Altruism and Spite in Tullock Contest

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Altruism and spite (or envy) in a two-party interaction refer to interdependent preferences where each party assigns, respectively, a positive or negative weight to the other party's material payoff. In general, the effect of such interdependence on the actual, material payoffs at equilibrium can go both ways. We show, however, that in a Tullock contest – both the classical model and a generalization where there can be uncertainty about the identity of the power-holding politician – the contestants can only gain from mutual altruism and be harmed by mutual spite. We establish this finding in two ways: by solving the models and explicitly finding their equilibria; and by applying general results linking comparative statics of altruism and spite with the stability of the equilibria. The second method has the advantages of being simpler as well as more general, in that it does not depend on the particularities of the models but only on certain qualitative properties of the success and cost functions. *JEL classification:* C72; D72

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1. Introduction

A Tullock contest (Tullock 2001), as understood in this paper, is an all-pay auction where two parties are participating in a lottery in which the winning probability of a contestant investing x against a rival investing y is given by the contest success function

$$P(x,y) = \frac{x}{x+y}, \tag{1}$$

with $P(0,0) = 1/2$. The investments may represent, for example, lobbying efforts directed at a politician or government official. The cost of lobbying is an increasing function of the investment and is independent of the contestant's success or failure. We allow for different cost functions for the two contestants, but assume that the prize from winning is the same, and is normalized to 1.

In any two-party contest, each side cares about the other side's probability of winning to the same extent that it cares about its own, simply because the two probabilities are complementary. Altruism and spite refer to cases where the opponent's success or failure has an additional, *direct* meaning. Mutual altruism means that each party sees the other's success as a partial substitute to its own success, and mutual spite, or envy, means the same for the other's failure. The former may hold, for example, if two firms hold stock in their competitor, and the latter may hold if a win for the competitor is harmful beyond the present contest. However, these examples, and the terms used, are only meant to be illustrative. The question we ask is, what would happen if each contestant behaved *as if* it attaches some weight r, which may be positive (but not greater than 1) or negative, to the competitor's success? Crucially, our concern is not with the *perceived* payoffs, which change automatically when the competitor's lot is taken into consideration, but with the effect of this change in preferences on the actual, *material* payoffs at equilibrium. Note that ours is emphatically not an evolutionary model. Unlike, e.g., Schmidt (2009, which see for additional references), we view r as an exogenous, common parameter, which quantifies the contestants' interdependence of preferences. We study the social consequences of this interdependence, not its origin or evolution. Thus, the comparison concerns different settings or circumstances, not different types within a population, each with its own kind of interdependent preferences, or different stages of evolution of a population.

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A simple but important observation is that the effect of altruism or spite does not depend on whether or not the contestants also care about their competitor's cost. As the latter cannot be directly controlled, its inclusion in the perceived payoff is inconsequential in that it has no effect on the choice of strategy and therefore also on the outcome. As the contestants' winning probabilities are complementary, this observation implies that adding the competitor's payoff multiplied by r to the own payoff has the same effect as multiplying the own winning probability by $1 - r$. This, in turn, means that the perceived payoff is effectively the material payoff with the *cost* multiplied by the inverse factor $1/(1 - r)$. The effect of this change in the relative importance of the cost may be expected to be a decrease in the equilibrium investment if $0 < r < 1$ (altruism) and an increase if $r < 0$ (spite). The former effect is socially beneficial, as it entails lower prize dissipation, and the latter is socially harmful. This is indeed what we find in Section 2, where we compute the exact effect of r on the investments and payoffs.

A similar argument applies to other contest settings. In Section 3, we consider the two-politician model of Epstein et al. (2007), where there is uncertainty about the identity of the power-holding politician. The two contestants receive correlated noisy signals about that identity and, at equilibrium, make positive investments in both politicians. As we show, here too altruism has a positive effect on the material payoffs at equilibrium and spite has a negative effect.

While the finding of positive comparative statics in a Tullock competition may not be particularly novel or surprising (see Schmidt 2009), it is not self-evident either. In fact, negative comparative statics may occur in strategic interactions as simple as Cournot competition (Milchtaich 2012) and congestion games (Milchtaich 2021). In these settings, two parties involved in a symmetric interaction may paradoxically be materially harmed by a small amount of mutual altruism and benefit from mutual spite. The fact that we do not find similar, negative comparative statics in our Tullock models is a reflection of particular properties of the payoff functions in these models, not of a simple universal truth. In the following subsections, we expand on this topic.

1.1 Comparative statics

Altruism and spite may be viewed from a wider perspective by considering them as special cases of internalization of a social payoff function (Milchtaich 2012). The degree of internalization of a given social payoff function f, which is any function of all participants' actions, is expressed by the *altruism coefficient* $r \leq 1$, which everyone involved in the interaction is assumed to share. Thus, the quantity that each player i seeks to maximize is not the *personal*, or *material*, payoff u_i, but the *modified*, or *perceived*, payoff

$$u_i^r \coloneqq (1 - r)u_i + rf. \tag{2}$$

These payoff functions define the *modified game*. The social payoff function relevant for our context is the aggregate payoff in a two-player game, $f = u_1 + u_2$, for which the modified payoff simplifies to $u_i^r = u_i + ru_j$, where j is the other player. Thus, r expresses the manner and the extent to which each player internalizes the other player's material payoff. A positive, zero or negative value reflects altruism, complete selfishness or spite (alternatively, envy), respectively.[1] Comparative statics of altruism and spite refer to the effect of an increase or decrease in r on the equilibrium value of the social payoff. Positive comparative statics means that altruism is socially beneficial and spite is socially harmful, and negative comparative statics means the opposite. In a symmetric setting, where everyone's equilibrium payoff is the same, the social effect coincides with the effect on the personal payoffs, which increase or decrease, respectively, when everyone becomes less selfish or spiteful.

[1] Note that we only bound r from above. While the assumption that the other's payoff is no more important than the own payoff is crucial, a similar restriction on the strength of spite is not needed.

1.2 The connection with stability

Whether internalization of a social payoff function increases or, paradoxically, decreases its equilibrium value is determined to a large extent by the stability or instability of the equilibria (or, in a symmetric setting, the equilibrium strategies) involved (Milchtaich 2012). This general connection between stability and comparative statics is not at all obvious, as stability is a property of a particular strategy profile in a single game, namely, the modified game with a specified altruism coefficient r, whereas comparative statics involve multiple games, each corresponding to a different value of the coefficient. Importantly, the connection mentioned is not with *dynamic* stability, which concerns the way the system evolves following a perturbation, but with *static* stability (Milchtaich 2023), which looks at the players *incentives* following a perturbation rather than the resulting changes of actions (which may vary according to the assumed law of motion). Static stability is a very general concept, relevant to any strategic game. It includes as special cases a number of other, familiar notions of stability, such as evolutionarily stable strategy (ESS) and continuously stable strategy (CSS), which are meaningful only in specific settings.

Static stability of the equilibria in our two models is examined in Section 4. The general results alluded to above, which relate stability with comparative statics, allow us to reach conclusions about the effect of changing the altruism coefficient in an easier and more general manner than we can do by explicitly computing the equilibria. The Appendix looks at the dynamic stability of these equilibria. It shows how two different, standard kinds of such stability differ from one another and from static stability in the context of Tullock contest.

2. Tullock model with a single politician

In the basic Tullock model described in the introduction, when the two contestants' investments are x_1 and x_2 their payoffs are given by

$$u_1(x_1, x_2) = P(x_1, x_2) - c_1(x_1)$$
$$u_2(x_1, x_2) = P(x_2, x_1) - c_2(x_2),$$

where P is the success function (1) and c_1 and c_2 are the cost functions. We assume that the latter are continuous and increasing, are zero at the origin, and are twice continuously differentiable outside the origin ($x_1, x_2 > 0$), where, for $i = 1,2$, they satisfy the inequalities[2] $c_i'(x_i) > 0$ and

$$c_i'(x_i) + x_i c_i''(x_i) > 0. \tag{3}$$

The social payoff is the aggregate payoff

$$f(x_1, x_2) \coloneqq u_1(x_1, x_2) + u_2(x_1, x_2) = 1 - c_1(x_1) - c_2(x_2). \tag{4}$$

For altruism coefficient $r \leq 1$, the modified payoffs (Eq. (2)) are both equal to $(1 + r)/2$ if $x_1 = x_2 = 0$ and otherwise they are given by[3]

$$\begin{aligned} u_1^r(x_1, x_2) &= \frac{x_1 + r x_2}{x_1 + x_2} - c_1(x_1) - r c_2(x_2) \\ u_2^r(x_1, x_2) &= \frac{x_2 + r x_1}{x_1 + x_2} - c_2(x_2) - r c_1(x_1). \end{aligned} \tag{5}$$

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[2] Inequality (3), which can be written also as ${}^{d}/_{dx_i}\,[x_i c_i'(x_i)] > 0$, means that the function $x_i c_i'(x_i)$ is strictly increasing. It is implied by (the first inequality $c_i'(x_i) > 0$ and) convexity, $c_i''(x_i) \geq 0$, but it is a significantly weaker condition (and holds, e.g., for all increasing power functions, $c_i(x_i) = x_i^d$, $d > 0$.)

[3] Note that we could drop the last term in each of the expressions on the right-hand side. Doing so would correspond to assuming that altruism only involves the opponent's chance of winning, and not the cost. The change would not affect the players' choice of strategy, because the dropped terms only depend on the opponent's strategy, and would also not affect the social payoff function f, which incorporates the players' personal payoffs rather than the modified ones.

Obviously, the unique equilibrium for $r = 1$ is $(0,0)$. For $r < 1$, there is no equilibrium where either investment is 0, and so any equilibrium is internal. An internal strategy profile (x_1^r, x_2^r) is an equilibrium only if it satisfies the first-order conditions

$$\begin{aligned} u_{1,1}^r(x_1^r, x_2^r) &\coloneqq \frac{\partial u_1^r}{\partial x_1}(x_1^r, x_2^r) = (1-r)\frac{x_2^r}{(x_1^r + x_2^r)^2} - c_1'(x_1^r) = 0 \\ u_{2,2}^r(x_1^r, x_2^r) &\coloneqq \frac{\partial u_2^r}{\partial x_2}(x_1^r, x_2^r) = (1-r)\frac{x_1^r}{(x_1^r + x_2^r)^2} - c_2'(x_2^r) = 0, \end{aligned} \tag{6}$$

which together imply that

$$x_1^r c_1'(x_1^r) = x_2^r c_2'(x_2^r).$$

(In the symmetric case, where $c_1 = c_2$, it follows from the last equality and the remark at footnote 2 that an equilibrium is necessarily symmetric, $x_1^r = x_2^r$.) The necessary second-order equilibrium conditions are weak inequalities. We assume, however, that these conditions hold as strict inequalities:

$$\begin{aligned} u_{1,11}^r(x_1^r, x_2^r) &\coloneqq \frac{\partial^2 u_1^r}{\partial x_1^2}(x_1^r, x_2^r) = -2(1-r)\frac{x_2^r}{(x_1^r + x_2^r)^3} - c_1''(x_1^r) = -2\frac{c_1'(x_1^r)}{x_1^r + x_2^r} - c_1''(x_1^r) < 0 \\ u_{2,22}^r(x_1^r, x_2^r) &\coloneqq \frac{\partial^2 u_2^r}{\partial x_2^2}(x_1^r, x_2^r) = -2(1-r)\frac{x_1^r}{(x_1^r + x_2^r)^3} - c_2''(x_2^r) = -2\frac{c_2'(x_2^r)}{x_1^r + x_2^r} - c_2''(x_2^r) < 0. \end{aligned} \tag{7}$$

The strictness assumption holds automatically if the cost functions are convex ($c_i'' \geq 0$), and it also follows automatically from assumption (3) if x_1^r and x_2^r are equal (as they necessarily are in the symmetric case). For later reference, note that, by (6), the mixed partial derivatives at the equilibrium point are given by

$$u_{1,12}^r(x_1^r, x_2^r) = (1-r)\frac{x_1^r - x_2^r}{(x_1^r + x_2^r)^3} = (x_1^r - x_2^r)\frac{c_1'(x_1^r)}{x_2^r(x_1^r + x_2^r)} = (x_1^r - x_2^r)\frac{c_2'(x_2^r)}{x_1^r(x_1^r + x_2^r)} \tag{8}$$

$$u_{2,21}^r(x_1^r, x_2^r) = (1-r)\frac{x_2^r - x_1^r}{(x_1^r + x_2^r)^3} = -u_{1,12}^r(x_1^r, x_2^r). \tag{9}$$

Example 1 In a symmetric setting, the players' cost functions are $c_i(x_i) = x_i^d$, $i = 1,2$, with $d > 0$. For $r < 1$, the solution of the first-order conditions (6) is

$$x_1^r = x_2^r = \left(\frac{1-r}{4d}\right)^{\frac{1}{d}}. \tag{10}$$

To check whether this solution is in fact an equilibrium, consider player 1's modified payoff when $x_2 = x_2^r$, which can be written as

$$\begin{aligned} u_1^r(x_1, x_2^r) = 1 - (1-r)\frac{x_2^r}{x_1 + x_2^r} - x_1^d - r(x_2^r)^d &= 1 - 4d\frac{(x_2^r)^{d+1}}{x_1 + x_2^r} - x_1^d - r(x_2^r)^d \\ &= 1 + (x_2^r)^d\left(-\frac{4d}{\frac{x_1}{x_2^r} + 1} - \left(\frac{x_1}{x_2^r}\right)^d\right) - r(x_2^r)^d. \end{aligned}$$

For $d \geq 1$, the function $-4d/(t+1) - t^d$ suggested by the expression in parentheses is unimodal, peaking at $t = 1$. For $0 < d < 1$, the function is decreasing for small $t \geq 0$, then increasing, and finally decreasing again from $t = 1$ and on. Therefore, player 1's best response to x_2^r (which is given by (10)) is either the same investment, which corresponds to $t = 1$, or zero. The former is true if and only if $-4d/(1+1) - 1^d \geq -4d/(0+1) - 0^d$, which holds if and only if $d \geq 0.5$. These are therefore the d values for which (10) is indeed an equilibrium.

In Example 1, the common equilibrium investment (10) is a decreasing function of the altruism coefficient r. It follows that, the more altruistic the players are, the higher is their personal, material payoff at equilibrium. Specifically, the latter is one-half the social payoff at equilibrium, which for every $r \leq 1$ is given by

$$f(x_1^r, x_2^r) = 1 - (x_1^r)^d - (x_2^r)^d = 1 + \frac{r-1}{2d}.$$

The next subsection examines comparative statics in the general (in particular, not necessarily symmetric) case.

2.1 Comparative statics with a single politician

The following proposition describes the effect of altruism and spite on the equilibrium strategies.

Proposition 1 For $r < 1$, the derivatives of the equilibrium strategies with respect to r are given by the matrix equation

$$H^r \begin{pmatrix} \frac{dx_1^r}{dr} \\ \frac{dx_2^r}{dr} \end{pmatrix} = \frac{1}{(x_1^r + x_2^r)^2} \begin{pmatrix} x_2^r \\ x_1^r \end{pmatrix}, \tag{11}$$

where

$$H^r = \begin{pmatrix} u_{1,11}^r(x_1^r, x_2^r) & u_{1,12}^r(x_1^r, x_2^r) \\ u_{2,21}^r(x_1^r, x_2^r) & u_{2,22}^r(x_1^r, x_2^r) \end{pmatrix}.$$

This coefficient matrix is negative definite (in the sense that its symmetric form $(1/2)(H^r + (H^r)^{\mathrm{T}})$ is so) and has a positive determinant, $|H^r| > 0$.

Proof By (6), for every $r < 1$

$$0 = u_{1,1}^r(x_1^r, x_2^r) = u_{1,1}(x_1^r, x_2^r) + r u_{2,1}(x_1^r, x_2^r).$$

Derivation with respect to r gives

$$\begin{aligned} 0 &= u_{1,11}(x_1^r, x_2^r)\frac{dx_1^r}{dr} + u_{1,12}(x_1^r, x_2^r)\frac{dx_2^r}{dr} + r u_{2,11}(x_1^r, x_2^r)\frac{dx_1^r}{dr} + r u_{2,12}(x_1^r, x_2^r)\frac{dx_2^r}{dr} + u_{2,1}(x_1^r, x_2^r) \\ &= u_{1,11}^r(x_1^r, x_2^r)\frac{dx_1^r}{dr} + u_{1,12}^r(x_1^r, x_2^r)\frac{dx_2^r}{dr} - \frac{x_2^r}{(x_1^r + x_2^r)^2}, \end{aligned}$$

where the second equality uses the identities

$$u_{1,1i}^r(x_1, x_2) = u_{1,1i}(x_1, x_2) + r u_{2,1i}(x_1, x_2),$$

$i = 1,2$. Similarly,

$$0 = u_{2,21}^r(x_1^r, x_2^r)\frac{dx_1^r}{dr} + u_{2,22}^r(x_1^r, x_2^r)\frac{dx_2^r}{dr} - \frac{x_1^r}{(x_1^r + x_2^r)^2}.$$

Eq. (11) presents these equalities in matrix form.

By (7) and (9), the diagonal elements of H^r are negative and the off-diagonal elements have opposite signs. Therefore, $\frac{1}{2}(H^r + (H^r)^{\mathrm{T}})$ is negative definite, as it has negative diagonal elements and zero off-diagonal ones, and

$$|H^r| = u_{1,11}^r(x_1, x_2) u_{2,22}^r(x_1, x_2) + \left(u_{1,12}^r(x_1^r, x_2^r)\right)^2 > 0.$$

∎

As the next theorem shows, the effect described by Proposition 1 spells positive comparative statics.

Theorem 1 For $r < 1$, the equilibrium strategies satisfy $dx_1^r/dr, dx_2^r/dr < 0$, and the equilibrium social payoff satisfies

$$\frac{d}{dr} f(x_1^r, x_2^r) > 0.$$

Proof Solving (11), and using (7), (8) and (9), we get

$$\frac{dx_1^r}{dr} = \frac{1}{|H^r|}\begin{vmatrix} \frac{x_2^r}{(x_1^r + x_2^r)^2} & u_{1,12}^r(x_1^r, x_2^r) \\ \frac{x_1^r}{(x_1^r + x_2^r)^2} & u_{2,22}^r(x_1^r, x_2^r) \end{vmatrix} = \frac{1}{(x_1^r + x_2^r)^2|H^r|}\begin{vmatrix} x_2^r & (x_1^r - x_2^r)\frac{c_2'(x_2^r)}{x_1^r(x_1^r + x_2^r)} \\ x_1^r & -2\frac{c_2'(x_2^r)}{x_1^r + x_2^r} - c_2''(x_2^r) \end{vmatrix}$$

$$= -\frac{c_2'(x_2^r) + x_2^r c_2''(x_2^r)}{(x_1^r + x_2^r)^2|H^r|} < 0,$$

$$\frac{dx_2^r}{dr} = \frac{1}{|H^r|}\begin{vmatrix} u_{1,11}^r(x_1^r, x_2^r) & \frac{x_2^r}{(x_1^r + x_2^r)^2} \\ u_{2,21}^r(x_1^r, x_2^r) & \frac{x_1^r}{(x_1^r + x_2^r)^2} \end{vmatrix} = \frac{1}{(x_1^r + x_2^r)^2|H^r|}\begin{vmatrix} -2\frac{c_1'(x_1^r)}{x_1^r + x_2^r} - c_1''(x_1^r) & x_2^r \\ -(x_1^r - x_2^r)\frac{c_1'(x_1^r)}{x_2^r(x_1^r + x_2^r)} & x_1^r \end{vmatrix}$$

$$= -\frac{c_1'(x_1^r) + x_1^r c_1''(x_1^r)}{(x_1^r + x_2^r)^2|H^r|} < 0,$$

where the inequalities follow from (3) and Proposition 1. By (4),

$$\frac{d}{dr}f(x_1^r, x_2^r) = -c_1'(x_1^r)\frac{dx_1^r}{dr} - c_2'(x_2^r)\frac{dx_2^r}{dr} > 0.$$

■

An alternative, shorter demonstration of positive comparative statics, which does not require explicit examination of the effect of r on the equilibrium strategies, is provided by the following general result. The result applies to any game where the players' strategy space are intervals (finite or otherwise) in the real line.

Proposition 2 (Milchtaich 2012, Proposition 8) For a two-player game with strategy spaces that are real intervals, and altruism coefficients r_0 and r_1 with $r_0 < r_1 \leq 1$, suppose that there is a continuously differentiable function assigning to each $r_0 < r < r_1$ an internal equilibrium (x_1^r, x_2^r) in the corresponding modified game with a neighborhood where the payoff functions have continuous second-order partial derivatives. For $r_0 < r < r_1$,

$$\frac{d}{dr}f(x_1^r, x_2^r) = -(1-r)\begin{pmatrix}\frac{dx_1^r}{dr} & \frac{dx_2^r}{dr}\end{pmatrix} H^r \begin{pmatrix}\frac{dx_1^r}{dr} \\ \frac{dx_2^r}{dr}\end{pmatrix}.$$

Obviously, replacing the matrix H^r with $(1/2)(H^r + (H^r)^{\mathrm{T}})$ leaves the quadratic form on the right-hand side unchanged. In our case, this expression is therefore equal to

$$-(1-r)\left(u_{1,11}^r(x_1^r, x_2^r)\left(\frac{dx_1^r}{dr}\right)^2 + u_{2,22}^r(x_1^r, x_2^r)\left(\frac{dx_2^r}{dr}\right)^2\right).$$

By (7), the expression is positive provided that at least one of $dx_1^r/\,dr$ and $dx_2^r/\,dr$ is not zero.

3. Tullock model with two politicians

A more general setting than the standard Tullock model considered in Section 2 is when there is incomplete information on the source of power in a contest (Epstein et al. 2007). Specifically, suppose that there are two politicians, or government officials, who are a priori equally likely to be the true decision-making politician d. The two contestants are uncertain about d's identity. Each of them receives a noisy signal s indicating that identity, which is correct ($s = d$) with probability $0 < p < 1$. (The cases $p = 0$ or 1 would correspond to the standard model.) The two players' signals may be conditionally dependent or independent, given d. We only assume that the correlation coefficient ρ

between them is the same regardless of whether d is the first or second politician and that $0 \leq \rho \leq 1$ (i.e., the signals are not *negatively* correlated). The parameters p and ρ determine the probability $\alpha = p^2 + \rho p(1-p)$ (> 0) that both players' signals are correct, the probability $\beta = (1-p)^2 + \rho p(1-p)$ (> 0) that they are both incorrect and the probability $\gamma = (1-\alpha-\beta)/2 = (1-\rho)p(1-p)$ that only one, specific player got a correct signal.

The winning probability of each player depends only on his own and his rival's investment in the decisive politician and is given by Tullock's contest success function. The cost depends on the investment in both politicians. To simply the analysis, we assume that the cost is just the sum of the two investments, which in particular makes our model symmetric. In a symmetric equilibrium, each player spends x on the politician his signal indicates is the decision-making one and y on the other politician. The equilibrium condition is that a player would not gain from unilaterally replacing x and y with any alternative values X and Y, thereby receiving the payoff

$$u(X,Y,x,y) \coloneqq \alpha P(X,x) + \beta P(Y,y) + \gamma P(X,y) + \gamma P(Y,x) - X - Y.$$

Since $\alpha, \beta > 0$, this condition implies, in particular, that a symmetric equilibrium must be internal, $x, y > 0$. A similar conclusion holds with any altruism coefficient $r < 1$, where, in an internal symmetric equilibrium, the payoff from deviating to alternative values X and Y is given by

$$\begin{aligned} u^r(X,Y,x,y) &= u(X,Y,x,y) + ru(x,y,X,Y) \\ &= \alpha\frac{X+rx}{X+x} + \beta\frac{Y+ry}{Y+y} + \gamma\frac{X+ry}{X+y} + \gamma\frac{Y+rx}{Y+x} - X - Y - rx - ry. \end{aligned} \quad (12)$$

The first-order equilibrium condition is that

$$\frac{\partial u^r(X,Y,x,y)}{\partial \mathrm{X}} = \alpha\frac{(1-r)x}{(X+x)^2} + \gamma\frac{(1-r)y}{(X+y)^2} - 1 = 0$$
$$\frac{\partial u^r(X,Y,x,y)}{\partial \mathrm{Y}} = \beta\frac{(1-r)y}{(Y+y)^2} + \gamma\frac{(1-r)x}{(Y+x)^2} - 1 = 0$$

at $(X,Y) = (x,y)$, which simplifies to

$$(1-r)\left(\frac{1}{4}\alpha(x+y)^2 + \gamma xy\right) = x(x+y)^2$$

$$(1-r)\left(\frac{1}{4}\beta(x+y)^2 + \gamma xy\right) = y(x+y)^2.$$

An equivalent pair of equations is obtained by subtracting and adding these two, which, using the notation

$$\delta \coloneqq \alpha - \beta,$$

gives

$$x - y = \frac{1}{4}(1-r)\delta \quad (13)$$

and

$$\begin{aligned} (x+y)^3 &= (1-r)\left(\frac{1}{4}(\alpha+\beta)(x+y)^2 + 2\gamma xy\right) = (1-r)\left(\frac{1}{4}(1-2\gamma)(x+y)^2 + 2\gamma xy\right) \\ &= \frac{1}{4}(1-r)((x+y)^2 - 2\gamma(x-y)^2) = \frac{1}{4}(1-r)(x+y)^2 - \frac{1}{32}(1-r)^3\gamma\delta^2. \end{aligned}$$

The last equation means that the third-order polynomial equation

$$z^3 - \frac{1}{4}z^2 + \frac{1}{32}\gamma\delta^2 = 0 \quad (14)$$

holds for

$$z = \frac{x+y}{1-r}. \quad (15)$$

Solving (13) and (15) for (x, y) gives the solution

$$(1-r)\left(\frac{1}{2}z+\frac{1}{8}\delta, \frac{1}{2}z-\frac{1}{8}\delta\right). \tag{16}$$

For this solution to be an internal strategy, both entries must be positive, which is the case if and only if $z > |\delta|/4$. As Epstein et al. (2007) show, there is a unique solution z of (14) that satisfies the last inequality, which is given by

$$z = \frac{1}{12} + \frac{1}{6}\cos\frac{\theta}{3}\,, \tag{17}$$

where $\theta = \arccos(1 - 27\gamma\delta^2)$. The next proposition shows that, for every r, the strategy (x^r, y^r) obtained by plugging (17) in (16) is indeed an equilibrium strategy. Moreover, the equilibrium is strict. That is, any unilateral deviation is actually harmful.

Proposition 3 For every $r \leq 1$, both players using strategy (x^r, y^r) is the unique symmetric equilibrium, and it is strict.

Proof For $r = 1$, the strategy in question is $(0,0)$ and the assertion is obvious. For $r < 1$, strategy (x^r, y^r) is the unique internal strategy that satisfies the first-order equilibrium conditions (6). These necessary conditions are also sufficient for a strict equilibrium because the payoff function u^r defined in (12) is strictly concave in the first two variables, as can be seen by examining the matrix of the corresponding second-order derivatives,

$$\begin{pmatrix} \frac{\partial^2 u^r(X,Y,x,y)}{\partial X^2} & \frac{\partial^2 u^r(X,Y,x,y)}{\partial X\,\partial Y} \\ \frac{\partial^2 u^r(X,Y,x,y)}{\partial Y\,\partial X} & \frac{\partial^2 u^r(X,Y,x,y)}{\partial Y^2} \end{pmatrix}$$

$$= -(1-r)\begin{pmatrix} \alpha\frac{2x}{(X+x)^3} + \gamma\frac{2y}{(X+y)^3} & 0 \\ 0 & \beta\frac{2y}{(Y+y)^3} + \gamma\frac{2x}{(Y+x)^3} \end{pmatrix}.$$

Since $\alpha, \beta > 0$ and $\gamma \geq 0$, this matrix is negative definite for all X, Y, x, y with $x, y > 0$. ■

3.1 Comparative statics with two politicians

At the symmetric equilibrium, where the investment of each player in the two politicians is given by (16), the personal, material payoff is $1/2 - x^r - y^r = 1/2 - (1-r)z$ (with the r-independnet $z > 0$ given by (17)). This finding means that comparative statics are positive.

Theorem 2 The investment in each politician at the symmetric equilibrium is a decreasing linear function of r, which reaches zero at $r = 1$. The equilibrium personal payoff is an increasing linear function, which reaches the socially efficient level of $1/2$ at $r = 1$.

4. Static stability

The notion of static stability of strategy profiles is introduced in Milchtaich (2023). Whereas equilibrium is defined by the condition that, at the strategy profile in question, no player can gain from a unilateral deviation, stability also examines the incentives associated with sequential moves, whereby several (possibly, all) players deviate one after the other.

Definition 1 In a two-player game, a strategy profile (x_1, x_2) is *stable* if it has a neighborhood where for every profile $(x_1', x_2') \neq (x_1, x_2)$

$$\frac{1}{2}\Big(u_1(x_1',x_2') - u_1(x_1,x_2') + u_1(x_1',x_2) - u_1(x_1,x_2) + u_2(x_1',x_2') - u_2(x_1',x_2) + u_2(x_1,x_2') - u_2(x_1,x_2)\Big) < 0. \tag{18}$$

A strategy profile is *globally stable* if a similar condition holds for *all* $(x_1', x_2') \neq (x_1, x_2)$.

The expression on the left-hand side of (18) is the sum of the players' changes of payoff as they move one-by-one from (x_1, x_2) to (x_1', x_2'), averaged over the two possible orders of move. The inequality thus expresses the condition that such a sequential move on average harms the deviating players.

A globally stable strategy profile is in particular a strict equilibrium. The converse is generally false. However, the following propositions identifies a special case where the converse does hold.

Proposition 4 In a two-player game, suppose that there exist real-valued functions h_1 and h_2 on the strategy spaces of player 1 and 2, respectively, such that for every strategy profile (x_1, x_2)

$$u_1(x_1, x_2) + u_2(x_1, x_2) = h_1(x_1) + h_2(x_2). \tag{19}$$

Then, every strict equilibrium in the game is globally stable.

Games as in the proposition are characterized by the property that the change to the aggregate payoff brought about by any unilateral deviation is independent of the other player's strategy.

Lemma 1 In a two-player game, functions h_1 and h_2 satisfying the identity (19) exist if and only if the aggregate payoff $f = u_1 + u_2$ satisfies the following identify: for all $(x_1, x_2), (x_1', x_2')$,

$$f(x_1', x_2') - f(x_1, x_2') = f(x_1', x_2) - f(x_1, x_2). \tag{20}$$

Proof If the first identify, (19), holds for some h_1 and h_2, then the second one also holds, as both sides of (20) are equal to $h_1(x_1') - h_1(x_1)$. Conversely, if the second identity holds, then for any fixed (x_1', x_2') the functions $h_1(x_1) \coloneqq f(x_1, x_2') - f(x_1', x_2')$ and $h_2(x_2) \coloneqq f(x_1', x_2)$ satisfy (19). ■

Proof of Proposition 4 A strict equilibrium (x, y) satisfies

$$u_1(x_1', x_2) - u_1(x_1, x_2) + u_2(x_1, x_2') - u_2(x_1, x_2) < 0$$

for all $(x_1', x_2') \neq (x_1, x_2)$. It therefore suffices to show that the expression on the left-hand side of the last inequality is equal to that in (18), equivalently,

$$0 = \frac{1}{2}\big(u_1(x_1', x_2') - u_1(x_1, x_2') - u_1(x_1', x_2) + u_1(x_1, x_2) + u_2(x_1', x_2') - u_2(x_1', x_2) - u_2(x_1, x_2') + u_2(x_1, x_2)\big).$$

This equality follows from Lemma 1, as the expression in parentheses is equal to the difference between the left- and right-hand side of (20). ■

Unlike global stability, the local condition of stability does not imply that the strategy profile is a strict equilibrium, or even an equilibrium. Conversely, being an equilibrium is not a sufficient condition for stability. An additional condition that makes an equilibrium stable, in a two-player game where each player's strategy space is an interval, is identified by the next proposition.

Proposition 5 In a two-player game where the strategy spaces are real intervals, let (x_1, x_2) be an internal equilibrium with a neighborhood where the players' payoff functions have continuous second-order derivatives. A sufficient condition for the equilibrium to be stable is that the matrix

$$H = \begin{pmatrix} u_{1,11}(x_1, x_2) & u_{1,12}(x_1, x_2) \\ u_{2,21}(x_1, x_2) & u_{2,22}(x_1, x_2) \end{pmatrix} \tag{21}$$

is negative definite. This condition holds if and only if $|H + H^{\mathrm{T}}| > 0$.

Proof The sufficient condition is a special case of Proposition 7 in Milchtaich (2012). The equivalence to $|H + H^{\mathrm{T}}| > 0$ holds because the diagonal elements of H are nonnegative by the second-order equilibrium conditions. As $(1/2)(H + H^{\mathrm{T}})$, the symmetric form of H, shares these elements, it is negative definite if and only if its determinant is positive. ■

Stability of the equilibria is strongly associated with positive *local* comparative statics and global stability is associated with positive *global* comparatives statics (Milchtaich 2012, 2021). The former kind of comparative statics concerns small, continuous changes to the altruism coefficient and the corresponding equilibria, whereas the latter allows for large, discrete changes and is therefore the stronger kind. Two of the results establishing such connections are the following theorems.

Theorem 3 (Milchtaich 2021, Theorem 9) For a two-player game with continuous payoff functions, and altruism coefficients r_0 and r_1 with $r_0 < r_1 \leq 1$, suppose that there is a continuous and finitely-many-to-one function (meaning that the inverse image of every point is a finite set) assigning to each $r_0 \leq r \leq r_1$ a strategy profile (x_1^r, x_2^r) such that the function $r \mapsto f(x_1^r, x_2^r)$ is continuously differentiable in $[r_0, r_1]$. A sufficient condition for the last function to be strictly increasing is that for every $r_0 < r < r_1$ the strategy profile (x_1^r, x_2^r) is stable in the corresponding modified game.

Theorem 4 (Milchtaich 2021, Theorem 7) For any two-player game, and altruism coefficients r and s with $r < s \leq 1$, if two distinct strategy profiles (x_1^r, x_2^r) and (x_1^s, x_2^s) are globally stable in the corresponding modified games, then

$$f(x_1^r, x_2^r) < f(x_1^s, x_2^s). \tag{22}$$

In the following, we apply these general theorems to the models studied in the previous sections, thereby obtaining alternative proofs to the positive comparative statics results established above as well as gaining appreciation of the generality of these results and a suggestion that they are robust to modest changes in the particularities of our settings.

4.1 Stability with a single politician

By Propositions 1 and 5, the equilibrium (x_1^r, x_2^r) in the basic Tullock model is stable for every $r < 1$. In view of Theorem 3, this finding suggests positive local comparative statics. And, indeed, this is what Theorem 1 establishes. In fact, as the latter theorem refers to the entire range of values of the altruism coefficient, it even establishes positive *global* comparative statics. We can, however, get a similar result much more easily by showing that the equilibria involved are globally stable. For this end, we strengthen our assumption concerning the cost functions c_1 and c_2 by assuming these are convex functions. It follows from this assumption that, for every $r < 1$, the modified payoff (Eq. (5)) of each player i is strictly concave in the player's own strategy x_i, which implies that the equilibrium (x_1^r, x_2^r) is strict (and also that the first-order conditions (6) are both necessary and sufficient for equilibrium). The conclusion clearly holds also for $r = 1$, where the equilibrium is $(0,0)$.

Corollary 1 With convex cost functions, the Tullock model with a single politician exhibits positive global comparative statics. That is, for all r and s with $r < s \leq 1$, the corresponding equilibria (x_1^r, x_2^r) and (x_1^s, x_2^s) satisfy inequality (22).

Proof As

$$u_1^r(x_1, x_2) + u_2^r(x_1, x_2) = (1 + r)\big(u_1(x_1, x_2) + u_2(x_1, x_2)\big) = (1 + r)\big(1 - c_1(x_1) - c_2(x_2)\big),$$

by Proposition 4 the strict equilibrium (x_1^r, x_2^r) is globally stable. A similar result holds for s, and the conclusion now follows from Theorem 4. ∎

4.2 Stability with two politicians

Our comparative statics result for the two-politician model, Theorem 2, is based on a lengthy analysis which pinpoints the equilibrium strategies. However, as in the single-politician case, such an exact identification is actually not required for establishing positive global comparative statics.

Corollary 2 The Tullock model with two politicians exhibits positive global comparative statics.

Proof The proof again relies on Theorem 4. In light of Proposition 3, we only need to show that the relevant form of the condition in Proposition 4 holds. With $x_1 = (X, Y)$ and $x_2 = (x, y)$, indeed

$$\begin{aligned} u^r(X, Y, x, y) + u^r(x, y, X, Y) &= (1 + r)(\alpha + \beta + \gamma + \gamma - X - Y - x - y) \\ &= (1 + r)\left(\frac{1}{2} - X - Y\right) + (1 + r)\left(\frac{1}{2} - x - y\right). \end{aligned}$$

∎

Note that this simple demonstration of positive comparative statics would apply also if the cost functions, which are linear in our model, were replaced by any (possibly, two different) convex functions and/or the contest success function P were replaced by any function that is strictly concave in the own investment. Corollary 2 is therefore a considerably more general, albeit less specific, result than Theorem 2.

5. Summary

The question of the material consequences of altruism and spite is not unique to Tullock competition; it is of relevance to essentially any strategic interaction. Most of the general results mentioned above are also not specific to two-party interactions but hold for any number n of players. The two Tullock models analyzed in this paper are simple enough to be solved analytically. However, this may not be so for other kinds of interactions. In more complicated cases, the identified connections between comparative statics and stability may potentially be used for establishing positive (or, sometimes, negative; see Milchtaich 2012) comparative statics without actually computing the equilibria.

As already indicated, a negative, paradoxical effect of the degree of internalization of a social payoff on the equilibrium level of that payoff is a theoretical possibility, which may occur even in symmetric settings and when everyone cares equally about social welfare. This paper demonstrates that this does not happen in Tullock contest, and for a good reason: the equilibria in this setting are (statically) stable. It is left for future research to find out whether other sorts of strategic interactions, in particular, contests and auctions of different sorts, also share this property, and what structural properties of these interactions guarantee it.

Appendix: Dynamic stability

The kind of stability that goes hand-in-hand with positive comparative statics is static stability, formalized in Definition 1 (for generalizations and extensions of which see Milchtaich 2023). The various kinds of the more commonly considered dynamic stability do not exhibit a similar connection (Milchtaich 2012, section 8).

Unlike static stability, which is fully expressible in terms of the players' payoff functions, dynamic stability also refers to a particular, extraneous law of motion. One such law, relevant to games where the strategy spaces are real intervals, posits that the rate of change of each player's strategy is proportional to the marginal payoff. In a two-player game, this means that there are constants $d_1, d_2 > 0$ such that

$$\begin{aligned} \frac{dx_1}{dt} &= d_1 u_{1,1}(x_1, x_2) \\ \frac{dx_2}{dt} &= d_2 u_{2,2}(x_1, x_2). \end{aligned} \tag{23}$$

With these dynamics, the condition for asymptotic stability of an interior equilibrium (x_1, x_2) is that, at that point, the (Jacobian) matrix

$$\begin{pmatrix} d_1 u_{1,11} & d_1 u_{1,12} \\ d_2 u_{2,21} & d_2 u_{2,22} \end{pmatrix}$$

is *stable*, that is, all its eigenvalues have negative real parts. The requirement that stability holds for any pair of positive coefficients d_1 and d_2 is known as *D-stability* of the matrix H defined in (21) (which is obtained from the matrix above by omitting the coefficients).

D-stability is implied by negative definiteness of H (but not the other way around; Milchtaich 2023). We may therefore conclude from Proposition 1 that, for every $r < 1$, an equilibrium (x_1^r, x_2^r) in the Tullock model with a single politician is asymptotically stable with respect to dynamics similar to (23) in which u is replaced with u^r, for all $d_1, d_2 > 0$. That is, starting at any point (x_1, x_2) close to (x_1^r, x_2^r), the system will converge to the equilibrium.

Convergence can also be inferred directly from Proposition 1. Write (x_1, x_2) as $(x_1^r + \xi, x_2^r + \eta)$. For small ξ and η, the linear approximations

$$\begin{aligned} u_{1,1}^r(x_1^r + \xi, x_2^r + \eta) &\approx \underbrace{u_{1,1}^r(x_1^r, x_2^r)}_{0} + u_{1,11}^r(x_1^r, x_2^r)\xi + u_{1,12}^r(x_1^r, x_2^r)\eta \\ u_{2,2}^r(x_1^r + \xi, x_2^r + \eta) &\approx \underbrace{u_{2,2}^r(x_1^r, x_2^r)}_{0} + u_{2,21}^r(x_1^r, x_2^r)\xi + u_{2,22}^r(x_1^r, x_2^r)\eta \end{aligned} \tag{24}$$

give the linearized dynamics

$$\begin{pmatrix} \frac{d\xi}{dt} \\ \frac{d\eta}{dt} \end{pmatrix} = \begin{pmatrix} d_1 u_{1,11}^r(x_1^r, x_2^r) & d_1 u_{1,12}^r(x_1^r, x_2^r) \\ d_2 u_{2,21}^r(x_1^r, x_2^r) & d_2 u_{2,22}^r(x_1^r, x_2^r) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Pre-multiplying with $(\xi/d_1, \eta/d_2)$, with get

$$\frac{d}{dt}\left(\frac{\xi^2}{2d_1} + \frac{\eta^2}{2d_2}\right) = \begin{pmatrix} \frac{\xi}{d_1} & \frac{\eta}{d_2} \end{pmatrix} \begin{pmatrix} \frac{d\xi}{dt} \\ \frac{d\eta}{dt} \end{pmatrix} = (\xi \;\; \eta) \begin{pmatrix} u_{1,11}^r(x_1^r, x_2^r) & u_{1,12}^r(x_1^r, x_2^r) \\ u_{2,21}^r(x_1^r, x_2^r) & u_{2,22}^r(x_1^r, x_2^r) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

By Proposition 1, the quadratic form on the right-hand side (whose matrix is H^r) is negative definite. This means that its value is negative as long as $(\xi, \eta) \neq 0$. It follows that the quadratic expression on the left-hand side decreases monotonically over time, and ξ and η converge to 0.

A different kind of dynamics involves alternate, discrete responses. Player 1 best-responds to the strategy of player 2, who then best-responds to 1's strategy, and so on. Asymptotic stability with respect to these "ping-pong" dynamics neither implies nor is implied by static stability (Milchtaich 2023). And the well-known geometric condition for asymptotic stability, which is that player 1's reaction curve is steeper than 2's curve, is not implied by and does not imply negative definiteness of H^r.

To see whether, for $r < 1$, the equilibrium (x_1^r, x_2^r) in the Tullock model with a single politician is asymptotic stable with respect to alternate best responses, consider again small deviations, ξ and η, from the equilibrium strategies. If ξ is a best response to η, then player 1's marginal payoff, which is approximated in the first equation in (24), is zero. If player 2 now best-responds to ξ by changing (his deviation from x_2^r) to η', then an expression similar to that in the second equation in (24) is zero, with η' replacing η. The two equalities give the following linearized dynamics for player 2's strategy:

$$\eta' = \frac{u_{1,12}^r(x_1^r, x_2^r) u_{2,21}^r(x_1^r, x_2^r)}{u_{1,11}^r(x_1^r, x_2^r) u_{2,22}^r(x_1^r, x_2^r)} \eta. \tag{25}$$

Convergence to the equilibrium holds if the absolute value of the quotient on the right-hand side is less than 1. Using (7), (8) and (9), we can write this condition as

$$(x_1^r - x_2^r)^2 \frac{c_1'(x_1^r)}{x_2^r(x_1^r + x_2^r)} \cdot \frac{c_2'(x_2^r)}{x_1^r(x_1^r + x_2^r)} < \left(2\frac{c_1'(x_1^r)}{x_1^r + x_2^r} + c_1''(x_1^r)\right)\left(2\frac{c_2'(x_2^r)}{x_1^r + x_2^r} + c_2''(x_2^r)\right).$$

By (3), there exist $\alpha, \beta > 0$ such that $c_1'(x_1^r) + x_1^r c_1''(x_1^r) = \alpha c_1'(x_1^r)$ and $c_2'(x_2^r) + x_2^r c_2''(x_2^r) = \beta c_2'(x_2^r)$. The last inequality is equivalent to

$$\left(\frac{x_1^r - x_2^r}{x_1^r + x_2^r}\right)^2 - \left(\frac{x_1^r - x_2^r}{x_1^r + x_2^r} + \alpha\right)\left(\frac{x_2^r - x_1^r}{x_1^r + x_2^r} + \beta\right) < 0.$$

The expression on the left-hand side is a strictly convex function of the quotient $(x_1^r - x_2^r)/(x_1^r + x_2^r)$, which lies in $(-1,1)$. Therefore, a sufficient condition for this expression to be negative is $1 - (\pm 1 + \alpha)(\mp 1 + \beta) \leq 0$, or $2 + |\alpha - \beta| \leq \alpha\beta$. In particular, the last inequality is a sufficient condition for asymptotic stability of the equilibrium with cost functions $c_1(x_1) = x_1^\alpha$ and $c_2(x_2) = x_2^\beta$. The condition is not necessary, however. Whenever we have a symmetric equilibrium, $x_1^r = x_2^r$, the right-hand side of (25) is zero by (8), which entails fast convergence to the equilibrium from any nearby point. This is so, in particular, in the case of cost functions that are power functions with identical exponents, $\alpha = \beta \geq 0.5$ (Example 1).

References

Epstein, G.S., Milchtaich, I., Nitzan, S., Schwarz, M.E. (2007). Ambiguous political power and contest efforts. *Public Choice* 132, 113–123.

Milchtaich, I. (2012). Comparative statics of altruism and spite. *Games and Economic Behavior* 75, 809–831.

Milchtaich, I. (2021). Internalization of social cost in congestion games. *Economic Theory* 71, 717–760.

Milchtaich, I. (2023). Rooted potential and static stability in games. Mimeo.

Schmidt, F. (2009). Evolutionary stability of altruism and envy in Tullock contests. *Economics of Governance* 10, 247–259.

Tullock, G. (2001). Efficient rent seeking. In: Lockard A.A., Tullock G. (eds) *Efficient Rent-Seeking: Chronicle of an Intellectual Quagmire*. Springer, Boston, MA.