COMPUTATION OF COMPLETELY MIXED EQUILIBRIUM PAYOFFS IN BIMATRIX GAMES

IGAL MILCHTAICH

Department of Economics, Bar-Ilan University,
Ramat Gan 52900, Israel
milchti@mail.biu.ac.il
http://faculty.biu.ac.il/~milchti

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A formula is presented for computing the equilibrium payoffs in a generic finite two-person game when the support of the equilibrium is known. Keywords: Equilibrium; Computation; Two-person games. *JEL Classification Number:* C72

Computing the (Nash) equilibrium payoffs in a given bimatrix game (i.e., a finite two-person game in strategic form) is a problem of considerable practical importance. One algorithm that can be used for this purpose is the Lemke-Howson algorithm (Lemke and Howson (1964); von Stengel (2002)), which is guaranteed to find one equilibrium. Another, more elementary, approach is to compute the equilibrium payoffs by: (1) "guessing" the support of the equilibrium, i.e., the set of pure strategies each player uses with positive probability (which can always be done by systematically checking all possibilities, if necessary); (2) finding a pair of mixed strategies with this support, such that all the pure strategies designated to either player give that player the same payoff against the other player's mixed strategy; (3) computing the corresponding payoffs; and, finally, (4) checking that none of the other pure strategies of either player gives that player a higher payoff. For an equilibrium with known support, the equilibrium payoffs are easier to compute. Indeed, it is shown below that, in the case of known support, steps (1), (2), and (4) can essentially be dispensed with. Examples of bimatrix games having equilibria with known supports include generalized rock–scissors–paper games, such as the (symmetric) game with payoff matrix

$$\begin{pmatrix} 0 & 5 & -2 \\ -2 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix}, \tag{1}$$

in which the unique equilibrium is symmetric and completely mixed; and 2×2 games with two strict equilibria, such as

$$\begin{pmatrix} 3,-1 & -3,-3 \\ -2, 1 & 1, 5 \end{pmatrix}, \tag{2}$$

which always also have a third equilibrium that is completely mixed. Certain types of zero-sum two-person games (namely, quasicyclic games and games with Minkowski-Leontief payoff matrices; see Vorob'ev (1994), Chapt. 4, Sec. 5) are also known to posses completely mixed equilibria. Other examples arise in studies of comparative statics, where the question is how the equilibrium payoffs change with changes in the underlying parameters. Typically, if the parameters do not change too much, a new equilibrium can be expected to exist that has the same support as the original one.

From a practical point of view, the problem of computing the equilibrium payoffs for equilibria with known supports essentially reduces to that of computing the equilibrium payoffs for completely mixed equilibria in bimatrix games with the same number of pure strategies for both players (and, hence, square payoff matrices). The reason is that, generically, different equilibria of a bimatrix game have distinct supports, and in each equilibrium, both players use the same number of pure strategies.[1] The simple formula presented below (Eq. (3)) can be used for computing the completely mixed equilibrium payoffs in such games. For each of the two players, the formula (normally) gives a unique number, which equals the player's payoff in every completely mixed equilibrium in the game (if it has one). It differs from other ways of computing the completely mixed equilibrium payoffs in that: (i) it presents them in a relatively simple closed form, which may be particularly important if the payoff matrices include unknown values or parameters; (ii) it does not involve finding the inverse or the adjoint matrices; and (iii) it enables simplifying the computation by exploiting certain regularities in the payoff matrices, such as elements that repeat several times. In the formula, $|B|$ denotes the determinant of square matrix B, and E is the square matrix with all entries 1.

Lemma. *Let A be a square matrix, and x and y distinct real numbers, with $|A - y\,E| \neq 0$. In any bimatrix game in which the payoff matrix of one of the players is given by A, and in any equilibrium in the game in which this player's strategy is completely mixed, the player's payoff λ is given by the formula*

$$\frac{\lambda - x}{\lambda - y} = \frac{|A - x\,E|}{|A - y\,E|}. \tag{3}$$

[1] There exist, however, bimatrix games in which the equilibria do not have these properties. An example is the reduced strategic form of the extensive game in which the first-moving player has to choose between playing matching pennies with the other player and receiving a sure payoff of ε, where $0 < \varepsilon < 1$. The first player's unique equilibrium strategy is choosing the latter alternative. The other player, by contrast, has a continuum of equilibrium strategies, which are all completely mixed.

Proof. Suppose, without loss of generality (see Remark 1 below), that the $n \times n$ matrix A is the payoff matrix of the row player. In any equilibrium in which this player's strategy is completely mixed, he would get the same expected payoff λ by choosing any of his pure strategies. Thus, the column player's strategy u is such that $A\,u = \lambda\,e$, where e is the column vector with all entries 1. Since the sum of the entries of (the column vector) u is 1, this equation is equivalent to $(A - \lambda\,E)\,u = 0$, and it therefore implies that $|A - \lambda\,E| = 0$. Now, the function $f(x) = |A - x\,E|$ is affine. Indeed,

$$f(x) = \begin{vmatrix} a_{11}-x & a_{12}-x & \cdots & a_{1n}-x \\ a_{21}-x & a_{22}-x & \cdots & a_{2n}-x \\ \vdots & \vdots & & \vdots \\ a_{n1}-x & a_{n2}-x & \cdots & a_{nn}-x \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}-x & a_{12}-x & \cdots & a_{1n}-x \\ a_{21}-a_{11} & a_{22}-a_{12} & \cdots & a_{2n}-a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1}-a_{11} & a_{n2}-a_{12} & \cdots & a_{nn}-a_{1n} \end{vmatrix}$$

$$= |A| - x \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_{21}-a_{11} & a_{22}-a_{12} & \cdots & a_{2n}-a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1}-a_{11} & a_{n2}-a_{12} & \cdots & a_{nn}-a_{1n} \end{vmatrix}.$$

Therefore, the quotient $(f(x) - f(\lambda))/(x - \lambda)$ has the same value for all $x \neq \lambda$. Since $f(\lambda) = 0$, this gives (3). □

For the purpose of actual computations, the numbers x and y in (3) should be chosen in such a way that the computation of the two determinants is as easy as possible. For example, if A is given by (1), then one may take $x = -2$ and $y = 0$. It is then easily seen that the right-hand side of (3) equals 14 and, therefore, the equilibrium payoff is given by $\lambda = 2/13$. In (2), the completely mixed equilibrium payoffs of both players may be computed by taking $x = -3$ and $y = 1$. This gives that both payoffs are equal to $-1/3$. These examples demonstrate the usefulness of this method for "small" games, which are the ones most often encountered in practice, and for which the equilibrium payoffs can be computed by hand.

Remarks

1. It makes no difference whether the player whose payoff matrix is A is the row or the column player. Indeed, replacing A by the transpose matrix A^{T} would not change the right-hand side of (3).

2. A bimatrix game with square payoff matrices can have more than one completely mixed equilibrium. However, it follows from the Lemma that, unless a given player's payoff matrix A is such that $|A - y\,E| = 0$ for all real numbers y, the player's payoff is the same in all the completely mixed equilibria.

3. If the matrix $A - y\,E$ is singular (i.e., $|A - y\,E| = 0$) for all values of y, the uniqueness of the completely mixed equilibrium payoffs need not hold. For example, in the symmetric 4×4 game with payoff matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix},$$

any $1/2 < \lambda < 1$ is a completely mixed symmetric equilibrium payoff (with the symmetric equilibrium strategy $(\lambda - 1/2, 1 - \lambda, 1 - \lambda, \lambda - 1/2)$). A bimatrix game that has more than one completely mixed equilibrium always has also at least one equilibrium that is not completely mixed (Raghavan (1970), Theorem 4).

4. If $|A - y\,E|$ is a *nonzero* constant, then completely mixed equilibria do not exist. Indeed, in this case, each of the player's completely mixed strategies is strictly dominated, since it is not a best response against any strategy of the other player.

5. Another formula for computing the completely mixed equilibrium payoff λ can be obtained from (3) by multiplying both sides by $\lambda - y$ and taking the limit as y tends to λ. This gives

$$\lambda = x - \frac{|A - x\,E|}{|\hat{A}|}, \tag{4}$$

where

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}.$$

It is not difficult to see that the determinant of the $(n + 1) \times (n + 1)$ matrix $\hat{A}$ is equal to the slope of the affine function f defined in the proof of the Lemma. Provided that this determinant is not zero, the formula (4) holds for all values of x. In particular, choosing $x = 0$ gives $\lambda = -|A| / |\hat{A}|$.

6. One application of the above lemma is the following result, the proof of which is left as an exercise to the reader:

For any $n \geq 2$ and any real numbers $a, b, c_1, c_2, \ldots, c_n$ with $a \neq b$ and $a, b < c_i$ for all i, the value λ of the two-person zero-sum game with payoff matrix

$$\begin{pmatrix} c_1 & b & b & \cdots & b \\ a & c_2 & b & \cdots & b \\ \vdots & & \ddots & & \vdots \\ a & a & \cdots & c_{n-1} & b \\ a & a & \cdots & a & c_n \end{pmatrix}$$

is given by the equation

$$\frac{\lambda - a}{\lambda - b} = \prod_{i=1}^{n} \frac{c_i - a}{c_i - b}.$$

It follows from this equation that the value of this game satisfies $\max\{a, b\} < \lambda < \min\{c_1, c_2, \ldots, c_n\}$.

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