On Some Saddle Point Matrices and Applications to Completely Mixed Equilibrium in Bimatrix Games

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Abstract

In this paper we focus on saddle point matrices with two vector blocks and their applications in game theory. Necessary and sufficient conditions for the existence and uniqueness of a completely mixed Nash equilibrium in a bimatrix game are presented. These conditions and formulas for computing the equilibrium and equilibrium payoffs are expressed by saddle point matrices.

Keywords: Saddle Point Matrix, Bimatrix Game, Completely Mixed Equilibrium

Introduction

The first part of the paper presents several properties of saddle point matrices with two vector blocks. In these block matrices, the top-left block is a real square matrix A, the top-right block is the column vector $\boldsymbol{e}$ with all entries 1, the bottom-left block is $\boldsymbol{e}$ transposed, and the bottom-right block has the single entry 0. If A is symmetric, the saddle point matrix can be interpreted as the bordered Hessian of a standard quadratic program over the standard simplex and it is usually called the Karush-Kuhn-Tucker matrix of the program. For a review of the many applications of such matrices, see [3].

Game theory models in mathematical language problems of strategic decision making arising in economics and other social sciences. A bimatrix, or finite two-player game is one of the fundamental constructs of game theory. In such a game, the payoff of each player is a bilinear function of the two players' mixed strategies. In a game-theoretic application of saddle point matrices, the top-left block is the payoff matrix of one of the two players in a bimatrix game.

The computational problem of finding a Nash equilibrium in a bimatrix game has received much attention (for example, see [9]). Completely mixed equilibria in such games, and especially in symmetric bimatrix games (sometimes called evolutionary games) have also been extensively studied. In a completely mixed equilibrium, all the entries in each player's mixed strategy are strictly positive. Blackwell [2] used such equilibria to give an alternative proof of the Perron-Frobenius theorem. Raghavan [8] showed that several characterizations of M-matrices can be deduced from the theory of completely mixed equilibria.

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Not every bimatrix game has a completely mixed Nash equilibrium. However, such equilibria always exist for certain classes of bimatrix games, for example, games with Minkowski-Leontief payoff matrices and quasicyclic games [11], games in which the payoff matrix is an M-matrix [4], and generalized rock-scissors-paper games [5].

The prevalence of bimatrix games, and the fact that for at least some classes of games a completely mixed equilibrium is guaranteed to exist, mean that a quick method for checking whether a given game has a completely mixed equilibrium and simple formulas for computing the equilibrium and equilibrium payoffs may come in handy. Such closed-form formulas may also be useful for games involving unknown values or parameters. For example, in studies of comparative statics one is often interested in the way the equilibrium changes with changing underlying parameters. If the game has a unique completely mixed equilibrium, this remains so as long as the parameters do not change too much. Note, however, that if the number of pure strategies is not the same for the two players, then in a generic bimatrix game a completely mixed equilibrium does not exist. For this reason, the analysis below only concerns games with an equal number of pure strategies, that is, square payoff matrices.

This paper complements and extends some earlier works of the authors. For bimatrix games, Milchtaich [6] presented a formula for computing the equilibrium payoffs when the support of the equilibrium is known. Ostrowski [7] presented formulas for computing the completely mixed strategy and the equilibrium payoff in a symmetric bimatrix game.

Saddle Point Matrices

In this section, some notation and basic properties of saddle point matrices are presented. The proofs are given in the Appendix.

The column vector and the square matrix with all entries 1 are denoted by $\boldsymbol{e} \in \mathbb{R}^n$ and $E \in \mathbb{R}^{n,n}$, respectively. Thus, $E = \boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}$. For a (real) matrix $A \in \mathbb{R}^{n,n}$, the saddle point matrix $(A, \boldsymbol{e})$ is defined by

$$(A, \boldsymbol{e}) = \begin{bmatrix} A & \boldsymbol{e} \\ \boldsymbol{e}^{\mathrm{T}} & 0 \end{bmatrix}. \tag{1}$$

(This is a shortened version of the notation (A, B, C, D) that is sometimes used to denote square partitioned matrices with four blocks.) If A is nonsingular, its Schur complement in (1) is given by

$$(A, \boldsymbol{e})/A = -\boldsymbol{e}^{\mathrm{T}} A^{-1} \boldsymbol{e}. \tag{2}$$

The matrices obtained from A by replacing the k-th column or row ($k = 1, 2, \dots, n$) with the column vector $\boldsymbol{e}$ or row vector $\boldsymbol{e}^{\mathrm{T}}$ are denoted by A_k or A^k, respectively. For square matrices A and B, the notation $A \cong B$ means that the matrices are congruent, i.e., $B = P^{\mathrm{T}} A P$ for some nonsingular matrix P.

Theorem 1. For a square matrix $A \in \mathbb{R}^{n,n}$ and scalars $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}\det(A, \boldsymbol{e}) &= \det A - \det(A+E) = \det(A-E) - \det A \\ &= -\sum_{i=1}^{n} \det A_i = -\sum_{i=1}^{n} \det A^i,\end{aligned} \tag{3}$$

$$\det(\alpha A + \beta E, \boldsymbol{e}) = \alpha^{n-1} \det(A, \boldsymbol{e}) \tag{4}$$

and

$$\det(\alpha A + \beta E) = \alpha^{n-1}(\alpha \det A - \beta \det(A, \boldsymbol{e})). \tag{5}$$

Theorem 2. For a nonsingular $A \in \mathbb{R}^{n,n}$,

$$\det(A, \boldsymbol{e}) = -(\boldsymbol{e}^{\mathrm{T}} A^{-1} \boldsymbol{e}) \det A \tag{6}$$

and

$$\operatorname{rank}(A, \boldsymbol{e}) = \begin{cases} n, & \text{if } \boldsymbol{e}^{\mathrm{T}} A^{-1} \boldsymbol{e} = 0 \\ n+1, & \text{if } \boldsymbol{e}^{\mathrm{T}} A^{-1} \boldsymbol{e} \neq 0 \end{cases} \tag{7}$$

If A is in addition symmetric, then moreover

$$(A, \boldsymbol{e}) \cong \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{e}^{\mathrm{T}} A^{-1} \boldsymbol{e} \end{bmatrix}. \tag{8}$$

If A is symmetric and positive definite, then $(A, \boldsymbol{e})$ is nonsingular and has one negative eigenvalue and n positive ones.

Bimatrix Games

A bimatrix, or finite two-player strategic game is given by an ordered pair of payoff matrices, $A = [a_{ij}]$ and $B = [b_{ij}]$, with equal dimensions. If the row and column players choose their i-th and j-th pure strategies, respectively, the row player's payoff is a_{ij} and the column player's payoff is b_{ij}. A mixed strategy is a column probability vector $\boldsymbol{x} = [x_i]$ that specifies the probability with which each pure strategy is played. If all the probabilities are strictly positive, $\boldsymbol{x}$ is said to be completely mixed. This paper only concerns games in which the number of pure strategies is the same for both players, so that A and B are square, $n \times n$ matrices, with $n \geq 2$.

A pair of mixed strategies $(\boldsymbol{x}, \boldsymbol{y})$ is a Nash equilibrium in a bimatrix game $[A, B]$ if

> $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{y} \geq \boldsymbol{p}^{\mathrm{T}} A \boldsymbol{y}$ for all strategies $\boldsymbol{p}$ for the row player and
>
> $\boldsymbol{x}^{\mathrm{T}} B \boldsymbol{y} \geq \boldsymbol{x}^{\mathrm{T}} B \boldsymbol{q}$ for all strategies $\boldsymbol{q}$ for the column player.

A completely mixed Nash equilibrium is one in which both players' strategies are completely mixed. In the special case of a symmetric bimatrix game, i.e., a game of the form $[A, A^{\mathrm{T}}]$, a mixed strategy $\boldsymbol{x}$ is a symmetric Nash equilibrium strategy if

> $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} \geq \boldsymbol{p}^{\mathrm{T}} A \boldsymbol{x}$ for all strategies $\boldsymbol{p}$.

Theorem 3. For $A, B \in \mathbb{R}^{n,n}$ with $\det(A, \boldsymbol{e}), \det(B, \boldsymbol{e}) \neq 0$, the bimatrix game $[A, B]$ has at most one completely mixed Nash equilibrium. The equilibrium strategies of the row player and the column player, respectively, are equal to the vectors $\boldsymbol{x}(B)$ and $\boldsymbol{y}(A)$ defined by

$$x_i(B) = -\frac{\det B^i}{\det(B, \boldsymbol{e})} \text{ and } y_i(A) = -\frac{\det A_i}{\det(A, \boldsymbol{e})} \quad (i = 1,2, \dots, n). \tag{9}$$

The equilibrium payoffs of the row player and the column player, respectively, are the scalars $u(A)$ and $u(B)$ defined by

$$u(A) = -\frac{\det A}{\det(A, \boldsymbol{e})} \text{ and } u(B) = -\frac{\det B}{\det(B, \boldsymbol{e})}. \tag{10}$$

In the special case $B = A^{\mathrm{T}}$, i.e., a symmetric bimatrix game $[A, A^{\mathrm{T}}]$, the completely mixed equilibrium is symmetric: both players use the strategy $\boldsymbol{y}(A)$ and receive the payoff $u(A)$.

If $(A, \boldsymbol{e})$ or $(B, \boldsymbol{e})$ are singular, the game $[A, B]$ may have multiple completely mixed Nash equilibria, possibly with different payoffs. For example, in the symmetric bimatrix game $[A, A^{\mathrm{T}}]$ with

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 \end{bmatrix},$$

the strategy $[1 - p, p - 1/2, 1 - p, p - 1/2]^{\mathrm{T}}$ is a completely mixed symmetric equilibrium strategy for any $p \in (1/2, 1)$, with the equilibrium payoff p. However, even if $(A, \boldsymbol{e})$ or $(B, \boldsymbol{e})$ are singular, multiplicity of completely mixed Nash equilibria is possible only if A or B, respectively, are also singular. Otherwise, a completely mixed equilibrium does not exist. For example, this is so for the symmetric bimatrix game $[A, A^{\mathrm{T}}]$ with

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix},$$

which satisfies

$$\det(A, \boldsymbol{e}) = \det \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 0 \text{ but } \det \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \neq 0.$$

Clearly, a completely mixed equilibrium does not exist in this game, in which one strategy is dominating.

Theorem 4. A necessary and sufficient condition for the existence of a unique completely mixed Nash equilibrium in a bimatrix game $[A, B]$ is that the following inequalities hold:

$$\det A_i \cdot \det A_j > 0 \text{ and } \det B^i \cdot \det B^j > 0 \quad (i, j = 1,2, \dots, n). \tag{11}$$

Appendix

Proof of Theorem 1. The first and second equalities in (3) are special cases of (5), obtained by setting $\alpha = 1$ and $\beta = \pm 1$ and rearranging. They therefore follow from the general case, which is considered below. The third equality is obtained by computing the determinant of $(A, \boldsymbol{e})$ using Laplace expansion about the last row and changing the order of columns in the resulting determinants, keeping in mind the resulting changes of signs. The fourth equality is similarly obtained by expanding $(A, \boldsymbol{e})$ about the last column.

Eq. (4) holds trivially if $\alpha = 0$. If $\alpha \neq 0$, then

$$\det(\alpha A + \beta E, \boldsymbol{e}) = \det\begin{bmatrix} \alpha A + \beta E & \boldsymbol{e} \\ \boldsymbol{e}^{\mathrm{T}} & 0 \end{bmatrix} = \det\begin{bmatrix} \alpha A & \boldsymbol{e} \\ \boldsymbol{e}^{\mathrm{T}} & 0 \end{bmatrix}$$
$$= \alpha^n \det\begin{bmatrix} A & \frac{1}{\alpha}\boldsymbol{e} \\ \boldsymbol{e}^{\mathrm{T}} & 0 \end{bmatrix} = \alpha^{n-1} \det\begin{bmatrix} A & \boldsymbol{e} \\ \boldsymbol{e}^{\mathrm{T}} & 0 \end{bmatrix} = \alpha^{n-1} \det(A, \boldsymbol{e}),$$

where the second equality is obtained by subtracting the last column multiplied by β from each of the other columns in $(\alpha A + \beta E, \boldsymbol{e})$ and the third and fourth equalities use the multilinearity of the determinant.

To prove (5), we differentiate its left-hand side with respect to β:

$$\frac{d}{d\beta} \det(\alpha A + \beta E) = \sum_{i=1}^{n} \det(\alpha A + \beta E)_i = \sum_{i=1}^{n} \det(\alpha A)_i = -\det(\alpha A, \boldsymbol{e}).$$

Since $\det(\alpha A + \beta E) = \det(\alpha A)$ for $\beta = 0$, this gives that

$$\det(\alpha A + \beta E) = \det(\alpha A) - \beta \det(\alpha A, \boldsymbol{e}) = \alpha^n \det A - \beta \alpha^{n-1} \det(A, \boldsymbol{e}),$$

where the second inequality uses (4). ∎

Proof of Theorem 2. For a nonsingular matrix A, Eq. (6) follows from the Schur determinant formula and (2). Eq. (7) follows immediately from (6), or alternatively from the Guttman rank additivity formula (see [12]). If A is in addition symmetric, direct computation (see also [10]) gives that

$$(A, \boldsymbol{e}) = P^{\mathrm{T}} \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{e}^{\mathrm{T}} A^{-1} \boldsymbol{e} \end{bmatrix} P,$$

where

$$P = \begin{bmatrix} I & A^{-1}\boldsymbol{e} \\ \mathbf{0} & 1 \end{bmatrix}.$$

This proves (8). If A is symmetric and positive definite, then the same holds for A^{-1}, and therefore $\boldsymbol{e}^{\mathrm{T}} A^{-1} \boldsymbol{e} > 0$. It then follows from (8) (see also [1]) that $(A, \boldsymbol{e})$ is nonsingular and has a single negative eigenvalue, which is $-\boldsymbol{e}^{\mathrm{T}} A^{-1} \boldsymbol{e}$. ∎

Proof of Theorem 3. In an equilibrium in $[A, B]$ in which the row player's strategy $\boldsymbol{x}$ is completely mixed, every pure strategy would give that player the same (equilibrium) payoff u against the column player's strategy $\boldsymbol{y}$. Thus,

$$A\boldsymbol{y} = u\boldsymbol{e}.$$

Since $\boldsymbol{y}$ is a probability vector,

$$\boldsymbol{e}^{\mathrm{T}}\boldsymbol{y} = 1.$$

In matrix form, these equations can be written as

$$\begin{bmatrix} A & \boldsymbol{e} \\ \boldsymbol{e}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ -u \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix}. \tag{12}$$

Since by assumption $\det(A, \boldsymbol{e}) \neq 0$, it follows almost immediately from (12) by Cramer's rule that

$$y_i = \frac{\det(A_i, \boldsymbol{e})}{\det(A, \boldsymbol{e})} \quad (i = 1,2, \dots, n)$$

and

$$-u = \frac{\det A}{\det(A, \boldsymbol{e})}.$$

The row player's equilibrium strategy and the column player's equilibrium payoff satisfy similar equations, in which A is replaced with B^{T}. This is because transposition of the payoff matrix B makes the row player a column player and vice versa. Since, by (3), for all i

$$\det(A_i, \boldsymbol{e}) = \det(A_i - E) - \det A_i = -\det A_i,$$

and

$$\det\left(B^{\mathrm{T}}\right)_i = \det\left(B^i\right)^{\mathrm{T}} = \det B^i,$$

this proves (9) and (10). ■

Proof of Theorem 4. Suppose that (11) holds. By (3), this implies that both saddle point matrices $(A, \boldsymbol{e})$ and $(B, \boldsymbol{e})$ are nonsingular and that the vectors $\boldsymbol{x}(B)$ and $\boldsymbol{y}(A)$ defined by (9) are strictly positive probability vectors. As shown in the proof of Theorem 3, $\boldsymbol{y}(A)$ and the scalar $u(A)$ defined in (10) together solve (12), and $\boldsymbol{x}(B)$ and $u(B)$ together solve a similar matrix equation in which A is replaced with B^{T}. This shows that $(\boldsymbol{x}(B), \boldsymbol{y}(A))$ is a completely mixed Nash equilibrium in $[A, B]$. By Theorem 3, it is in fact the unique such equilibrium.

It remains to prove the necessity of condition (11). If the game has a completely mixed equilibrium $(\boldsymbol{x}, \boldsymbol{y})$, then as shown in the proof of Theorem 3, $\boldsymbol{y}$ and the row player's equilibrium payoff u solve (12). If the equilibrium is unique, then $\boldsymbol{y}$ and u moreover constitute an isolated solution of (12). This is because replacing them with any other pair of a strictly positive vector $\boldsymbol{y}'$ and a scalar u' that satisfy (12) would give a different completely mixed Nash equilibrium $(\boldsymbol{x}, \boldsymbol{y}')$, which by assumption does not exist. The existence of an isolated solution implies that $\det(A, \boldsymbol{e}) \neq 0$. Similar considerations give that $\det(B, \boldsymbol{e}) \neq 0$. Therefore, by Theorem 3, $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}(B), \boldsymbol{y}(A))$. Condition (11) now follows from (9). ■

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