Crowding Games are Sequentially Solvable*

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Abstract

A sequential-move version of a given normal-form game Γ is an extensive-form game of perfect information in which each player chooses his action after observing the actions of all players who precede him and the payoffs are determined according to the payoff functions in Γ. A normal-form game Γ is sequentially solvable if each of its sequential-move versions has a subgame-perfect equilibrium in pure strategies such that the players' actions on the equilibrium path constitute an equilibrium of Γ.

A crowding game is a normal-form game in which the players share a common set of actions and the payoff a particular player receives for choosing a particular action is a nonincreasing function of the total number of players choosing that action. It is shown that every crowding game is sequentially solvable. However, not every pure-strategy equilibrium of a crowding game can be obtained in the manner described above. A sufficient, but not necessary, condition for the existence of a sequential-move version of the game that yields a given equilibrium is that there is no other equilibrium that Pareto dominates it.

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A *crowding game* (or a congestion game with player-specific payoff functions; Milchtaich, 1996) is an n-person normal-form game Γ where each player i chooses one action j from a common finite set A of actions, and receives a payoff which is a nonincreasing function S_{ij} of the total number n_j of players choosing j. $(n_j)_{j\in A}$ is called the *congestion vector*. The actions chosen by the players constitute a (pure-strategy Nash) *equilibrium* of Γ if the action σ_i chosen by each player i is a best response for that player against the actions chosen by the other players. Formally, an *action profile* $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in A^n$ is an equilibrium if and only if, for every player i and action j, $S_{i\sigma_i}(n_{\sigma_i}) \geq S_{ij}(n_j + 1)$. Every symmetric crowding game is also a congestion game, as defined in Rosenthal (1973; see below). The class of all congestion games coincides with the class of potential games (Monderer and Shapley, 1996). This implies, in particular, that every such game has at least one Nash equilibrium in pure strategies. Nonsymmetric crowding games, however, generally do not admit a potential function.

Given any n-person normal-form game Γ, a *sequential-move* version of Γ is an extensive-form game of perfect information in which the n players are put in some fixed order, each player chooses his action after observing the choices of all players who precede him, and the payoffs are determined according to the payoff functions in Γ. There are $n!$ ways of ordering the players, and each order defines a different sequential-move version of Γ. As is well known, an extensive-form game of perfect information always has at least one subgame-perfect equilibrium in pure strategies. We will say that the (simultaneous-move) game Γ is *sequentially solvable* if each of its sequential-move versions has at least one subgame-perfect equilibrium in pure strategies such that the players' actions on the equilibrium path constitute a (Nash) equilibrium of Γ.[1] The main result of this paper is the following:

Theorem 1. *Every crowding game is sequentially solvable.*

It follows, in particular, from Theorem 1 that every crowding game has a Nash equilibrium in pure strategies. For a computationally efficient algorithm for finding such an equilibrium, and for a discussion of convergence to an equilibrium, see Milchtaich (1996). At least one of the (pure-strategy) equilibria of every crowding game is a strong equilibrium (Konishi et al., 1997a). This is not generally true for

[1] An alternative term that comes to mind is "Stackelberg solvable". However, this term was used in another meaning by d'Aspremont and Gérard-Varet (1980). Stackelberg solvability, as defined by these authors, neither implies nor is implied by sequential solvability.

congestion games. See, however, Holzman and Law-Yone (1997), were sufficient conditions for the existence of a strong equilibrium in such games are obtained.

Sequential-move equilibria

By Theorem 1, every sequential-move version of a given crowding game Γ has a subgame-perfect equilibrium in pure strategies such that the action profile played on the equilibrium path is an equilibrium of Γ. Such an action profile will be said to be a *sequential-move* equilibrium of Γ. The question naturally arises, whether *all* equilibria of a given crowding game are sequential-move equilibria. The following example shows that the answer, in general, is "no."

Consider a two-person, two-action crowding game where $S_{11}(1) > S_{12}(1) > 0$, $S_{22}(1) > S_{21}(1) > 0$, and $0 > S_{ij}(2)$ for every i and j. This game has two equilibria: $(1,2)$ and $(2,1)$. However, only the first equilibrium is a sequential-move equilibrium; there is no subgame-perfect equilibrium of a sequential-move version of this game in which the second action profile is played.

The second equilibrium of the above game is Pareto dominated by the first one.[2] As we will presently see, this is no accident: every equilibrium of a crowding game which is *not* Pareto dominated by another equilibrium of the game is a sequential-move equilibrium.

In every crowding game, an equilibrium which is not Pareto dominated by another equilibrium is strong in the sense of Aumann (1959): there exists no group of players who can increase their payoffs by simultaneously changing their actions, if the rest of the players do not change their actions. This follows from the fact that if an equilibrium σ is not strong, so that there is an action profile $\sigma' \neq \sigma$ such that, for every player i for which $\sigma'_i \neq \sigma_i$, player i's payoff when σ' is played is strictly higher than his payoff when σ is played, then σ' is an equilibrium and it Pareto dominates σ. To prove this fact, note first that the congestion vector $(n'_j)_{j\in A}$ corresponding to σ' is equal to the congestion vector $(n_j)_{j\in A}$ corresponding to σ. Indeed, $n'_j > n_j$ for some j would imply that there is a player, i, for which $\sigma'_i = j$ and $\sigma_i \neq j$. But it follows from our assumption concerning σ' and from the monotonicity of S_{ij} that in such a case $S_{i\sigma_i}(n_{\sigma_i}) < S_{ij}(n'_j) \leq S_{ij}(n_j+1)$—a contradiction to the assumption that σ is an equilibrium. Since the two congestion vectors are equal, for every player i for

[2] An action profile is *Pareto dominated* by another action profile if there is no player whose payoff is strictly higher when the first action profile is played and there is at least one player whose payoff is strictly higher when the second action profile is played.

which $\sigma'_i = \sigma_i$, player i's payoff when σ' is played is equal to his payoff when σ is played. It follows that σ' Pareto dominates σ. Also, since σ is an equilibrium, $S_{i\sigma'_i}(n'_{\sigma'_i}) \geq S_{i\sigma_i}(n_{\sigma_i}) \geq S_{ij}(n_j+1) = S_{ij}(n'_j+1)$ for every player i and action j. Hence, σ', too, is an equilibrium.

For generic[3] crowding games the converse is also true: an equilibrium σ that is Pareto dominated by another equilibrium σ' is not strong. The reason is that, in such a case, for every player i for which $\sigma'_i \neq \sigma_i$, player i's payoff when σ' is played cannot be equal to, and is therefore strictly higher than, his payoff when σ is played.

Theorem 2. *Every strong equilibrium of a crowding game* Γ *is a sequential-move equilibrium of* Γ.

The converse of Theorem 2 is false; not every sequential-move equilibrium of a crowding game is strong. Consider, for example, the sequential-move version of a three-person, three-action generic crowding game where $S_{11}(1) > S_{12}(1) > S_{12}(2) > 0$, $S_{22}(1) > S_{23}(1) > 0$, $S_{32}(1) > S_{31}(1) > S_{31}(2) > 0$, and $0 > S_{ij}(m)$ otherwise and the players move in the order 1, 2, 3. If player 1 plays 1, then player 2 will play 2 and player 3 will play 1. If player 1 plays 2, then player 2 will play 3 and player 3 will play 1. Player 1's payoff is negative in the former case and positive in the latter case. The unique sequential-move equilibrium corresponding to this sequential-move version is therefore $(2,3,1)$. However, this equilibrium is not strong; and it is Pareto dominated by the other equilibrium of the simultaneous-move game, $(1,3,2)$.

When a crowding game has only one sequential-move equilibrium, that equilibrium may be said to be *commitment robust* (Rosenthal, 1991). It follows as a corollary from Theorem 2 that a necessary condition for an equilibrium of a crowding game to be commitment robust is that it Pareto dominates all other equilibria. However, as the last example shows, this condition is not sufficient for commitment robustness.

[3] A crowding game will be said to be *generic* if, for every i, j, j', and $1 \leq m, m' \leq n$, $j \neq j'$ implies $S_{ij}(m) \neq S_{ij'}(m')$.

Other games

Rosenthal (1973) introduced a class of (congestion) games that are similar in some respects to the games considered here. In this class of games, the set of strategies of each player is not the set A of actions but some subset of the power set of A; each player chooses a *combination* of actions. A player's payoff is the sum of the payoffs associated with each of the actions included in his choice. The payoff associated with action j is an arbitrary function c_j of the number n_j of players who include j in their choice. Rosenthal (1973) proved that each game in this class has a Nash equilibrium in pure strategies. Nevertheless, the following example shows that in this class of games Theorems 1 and 2 do not hold.

Consider a three-person symmetric game where the set of actions is $A = \{1,2,3,4\}$ and the common set of strategies is $\{\{1\},\{2\},\{3\},\{1,4\},\{2,3,4\}\}$. Suppose that $c_1(1) = 10$, $c_1(2) = 0$, $c_2(1) = 3$, $c_3(1) = 2$, $c_4(1) = -1$, and $c_j(m) < -10$ otherwise. A sequential-move version of this game has a unique subgame-perfect equilibrium, in which the first player plays $\{1,4\}$, the second player plays $\{2\}$, and the third player plays $\{3\}$. This, however, is not an equilibrium of the simultaneous-move game: the first player can increase his payoff from 9 to 10 by unilaterally shifting to $\{1\}$. And $(\{1\},\{2\},\{3\})$ is a strong equilibrium of the simultaneous-move game. But if the first player played $\{1\}$ in the sequential-move version of the game then the second player would play $\{2,3,4\}$ and the third player would play $\{1\}$, and so the payoff of the first player would in this case be zero.

Another closely related class of games in which the main result of the present paper does not hold is games in which crowding has a *positive* effect on players' payoffs (Konishi et al., 1997b). Consider, for example, a three-player, three-action game where the following inequalities hold: $S_{11}(3) > S_{12}(3) > S_{12}(2) > S_{11}(2) > S_{11}(1) > S_{12}(1) > 0$, $S_{22}(3) > S_{23}(3) > S_{23}(2) > S_{22}(2) > S_{22}(1) > S_{23}(1) > 0$, $S_{33}(3) > S_{31}(3) > S_{31}(2) > S_{33}(2) > S_{33}(1) > S_{31}(1) > 0$, and $0 > S_{ij}(m)$ otherwise. As can be readily verified, this game does not even have a pure-strategy Nash equilibrium.

Proofs

It suffices to prove Theorems 1 and 2 for generic crowding games. The reason is that, for every crowding game Γ, every sequential-move version $\hat{\Gamma}$ of Γ, and every generic crowding game Γ' close enough to Γ,[4] the unique subgame-perfect equilibrium of the sequential-move version of Γ' in which the order of players is the same as in $\hat{\Gamma}$ is also a subgame-perfect equilibrium of $\hat{\Gamma}$, and every equilibrium of Γ' is also an equilibrium of Γ. Also, for every strong equilibrium σ of Γ, Γ' can be chosen in such a way that σ is also a strong equilibrium of Γ'. Notice, however, that a sequential-move version of a non-generic crowding game may possess pure-strategy subgame-perfect equilibria whose equilibrium paths do not correspond to equilibria of the simultaneous-move game.

Every subgame of a sequential-move version $\hat{\Gamma}$ of a given generic crowding game Γ is itself a sequential-move version of some generic crowding game. Specifically, suppose that the players move in the order $1,2,\ldots,n$. The subgame of $\hat{\Gamma}$ determined by a given path $(\sigma_1,\sigma_2,\ldots,\sigma_i)$ $(1 \le i \le n)$ is a sequential-move version of a unique $(n-i)$-person generic crowding game, which will be called the *subgame* of Γ determined by $(\sigma_1,\sigma_2,\ldots,\sigma_i)$. The equilibrium path $(\sigma_{i+1},\sigma_{i+2},\ldots,\sigma_n)$ of the unique subgame-perfect equilibrium of the subgame of $\hat{\Gamma}$ determined by $(\sigma_1,\sigma_2,\ldots,\sigma_i)$ will be called the *subgame equilibrium path* determined by $(\sigma_1,\sigma_2,\ldots,\sigma_i)$. The congestion vector corresponding to the action profile $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ will be referred to as the congestion vector determined by $(\sigma_1,\sigma_2,\ldots,\sigma_i)$. Note that, since in a crowding game a player's payoff is affected only by the number of other players playing each action, the same subgame, subgame equilibrium path, and congestion vector are determined by $(\sigma_{\tau(1)},\sigma_{\tau(2)},\ldots,\sigma_{\tau(i)})$, where τ is an arbitrary permutation of $\{1,2,\ldots,i\}$. The proof of Theorem 1 depends on the following lemma:

Lemma. *In a generic crowding game, let $(n_j)_{j\in A}$ be the congestion vector determined by (σ_1) and let $(n'_j)_{j\in A}$ be the congestion vector determined by (σ'_1) $(\sigma_1,\sigma'_1 \in A)$. Then, for every j, $n'_j \le n_j+1$, and equality holds for at most one j, which is* not σ_1.

Proof of Theorem 1. Let Γ be a generic crowding game. It suffices to show that the action profile σ played on the equilibrium path of the unique subgame-perfect equilibrium of the sequential-move version of Γ in which the players move in the order $1,2,\ldots,n$ is an equilibrium of Γ. If σ is not an equilibrium then, for some i and j, $S_{ij}(n_j+1) > S_{i\sigma_i}(n_{\sigma_i})$, where $(n_k)_{k\in A}$ is the congestion vector correspond-

[4] It suffices to assume that strict preferences are not reversed in Γ', i.e., that $S_{ij}(m) > S_{ij'}(m')$ implies $S'_{ij}(m) \ge S'_{ij'}(m')$ for every i, j, j', m, and m', where $(S'_{ij})_{j\in A}$ are the functions that determine player i's payoff in Γ'.

ing to σ. Without loss of generality, $i = 1$. (If not, then consider not Γ but the subgame determined by $(\sigma_1, \sigma_2, \ldots, \sigma_{i-1})$.) Set $\sigma'_1 = j$, and let $(n'_k)_{k \in A}$ be the congestion vector determined by (σ'_1). By the lemma, and the monotonicity of S_{1j}, $S_{1j}(n'_j) \geq S_{1j}(n_j + 1) > S_{1\sigma_1}(n_{\sigma_1})$. But this contradicts the assumption that σ is played on the equilibrium path: player 1 could have achieved more by playing j (instead of σ_1). $\square$

Proof of the lemma. The proof proceeds by induction on the number n of players. The lemma is evidently true if $n = 1$. Assume that it holds true for every $(n-1)$-person generic crowding game, and let an n-person ($n \geq 2$) generic crowding game be given. It is always possible to view the game as a subgame of an $(n+1)$-person generic crowding game, more specifically the subgame determined by (σ_0), the play of the 0-th player. Moreover, it may be assumed that σ_1 and σ'_1 are different from σ_0, and that σ_0 is such that $S_{i\sigma_0}(1) < S_{ij}(n+1)$ for every i and $j \neq \sigma_0$. The latter assumption implies that, in every equilibrium of every subgame of the $(n+1)$-person game, no player plays σ_0.

Let $(n^1_j)_{j \in A}$ be the congestion vector determined by (σ_0, σ_1), and let $(n^{1'}_j)_{j \in A}$ be the congestion vector determined by (σ_0, σ'_1). For every $j \neq \sigma_0$, $n^1_j = n_j$ and $n^{1'}_j = n'_j$. If $(\sigma_2, \ldots, \sigma_n)$ is the subgame equilibrium path determined by (σ_0, σ_1) and $(\sigma'_2, \ldots, \sigma'_n)$ is the subgame equilibrium path determined by (σ_0, σ'_1) then $(n^1_j)_{j \in A}$ is also the congestion vector determined by $(\sigma_0, \sigma_2, \sigma_1)$ and $(n^{1'}_j)_{j \in A}$ is also the congestion vector determined by $(\sigma_0, \sigma'_2, \sigma'_1)$. If $\sigma_2 = \sigma'_2$ then it follows from the induction hypothesis, applied to the $(n-1)$-person subgame determined by (σ_0, σ_2), that

$$n^{1'}_j \leq n^1_j + 1 \text{ for every } j, \text{ equality holds for at most one } j, \text{ but } n^{1'}_{\sigma_1} \leq n^1_{\sigma_1}. \quad (1)$$

It also follows from the induction hypothesis that, symmetrically,

$$n^1_j \leq n^{1'}_j + 1 \text{ for every } j, \text{ equality holds for at most one } j, \text{ but } n^1_{\sigma'_1} \leq n^{1'}_{\sigma'_1}. \quad (1')$$

We have to show that (1) and (1′) also hold if $\sigma_2 \neq \sigma'_2$. In what follows, we will assume that the last inequality holds. In the course of the proof we will consider several subgames, each one determined by a particular triplet of actions, as well as the congestion vectors determined by these triplets. The triplets, preceded by the index of the congestion vector they determine, are shown in Fig. 1. A straight line in this figure indicates that the two triplets connected by the line differ in exactly one of the three actions. The roman number near the line refers to the equation which, in the sequel, is deduced from this relation between the two triplets by means of the induction hypothesis, applied to the subgame determined by the two common actions.

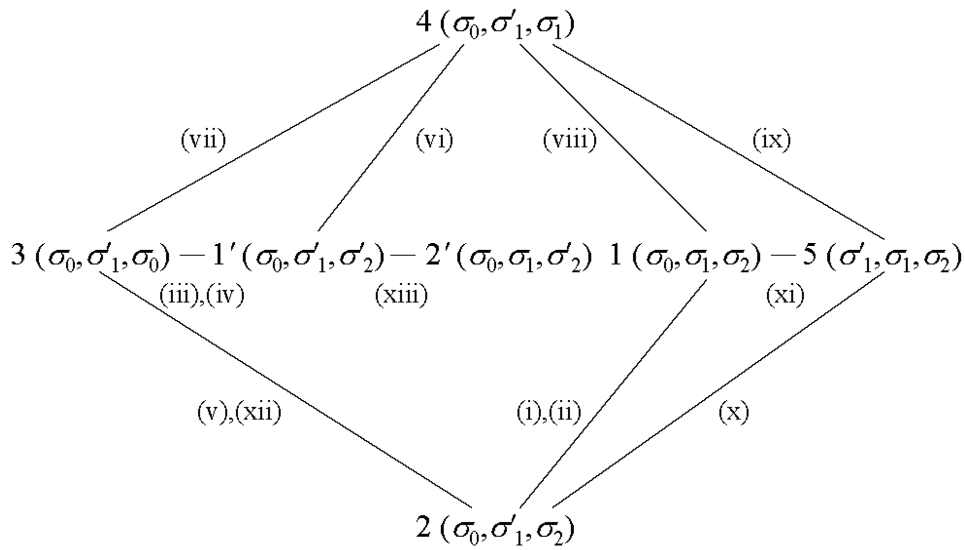

Figure 1: For an explanation see text.

Let $(n_j^2)_{j\in A}$ be the congestion vector determined by $(\sigma_0, \sigma'_1, \sigma_2)$. This is the congestion vector that would result if player 2 did not change his action as a response to the change of player 1's action from σ_1 to σ'_1. $(n_j^2)_{j\in A}$ is also the congestion vector determined by $(\sigma_0, \sigma_2, \sigma'_1)$. It hence follows from the induction hypothesis that either

$$n_{\sigma_2}^2 = n_{\sigma_2}^1 + 1 \text{ and } n_j^2 \leq n_j^1 \text{ for every } j \neq \sigma_2 \tag{ia}$$

or

$$n_{\sigma_2}^2 \leq n_{\sigma_2}^1. \tag{ib}$$

We will consider each of these two cases separately. It also follows from the induction hypothesis that

$$n_{\sigma'_1}^1 \leq n_{\sigma'_1}^2 \tag{ii}$$

and $n^2_{\sigma_1} \leq n^1_{\sigma_1}$. In light of the latter inequality, (ia) is possible only if $\sigma_2 \neq \sigma_1$.

CASE (ia). Let $(n^3_j)_{j\in A}$ be the congestion vector determined by $(\sigma_0, \sigma_1', \sigma_0)$. Since the congestion vector determined by $(\sigma_0, \sigma_1', \sigma_2')$ is $(n^{1'}_j)_{j\in A}$, by the induction hypothesis

$$n^{1'}_j \leq n^3_j + 1 \text{ for every } j, \text{ and equality holds for at most one } j. \tag{iii}$$

Since $n^3_{\sigma_0} = n^{1'}_{\sigma_0} + 1$, by the induction hypothesis

$$n^3_j \leq n^{1'}_j \text{ for every } j \neq \sigma_0. \tag{iv}$$

Similarly, since $n^3_{\sigma_0} = n^2_{\sigma_0} + 1$,

$$n^3_j \leq n^2_j \text{ for every } j \neq \sigma_0. \tag{v}$$

Let $(n^4_j)_{j\in A}$ be the congestion vector determined by $(\sigma_0, \sigma_1', \sigma_1)$. By the induction hypothesis,

$$n^{1'}_{\sigma_1} \leq n^4_{\sigma_1}. \tag{vi}$$

Since $n^3_{\sigma_0} = n^4_{\sigma_0} + 1$,

$$n^3_j \leq n^4_j \text{ for every } j \neq \sigma_0. \tag{vii}$$

Since $(n^4_j)_{j\in A}$ is also the congestion vector determined by $(\sigma_0, \sigma_1, \sigma_1')$, and since the congestion vector determined by $(\sigma_0, \sigma_1, \sigma_2)$ is $(n^1_j)_{j\in A}$, by the induction hypothesis

$$n^4_{\sigma_2} \leq n^1_{\sigma_2}. \tag{viii}$$

It follows from (ia), (v), (vii), and (viii) that $n^3_j \leq n^1_j$ for every $j \neq \sigma_0$. Hence, by (iii), $n^{1'}_j \leq n^1_j + 1$ for every j and equality holds for at most one j. Since $\sum_j n^{1'}_j = \sum_j n^1_j = n+1$, it follows that $n^1_j \leq n^{1'}_j + 1$ for every j and equality holds for at most one j. To complete the proof of (1) and (1′) in the case that (ia) holds, it remains to show that $n^{1'}_{\sigma_1} \leq n^1_{\sigma_1}$ and $n^1_{\sigma_1'} \leq n^{1'}_{\sigma_1'}$.

Let $(n^5_j)_{j\in A}$ be the congestion vector determined by $(\sigma_1', \sigma_1, \sigma_2)$. Since $n^4_{\sigma_0} = n^5_{\sigma_0} + 1$,

$$n^4_{\sigma_1} \leq n^5_{\sigma_1}. \tag{ix}$$

Since $n^2_{\sigma_0} = n^5_{\sigma_0} + 1$,

$$n^2_{\sigma_2} \leq n^5_{\sigma_2}, \tag{x}$$

and hence $n^5_{\sigma_2} \geq n^1_{\sigma_2} + 1$ by (ia). Since, as shown above, (ia) implies $\sigma_2 \neq \sigma_1$, by the induction hypothesis

$$n^5_{\sigma 1} \leq n^1_{\sigma 1}. \tag{xi}$$

In conjunction with (vi) and (ix) this gives $n^{1'}_{\sigma_1} \leq n^1_{\sigma_1}$.

By (ia), (vii), and (viii), $n^2_{\sigma_2} \geq n^3_{\sigma_2} + 1$. Therefore, if $\sigma'_1 \neq \sigma_2$ then by the induction hypothesis

$$n^2_{\sigma'_1} \leq n^3_{\sigma'_1}, \tag{xii}$$

and hence $n^1_{\sigma'_1} \leq n^{1'}_{\sigma'_1}$ by (ii) and (iv). If $\sigma'_1 = \sigma_2$ then by the induction hypothesis $n^2_{\sigma'_1} = n^3_{\sigma'_1} + 1$, and by (ia) $n^2_{\sigma'_1} = n^1_{\sigma'_1} + 1$. It hence follows from (iv) that in this case, too, $n^1_{\sigma'_1} \leq n^{1'}_{\sigma'_1}$.

CASE (ib). Since the action of player 2 in the subgame equilibrium path determined by (σ_0, σ'_1) is σ'_2, it must be that $S_{2\sigma'_2}(n^{1'}_{\sigma'_2}) \geq S_{2\sigma_2}(n^2_{\sigma_2})$. Since we have assumed that $\sigma'_2 \neq \sigma_2$, the assumed genericity of the game implies that this inequality is in fact strict. Similarly, since the action of player 2 in the subgame equilibrium path determined by (σ_0, σ_1) is σ_2, it must be that $S_{2\sigma_2}(n^1_{\sigma_2}) > S_{2\sigma'_2}(n^{2'}_{\sigma'_2})$, where $(n^{2'}_j)_{j \in A}$ is the congestion vector determined by $(\sigma_0, \sigma_1, \sigma'_2)$. Note that, formally, the last triplet is obtained from the triplet $(\sigma_0, \sigma'_1, \sigma_2)$ that determines $(n^2_j)_{j \in A}$ by taking the prime off σ_1 and adding a prime to σ_2; hence the index $2'$. By (ib) and the monotonicity of $S_{2\sigma_2}$, $S_{2\sigma'_2}(n^{1'}_{\sigma'_2}) > S_{2\sigma_2}(n^2_{\sigma_2}) \geq S_{2\sigma_2}(n^1_{\sigma_2}) > S_{2\sigma'_2}(n^{2'}_{\sigma'_2})$. Hence, by the monotonicity of $S_{2\sigma'_2}$, $n^{1'}_{\sigma'_2} < n^{2'}_{\sigma'_2}$. It follows, by the induction hypothesis, that

$$n^{2'}_{\sigma'_2} = n^{1'}_{\sigma'_2} + 1 \text{ and } n^{2'}_j \leq n^{1'}_j \text{ for every } j \neq \sigma'_2. \tag{xiii}$$

Formally, (xiii) is a "primed" version of (ia). More precisely, the transformation that interchanges σ_1 with σ'_1 and σ_2 with σ'_2 transforms (ia) into (xiii). Hence, our analysis of case (ia) implies at once that the formal transforms of (1) and (1$'$) hold true. But these transforms are (1$'$) and (1), respectively. Thus, (1) and (1$'$) hold in the present case, too. $\square$

Proof of Theorem 2. The proof proceeds by induction on the number n of players. The theorem is evidently true if $n = 1$. Assume that it holds true for every $(n-1)$-person generic crowding game, and let σ be a strong equilibrium of an

n-person ($n \geq 2$) generic crowding game. Consider the directed graph, with n vertices corresponding one-to-one to the players in the game, where an arc is going from vertex i to vertex i' if and only if player i *envies* player i' when σ is played, that is, if and only if $S_{i\sigma_i}(n_{\sigma_i}) < S_{i\sigma_{i'}}(n_{\sigma_{i'}})$, where $(n_j)_{j\in A}$ is the congestion vector corresponding to σ. The assumption that σ is a strong equilibrium implies that this graph has no cycles. Indeed, if the graph had a cycle then the players involved could change their actions—each player shifting to the action played by the player he envies—and they all would gain. Therefore, there exists a player, i, who envies no other player; $S_{i\sigma_i}(n_{\sigma_i}) > S_{ij}(\max\{1, n_j\})$ holds for every $j \neq \sigma_i$. Without loss of generality, $i = 1$. $(\sigma_2, \sigma_3, \ldots, \sigma_n)$ is a strong equilibrium of the subgame determined by (σ_1). Hence, by the induction hypothesis, it is also a sequential-move equilibrium of that subgame, say the sequential-move equilibrium corresponding to the sequential-move version in which the players move in the order $2, 3, \ldots, n$. To prove that σ is a sequential-move equilibrium of the n-person game, it suffices to show that, in the sequential-move version of that game in which the players move in the order $1, 2, \ldots, n$, player 1 cannot gain from choosing an action j different from σ_1. Let $(n'_k)_{k\in A}$ be the congestion vector determined by (j). By the lemma, $n'_j \geq n_j$ (and, of course, $n'_j \geq 1$). It therefore follows from our assumption concerning player 1 that $S_{1j}(n'_j) \leq S_{1j}(\max\{1, n_j\}) < S_{1\sigma_1}(n_{\sigma_1})$. Thus, playing σ_1 is indeed the best option for player 1. □

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