrium distribution if and only if σ is a strict equilibrium.*

PROOF. For $j \in \mathcal{A}$, define a nonnegative (Borel) measure ν_j on $\mathcal{T}$ by $\nu_j(B) = \eta(B \times \{j\})$. The probability measure η is equal to the type–action distribution of g and a strategy profile σ if and only if, for every j,

$$\nu_j(B) = \int_{g^{-1}(B)} \sigma_j(i) d\mu(i) \qquad (B \in \mathcal{B}(\mathcal{T})). \tag{4}$$

It is an equilibrium distribution if and only if, for every j, $\nu_j(\mathcal{T}_j) = 0$, where $\mathcal{T}_j = \{\tau \in \mathcal{T} \mid \tau_j(\nu_j(\mathcal{T})) < \max_{k \neq j} \tau_k(\nu_k(\mathcal{T}))\}$. A strategy profile σ is an equilibrium of g if and only if, for every j, $\sigma_j(i) = 0$ for almost every $i \in I_j$, where $I_j = \{i \in I \mid g_{ij}(\int \sigma_j) < \max_{k \neq j} g_{ik}(\int \sigma_k)\}$. If (4) holds, then $\nu_j(\mathcal{T}) = \int \sigma_j$, and therefore $I_j = g^{-1}(\mathcal{T}_j)$. Then (again by (4)), $\nu_j(\mathcal{T}_j) = 0$ if and only if $\sigma_j(i) = 0$ for almost every $i \in I_j$. This proves that if η is equal to the type–action distribution of g and a strategy profile σ, then η is an equilibrium distribution if and only if σ is an equilibrium of g. In this case, almost the same proof (replacing the strict inequalities in the definitions of $\mathcal{T}_j$ and I_j by weak inequalities) shows that η is strict if and only if σ is strict.

Suppose now that η is an equilibrium distribution. Then, since $\sum_j \nu_j = \mu \circ g^{-1}$, for every $j \in \mathcal{A}$ there is a nonnegative version of the Radon-Nykodim derivative $d\nu_j/d(\mu \circ g^{-1})$ that is equal to zero in $\mathcal{T}_j$, such that $\sum_j (d\nu_j/d(\mu \circ g^{-1})) = 1$ identically. By the change-of-variable formula, and by the definition of the Radon-Nykodim derivative, the function $\sigma_j = (d\nu_j/d(\mu \circ g^{-1})) \circ g$ satisfies (4). Hence, η is the type–action distribution of g and the strategy profile σ defined by $\sigma(i) = (\sigma_1(i), \sigma_2(i), \ldots, \sigma_\alpha(i))$. Since, for every $j \in \mathcal{A}$, $\sigma_j = 0$ in $g^{-1}(\mathcal{T}_j)$ and $g^{-1}(\mathcal{T}_j) = I_j$, σ is an equilibrium of g. $\square$

It follows from Lemma 5.1 that two large crowding games with equal type distributions also have equal equilibrium action distributions. In fact, the following stronger result holds: The supremum-norm distance between the equilibrium action distributions of any two large crowding games (even games with different sets of representative players or different population measures) does not exceed the supremum-norm distance between their type distributions. (The supremum-norm distance $\|\lambda - \lambda'\|$ between two probability measures λ and λ' on the same measurable space $(X, \mathcal{B})$ is defined as $\sup_{B \in \mathcal{B}} |\lambda(B) - \lambda'(B)|$.) This result is an immediate corollary of Lemma 5.1 and the following proposition.

PROPOSITION 5.2. *If $\hat{\eta}, \tilde{\eta} \in \Delta(\mathcal{T} \times \mathcal{A})$ are equilibrium distributions, then $\|\hat{\eta}_{\mathcal{A}} - \tilde{\eta}_{\mathcal{A}}\| \leq \|\hat{\eta}_{\mathcal{T}} - \tilde{\eta}_{\mathcal{T}}\|$.*

PROOF. Substituting $\hat{p}_j$ for $\hat{\eta}_{\mathcal{A}}(\{j\})$ and $\tilde{p}_j$ for $\tilde{\eta}_{\mathcal{A}}(\{j\})$, it suffices to show that

$$\sum_{\substack{j \in \mathcal{A} \\ \hat{p}_j > \tilde{p}_j}} (\hat{p}_j - \tilde{p}_j) \leq (\hat{\eta}_{\mathcal{T}} - \tilde{\eta}_{\mathcal{T}}) \left(\left\{ \tau \in \mathcal{T} \mid \max_{k \in \mathcal{A}} \tau_k(\hat{p}_j) < \max_{k \in \mathcal{A}} \tau_k(\tilde{p}_j) \right\} \right). \tag{5}$$

The difference between the right- and left-hand sides of (5) is

$$\begin{aligned}
& (\hat{\eta} - \tilde{\eta}) \left(\left\{ (\tau, j) \in \mathcal{T} \times \mathcal{A} \mid \max_{k \in \mathcal{A}} \tau_k(\hat{p}_j) < \max_{k \in \mathcal{A}} \tau_k(\tilde{p}_j) \right\} \right) - (\hat{\eta} - \tilde{\eta}) \left(\{ (\tau, j) \in \mathcal{T} \times \mathcal{A} \mid \hat{p}_j > \tilde{p}_j \} \right) \\
= \; & (\hat{\eta} - \tilde{\eta}) \left(\left\{ (\tau, j) \mid \hat{p}_j \leq \tilde{p}_j \text{ and } \max_{k \in \mathcal{A}} \tau_k(\hat{p}_j) < \max_{k \in \mathcal{A}} \tau_k(\tilde{p}_j) \right\} \right) \\
& - (\hat{\eta} - \tilde{\eta}) \left(\left\{ (\tau, j) \mid \hat{p}_j > \tilde{p}_j \text{ and } \max_{k \in \mathcal{A}} \tau_k(\tilde{p}_j) \leq \max_{k \in \mathcal{A}} \tau_k(\hat{p}_j) \right\} \right) \\
\geq \; & -\tilde{\eta} \left(\left\{ (\tau, j) \mid \tau_j(\tilde{p}_j) \leq \tau_j(\hat{p}_j) < \max_{k \in \mathcal{A}} \tau_k(\tilde{p}_j) \right\} \right) - \hat{\eta} \left(\left\{ (\tau, j) \mid \tau_j(\hat{p}_j) < \tau_j(\tilde{p}_j) \leq \max_{k \in \mathcal{A}} \tau_k(\hat{p}_j) \right\} \right).
\end{aligned}$$

Since $\hat{\eta}$ and $\tilde{\eta}$ are equilibrium distributions, the last two expressions are equal to zero. □

Another corollary of Lemma 5.1 is that every probability measure on $\mathcal{T}$ that is the type distribution of some large crowding game g has an equilibrium distribution and a unique equilibrium action distribution which coincides with that of g. This implies that *every* element of $\Delta(\mathcal{T})$ has at least one equilibrium distribution and a unique equilibrium action distribution. The reason is that there exist a set of representative players and a population measure which make every probability measure on (the separable and topologically complete metric space) $\mathcal{T}$ the type distribution of some large crowding game. Indeed, by Skorokhod's representation theorem (Billingsley, 1971, p. 8), if I is the unit interval $[0, 1]$ and μ is Lebesgue measure on I, then for every $\nu \in \Delta(\mathcal{T})$ there is a large crowding game g such that $\nu = \mu \circ g^{-1}$. Furthermore, a sequence $\{\nu^{(n)}\}$ converges (weakly) to ν if and only if there is a sequence $\{g^{(n)}\}$ of large crowding games that satisfies $\nu^{(n)} = \mu \circ (g^{(n)})^{-1}$ for every n and converges almost everywhere to g. It follows that, for this set of representative players and this population measure, the function $g \mapsto \mu \circ g^{-1}$, which sends each large crowding game to its type distribution, is continuous, open, and onto $\Delta(\mathcal{T})$. In view of this fact, *all* elements of $\Delta(\mathcal{T})$ may be referred to as "type distributions."

A type distribution that has a strict equilibrium distribution does not have any other equilibrium distribution. For if it did, then, by Lemma 5.1 and the above remarks, it would be the type distribution of some large crowding game with two distinct equilibria, one of them strict. However, by Lemma 4.1, this is impossible. Therefore, to prove that a type distribution generically has *precisely* one equilibrium distribution, it suffices to show that, generically, a type distribution has a strict equilibrium distribution.

THEOREM 5.3. *Every type distribution has at least one equilibrium distribution. The set of all type distributions that are nonatomic and have a strict, and hence a unique, equilibrium distribution is a dense G_δ in $\Delta(\mathcal{T})$.*

PROOF. Let I be the unit interval, and μ Lebesgue measure on I. In the following, $\mathcal{G}$ refers to the space of large crowding games defined over this set of representative players and this population measure. By Lemma 3.4, the function $f : \mathcal{G} \to \mathbb{R}$, defined by $f(g) = \int \#BR(i, g, e(g)) d\mu(i)$, is upper semicontinuous. If $g, \hat{g} \in \mathcal{G}$ are such that $\mu \circ g^{-1} = \mu \circ \hat{g}^{-1}$, then, by the remark that follows Lemma 5.1, $e(g) = e(\hat{g})$, and therefore $f(g) = \sum_{j \in \mathcal{A}} (\mu \circ g^{-1})(\{\tau \in \mathcal{T} \mid \tau_j(e_j(g)) = \max_{k \in \mathcal{A}} \tau_k(e_k(g))\}) = f(\hat{g})$. It follows that there exists a unique function $\hat{f} : \Delta(\mathcal{T}) \to \mathbb{R}$ such that $f(g) = \hat{f}(\mu \circ g^{-1})$ for every $g \in \mathcal{G}$. Since, as shown above, the function that sends each large crowding game to its type distribution is continuous, open, and onto $\Delta(\mathcal{T})$, the function $\hat{f}$ is upper semicontinuous, and is continuous at $\mu \circ g^{-1}$ if and only if f is continuous at g (see Kuratowski, 1966, § 13, XV). By Lemma 4.2 and the fact that Lebesgue measure is nonatomic, f is continuous at a large crowding game g if and only if that game has a strict equilibrium. It therefore follows from Lemma 5.1 that $\hat{f}$ is continuous at $\mu \circ g^{-1}$ if and only if that type distribution has a strict equilibrium distribution. The set of all type distributions with a strict equilibrium distribution is therefore a dense G_δ. Since $\mathcal{T}$ is separable, topologically complete, and dense in itself, the set of all nonatomic probability measures on $\mathcal{T}$ is also a dense G_δ in $\Delta(\mathcal{T})$ (Parthasarathy, 1967, Chapter II, Corollary 8.1). The intersection of these two sets is again a dense G_δ. $\square$

The closedness of the graph and, under certain conditions, the upper semicontinuity of the equilibrium correspondences on certain topological spaces of large games, modeling the anonymous interactions of large numbers of players, is the main result (Theorem 3) of Green (1984). For the particular case under consideration, namely, for the equilibrium distribution correspondence, it is not difficult to prove these results directly. The proof of the following proposition is given at the end of Section 6.

Proposition 5.4. *The correspondence that sends each type distribution to the set of its equilibrium distributions is compact- and convex-valued and upper semicontinuous and has a closed graph. The set of all equilibrium distributions is closed in $\Delta(\mathcal{T} \times \mathcal{A})$.*

It follows as a corollary from Proposition 5.4 that the function $\Delta(\mathcal{T}) \to \Delta(\mathcal{A})$ that sends each type distribution to its unique equilibrium action distribution is continuous (with respect to the topology of weak convergence of measures). This result can also be deduced from Lemma 3.4. Note that, by Proposition 5.2, this function is also Lipschitz continuous with respect to the supremum-norm distances for probability measures on $\mathcal{T}$ and on $\mathcal{A}$.

6. Finite crowding games and the convergence of their sets of equilibria. For a positive integer n, an *n-person crowding game* is defined over a game structure that consists of a finite set I of representative players, a population measure μ, which is a probability measure on I such that $\mu(\{i\})\,n$ is a positive integer for every representative player i (this integer is interpreted as the number of identical players represented by i), and a finite set $\mathcal{A}$ of actions. An n-person crowding game g assigns a continuous and strictly decreasing payoff function $g_{ij} : [0,1] \to \mathbb{R}$ to every representative player i and action j. (This definition of an n-person crowding game is less general than the one in Milchtaich, 1996a, 1998, where the payoff functions are only assumed to be nonincreasing. It is tailored to the application at hand.) An *equilibrium* of an n-person crowding game g is a strategy profile $\sigma : I \to \Delta(\mathcal{A})$ such that, for every i and every j, $\sigma_j(i)\mu(\{i\})n$ is an integer (interpreted as the number of players represented by i who choose action j), and if $\sigma_j(i){>}0$, then $g_{ij}(\int \sigma_j) \geq \max_{k \neq j} g_{ik}(\int \sigma_k + 1/n)$. It is shown in Milchtaich (1996a, Theorem 2) that the first part of Theorem 3.1 also holds for finite, n-person crowding games.

Theorem 6.1. *Every n-person crowding game has at least one equilibrium.*

For every n-person crowding game there is a corresponding large crowding game, which is defined over the same game structure and has the same payoff functions as the finite game. This large crowding game can be seen as the limit of the sequence of m-replicas of the n-person game. For a positive integer m, the m-replica of an n-person crowding game $g^{(1)}$ is the mn-person crowding game $g^{(m)}$ that is defined over the same game structure and has the same payoff functions as $g^{(1)}$.

More generally, suppose that g is a large crowding game that is defined over a game structure such that the set of representative players is finite and the population measure assigns positive, rational mass to all of them. For $m = 1, 2, \ldots$, let n_m be a positive integer and let $g^{(m)}$ be an n_m-person crowding game that is defined over the same game structure as g. If $g_{ij}^{(m)}(x) \to g_{ij}(x)$ for every i, j, and x, and if $n_m \to \infty$, then the sequence $\{g^{(m)}\}$ will be said to *converge* to g. The following proposition shows that, in this case, if g has a unique equilibrium, then the sets of equilibria of $g^{(m)}$ converge (in the topological sense) to the singleton consisting of the unique equilibrium of g. In particular, if $g^{(1)}$ is a finite crowding game such that the corresponding large crowding game g has a unique equilibrium (which, by Theorem 4.3, is generically the case) then, if m is large enough, all the equilibria of the m-replica of $g^{(1)}$ are arbitrarily close to the unique equilibrium of g.

Proposition 6.2. *Let g be a large crowding game with a unique equilibrium. For $m = 1, 2, \ldots$, let $g^{(m)}$ be an n_m-person crowding game that is defined over the same game structure as g and let $\sigma^{(m)}$ be an equilibrium of $g^{(m)}$. If $\{g^{(m)}\}$ converges to g, then the sequence $\{\sigma^{(m)}\}$ converges (pointwise) to the unique equilibrium of g.*

Proof. By passing, if necessary, to a subsequence, it may be assumed that the sequence $\sigma^{(1)}, \sigma^{(2)}, \ldots$ is convergent. We have to show that the limit, σ, is an equilibrium of g. For every i and every j, $\sigma_j(i)>0$ only if $\sigma_j^{(m)}(i)>0$, and hence $g_{ij}^{(m)}(\int \sigma_j^{(m)}) \geq \max_{k\neq j} g_{ik}^{(m)}(\int \sigma_k^{(m)}+1/n_m)$, for all m large enough. Therefore, $\sigma_j(i)>0$ implies $g_{ij}(\int \sigma_j) \geq \max_{k\in\mathcal{A}} g_{ik}(\int \sigma_k)$. $\square$

If $g^{(1)}, g^{(2)}, \ldots$ is the sequence of m-replicas of a finite crowding game $g^{(1)}$, and the large crowding game g is the limit of this sequence, then the corresponding sequence of type distributions converges to the type distribution of g. In fact, $\mu \circ (g^{(m)})^{-1} = \mu \circ g^{-1}$ for all m. Consider now *any* sequence of either finite or large crowding games whose type distributions converge to some limit ν. The next proposition will show that if ν has a unique equilibrium distribution (which, by Theorem 5.3, is generically the case), and if the games in the sequence are all large crowding games or are such that the number of players tends to infinity, then the sets of type–action distributions of these games and their equilibria converge (in the topological sense) to the singleton consisting of the unique equilibrium distribution of ν.

For $\epsilon \geq 0$, a probability measure η on $\mathcal{T}\times\mathcal{A}$ that is supported in the set $\{(\tau,j)\in\mathcal{T}\times\mathcal{A} \mid \tau_j(\eta_{\mathcal{A}}(\{j\})) \geq \max_{k\in\mathcal{A}} \tau_k(\min\{1, \eta_{\mathcal{A}}(\{k\})+\epsilon\})\}$ will be called an *ϵ-equilibrium distribution.* If $\nu \in \Delta(\mathcal{T})$ is the marginal on $\mathcal{T}$ of an ϵ-equilibrium distribution η, then η will be said to be an ϵ-equilibrium distribution *of* ν. An equilibrium distribution (which is a 0-equilibrium distribution) is also an ϵ-equilibrium distribution for every $\epsilon>0$. Therefore, for all $\epsilon \geq 0$, every $\nu \in \Delta(\mathcal{T})$ has at least one ϵ-equilibrium distribution. The type–action distribution of an n-person crowding game g and an equilibrium σ of g is an $1/n$-equilibrium distribution, more specifically, an $1/n$-equilibrium distribution of the type distribution $\mu\circ g^{-1}$ of g.

Proposition 6.3. *Let $\nu^{(1)}, \nu^{(2)}, \ldots$ be a sequence in $\Delta(T)$ that converges to a limit ν, and let $\epsilon_1, \epsilon_2, \ldots$ be a sequence of nonnegative real numbers that converges to a limit ϵ ($<\infty$). For $n \geq 1$, let $\eta^{(n)}$ be an ϵ_n-equilibrium distribution of $\nu^{(n)}$. If ν has a unique ϵ-equilibrium distribution, then the sequence $\{\eta^{(n)}\}$ converges to it.*

Proof. It follows from the relative compactness of $\{\nu^{(n)}\}$ and the compactness of (the finite set) $\mathcal{A}$ that $\{\eta^{(n)}\}$ is relatively compact (see Billingsley, 1968, Section 6), and thus has a converging subsequence. It may therefore be assumed, without loss of generality, that the sequence itself converges. We have to show that the limit, $\eta^{(0)}$, is an ϵ-equilibrium distribution.

Let ϵ_0 be equal to ϵ and, for $n \geq 0$, let the function $h_n : \mathcal{T}\times\mathcal{A} \to \mathbb{R}$ be defined by $h_n(\tau, j) = \tau_j(\eta_{\mathcal{A}}^{(n)}(\{j\})) - \max_{k\in\mathcal{A}} \tau_k(\min\{1, \eta_{\mathcal{A}}^{(n)}(\{k\})+\epsilon_n\})$. If $\{(\tau^{(n)}, j^{(n)})\}$ is a sequence in $\mathcal{T}\times\mathcal{A}$ that converges to some limit (τ, j), then $h_n(\tau^{(n)}, j^{(n)}) \to h_0(\tau, j)$. It follows, by theorem 5.5 of Billingsley (1968), that $\eta^{(n)}\circ h_n^{-1} \to \eta^{(0)}\circ h_0^{-1}$, and therefore $\limsup_n (\eta^{(n)}\circ h_n^{-1})([0,\infty)) \leq (\eta^{(0)}\circ h_0^{-1})([0,\infty))$. The condition that $\eta^{(n)}$ is an ϵ_n-equilibrium distribution is equivalent to $(\eta^{(n)}\circ h_n^{-1})([0,\infty)) = 1$. This condition is assumed to hold for all $n \geq 1$. It follows that it also holds for $n = 0$. $\square$

Proof of Proposition 5.4. It is shown as part of the proof of Proposition 6.3 that if $\nu^{(1)}, \nu^{(2)}, \ldots$ is a convergent sequence in $\Delta(\mathcal{T})$ and, for every n, $\eta^{(n)}$ is an equilibrium distribution of $\nu^{(n)}$, then $\{\eta^{(n)}\}$ has a subsequence that converges to an equilibrium distribution (of $\lim_n \nu^{(n)}$). This proves that the set of all equilibrium distributions is closed. Also, the special case in which $\nu^{(1)} = \nu^{(2)} = \cdots$ shows that the correspondence that sends each type distribution to the set of its equilibrium distributions is compact-valued. The general case shows that it is upper semicontinuous (Hildenbrand, 1974, p. 24) and has a closed graph. The fact that this correspondence is convex-valued follows from Lemma 5.1 and the fact that the set of equilibria of a large crowding game is convex. $\square$

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