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A Multivariate Stochastic Unit Root Model with an Application to Derivative Pricing

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Abstract

This paper extends recent findings of Lieberman and Phillips (2014) on stochastic unit root (SUR) models to a multivariate case. The extensions are useful because they lead to a generalization of the Black-Scholes formula for derivative pricing. In place of the standard assumption that the price process follows a geometric Brownian motion, we derive a new form of the Black-Scholes equation that allows for a multivariate time varying coefficient element in the price equation. The corresponding formula for the value of a European-type call option is obtained and shown to extend the existing option price formula in a manner that embodies the effect of a stochastic departure from a unit root. An empirical application reveals that the new model is consistent with excess skewness and kurtosis in the price distribution relative to a lognormal distrubution and that data generated from the model are consistent with a volatility smile.

Key words and phrases: Autoregression; Derivative; Nonlinear diffusion; Options; Similarity; Stochastic unit root; Time-varying coefficients.

JEL Classification: C22

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1 Introduction

Unit root and local unit root time series models have attracted much attention in the last few decades, providing a wellspring of work that has been found useful in applied research in many disciplines, including finance. The prototype model

$$\begin{aligned} Y_1 &= \varepsilon_{1,} \\ Y_t &= \mu + \beta Y_{t-1} + \varepsilon_t,\ t = 2, ..., n,\ \varepsilon_t \overset{iid}{\sim} \left(0, \sigma_\varepsilon^2\right),\ t = 1, ..., n, \end{aligned} \tag{1}$$

has substantial flexibility and, when the autoregressive parameter β is in the vicinity of unity, data generated from the model take many plausible forms that include stationary, trend stationary, random wandering, and explosive possibilities. A key mechanism in determining the large sample limit form of the process is the invariance principle for standardized versions of partial sums of the innovations $S_{\lfloor nr \rfloor} = \sum_{t=1}^{\lfloor nr \rfloor} \varepsilon_t$, where $\lfloor nr \rfloor$ is the integer part of nr. The simplest case involves the Donsker result

$$\frac{1}{\sigma_\varepsilon \sqrt{n}} \sum_{t=1}^{\lfloor nr \rfloor} \varepsilon_t \Rightarrow W(r),\ r \in [0, 1], \tag{2}$$

where $W(r)$ is standard Brownian motion and $\Rightarrow$ denotes weak convergence, but much more general results are known to hold (e.g., Phillips, 1987a; see Giraitis et al, 2012, for a recent discussion). As is well known, the limit theory has implications for standardized versions of the output process Y_t when β is in the vicinity of unity.

To illustrate, let n be the number of subintervals into which a T-year period is subdivided, and let μ_A and $\sigma_{\varepsilon,A}$ denote the mean and standard deviation in annualized terms, so that

$$\mu_A = \frac{n}{T}\mu \text{ and } \sigma_{\varepsilon,A} = \sqrt{\frac{n}{T}}\sigma_\varepsilon. \tag{3}$$

Then, when $\beta = 1$, we have $Y_{\lfloor nr \rfloor} = \lfloor nr \rfloor \mu + \sum_{t=1}^{\lfloor nr \rfloor} \varepsilon_t$, leading directly to

$$Y_{\lfloor nr \rfloor} \Rightarrow Y(r) = T\mu_A r + \sqrt{T}\sigma_{\varepsilon,A} W(r). \tag{4}$$

It is difficult to over-emphasize the role that this last formula plays in the literature. For instance, the celebrated Black-Scholes formula (Black and Scholes, 1973, henceforth BS) for option pricing, critically depends on the assumption that stock prices, $S(r)$, follow a geometric Brownian motion, viz.,

$$\frac{dS(r)}{S(r)} = T\mu_A dr + \sqrt{T}\sigma_{\varepsilon,A} dW(r) = dY(r). \tag{5}$$

A tacit assumption that leads to the limit theory embedded in (4) and (5) is that the coefficient β of Y_{t-1} in (1) is fixed and equals unity for all t. For some data sets and models, this assumption may be reasonable *on average*, but it is often likely to be restrictive. Recognition of this limitation has led to the consideration of local unit root (LUR) models where β is fixed (within an array framework) but lies in the vicinity of unity (Chan and Wei, 1987; Phillips, 1987b; Phillips and Magdalinos, 2007). A more realistic working hypothesis might relax the requirement that the coefficient be fixed and allow for some time variation and possible dependencies on other stochastic variables. Phillips and Yu (2011) explored some time variation in the localizing coefficient to collapse in financial markets and bubble migration but used a deterministic β. The stochastic localizing coefficient we use here has the form

$$\begin{aligned} Y_1 &= \mu + \varepsilon_1, \\ Y_t &= \mu + \beta_t(a_n) Y_{t-1} + \varepsilon_t, \; t = 2, ..., n, \end{aligned} \tag{6}$$

where

$$\beta_t(a_n) = \exp\left(\frac{a_n' u_t}{\sqrt{n}}\right) \tag{7}$$

and u_t is a $K \times 1$ vector and is the source of the variation in the autoregressive coefficient. In applications, u_t will typically stand for a vector of returns on market indices and/or related stocks, but it need not be the case. We assume that partial sums of $\eta_t = (u_t, \varepsilon_t)'$ satisfy the invariance principle

$$n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} \eta_t \Rightarrow B(r) \equiv \mathrm{BM}(\Sigma), \; \Sigma = \begin{pmatrix} \Sigma_u & \Sigma_{u\varepsilon} \\ \Sigma_{u\varepsilon}' & \sigma_\varepsilon^2 \end{pmatrix}, \tag{8}$$

where $B = (B_u, B_\varepsilon)'$ is a vector Brownian motion with Σ positive definite and component $K \times K$ submatrix $\Sigma_u > 0$ and scalar $\sigma_\varepsilon^2 > 0$. The parameters

μ and Σ relate to the instantaneous drift and covariance matrix of the continuous process and are to be distinguished from their annual- and T-year counterparts.

The model (6)-(7) is a multivariate version of a stochastic unit root model (STUR) model introduced in Lieberman and Phillips (2014) and belongs to the general class of time varying coefficients (TVC) models. That paper explored the connection of the STUR model to recent developments in the literature, including similarity models. The term 'similarity' originated from the theory of empirical similarity, developed in Gilboa *et. al.* (2006), and under which the value of $\beta_t(a_n)$ is dictated by the degree of similarity between Y_t and Y_{t-1}, as measured by the input u_t to the exponent of (7). The main feature of the STUR model is that for any given t, the coefficient $\beta_t(a_n)$ can be less than-, equal to-, or greater than unity, with a time specific value that is determined by u_t.

This paper derives the stochastic limit theory of the STUR model (6)-(7) and uses this limit theory to generalize the classic stochastic differential equation (5) so that it embodies the limit of (6)-(7). The special case where $a = 0$ produces the limit process (4) and so the new limit theory provides an extension of (5) to include a TVC feature. Within this framework, a further contribution of the paper is to derive the Black Scholes sde for derivative pricing and the Black Scholes price of a European call option under the new scheme.

The idea of modifying the base model (5) to enhance realism is by no means new. Two main streams of extension appear in the literature. The first is the stochastic volatility (SV) model (e.g., Hull and White (1987), Heston (1993)). In that model, if the volatility process is not correlated with $W(r)$, the process is consistent with a symmetric volatility smile (Renault and Touzi (1996)), whereas if there is a negative correlation between the two, the process will be consistent with an asymmetric volatility skew, which is often claimed to be empirically better suited to stock options (see, for instance, Hull, 2009). One criticism of the SV model is that unlike the BS model, it is incomplete, so that one cannot take every reasonable contingent claim and perfectly hedged it by a self-financing portfolio consisting of stocks and bonds.

In the second stream of literature it is suggested to replace the standard Brownian motion driver process in (5) by a fractional Brownian motion (FBM), B^H, with a Hurst parameter H. See, for instance, Hu and Øksendal (2000) and Biagini *et. al.* (2008). While the FBM model is reported to fit certain data sets better than the base model (5), B^H has correlated increments and is not a semimartingale so that classic Itô calculus is inapplicable and the model introduces arbitrage possibilities, see Bjork and Hult (2005). In contrast, the extension based on a similarity STUR model produces an arbitrage free model and standard Itô calculus is applicable.

In sum, the BS model (5) is simple, tractable and possesses some desirable features such as completeness and no-arbitrage but suffers limitations such as no implied volatility smile and the absence of heavy tails. These features of the BS model suggest that there is value in an extension of the model that captures its main advantages while overcoming its main empirical shortcomings. The model presented here seeks to achieve this goal.

The plan for the remainder of the paper is as follows. Section 2 develops the limit theory for the multivariate STUR model and provides the associated sde. Section 3 derives the BS sde corresponding to our model and the price process associated with it. The BS European-type option pricing sde is generalized in Section 4, specializing to a formula for the value of a European call option in Section 5. An empirical application is conducted in Section 6, showing that our model is consistent with excess skewness, kurtosis, and an implied volatility smile, in contrast to the lognormality and flat volatility implied by the BS model.

2 Continuous Limit of the STUR Model

By backward substitution, the model (6) gives the following solution from initialization at $Y_1 = \mu + \varepsilon_1$

$$Y_2 = (\beta_2 + 1)\,\mu + (\beta_2 \varepsilon_1 + \varepsilon_2)\,, \; Y_3 = (\beta_3\beta_2 + \beta_3 + 1)\,\mu + (\beta_3\beta_2\varepsilon_1 + \beta_3\varepsilon_2 + \varepsilon_3)\,,$$

and generally for any $t \geq 2$,

$$\begin{aligned} Y_t &= \left(\sum_{s=1}^{t-1} \left(\prod_{j=s+1}^{t} \beta_j \right) + 1 \right) \mu + \sum_{s=1}^{t-1} \left(\prod_{j=s+1}^{t} \beta_j \right) \varepsilon_s + \varepsilon_t \\ &= \left(\sum_{s=1}^{t-1} \left(\prod_{j=s+1}^{t} \beta_j \right) + 1 \right) \mu + \sum_{s=1}^{t-1} e^{\frac{a'}{\sqrt{n}} \sum_{j=s+1}^{t} u_j} \varepsilon_s + \varepsilon_t \qquad (9) \\ &=: h_t(\beta)\,\mu + Y_t^*, \text{ say.} \qquad (10) \end{aligned}$$

In what follows it is convenient to expand the probability space as necessary to ensure that the convergence in (8) is in probability so that $n^{-1/2}\sum_{t=1}^{\lfloor nr \rfloor}\eta_t \to_p B(r)$. Standardizing Y_t^* we then have the following result.

Lemma 1 *In a suitably expanded probability space as $n \to \infty$*

$$n^{-1/2}Y_{\lfloor nr \rfloor}^* \to_p e^{a'B_u(r)}\left(\int_0^r e^{-a'B_u(p)}dB_\varepsilon(p) - a'\Sigma_{u\varepsilon}\int_0^r e^{-a'B_u(p)}dp\right) := G_a(r), \tag{11}$$

and

$$\frac{1}{n}\left(\sum_{s=1}^{\lfloor nr \rfloor -1}\left(\prod_{j=s+1}^{\lfloor nr \rfloor}\beta_j\right)+1\right)\mu \to_p \mu e^{a'B_u(r)}\int_0^r e^{-a'B_u(p)}dp. \tag{12}$$

Using the differentials $d\left(e^{a'B_u(r)}\right) = e^{a'B_u(r)}\left\{a'dB_u(r) + \frac{1}{2}a'\Sigma_u a dr\right\}$ and

$$d\left(\int_0^r e^{-a'B_u(p)}dB_\varepsilon(p) - a'\Sigma_{u\varepsilon}\int_0^r e^{-a'B_u(p)}dp\right) = e^{-a'B_u(r)}\left(dB_\varepsilon(r) - a'\Sigma_{u\varepsilon}dr\right)$$

we find that $G_a(r)$ follows the stochastic differential equation (sde)

$$\begin{aligned} dG_a(r) &= e^{a'B_u(r)}\left(\int_0^r e^{-a'B_u(p)}dB_\varepsilon(p) - a'\Sigma_{u\varepsilon}\int_0^r e^{-a'B_u(p)}dp\right)\left\{a'dB_u(r) + \frac{1}{2}a'\Sigma_u a dr\right\} \\ &\quad + dB_\varepsilon(r) - a'\Sigma_{u\varepsilon}dr \\ &= G_a(r)\,a'dB_u(r) + dB_\varepsilon(r) + \left[\frac{a'\Sigma_u a}{2}G_a(r) - a'\Sigma_{u\varepsilon}\right]dr, \end{aligned} \tag{13}$$

which has the form of a nonlinear diffusion driven by vector Brownian motion (B_u, B_ε). Observe that when $a = 0$, $G_a(r)$ reduces simply to the Brownian motion $B_\varepsilon(r)$.

It follows from (10) - (12) that

$$Y_n(r) := Y_{\lfloor nr \rfloor} = h_{\lfloor nr \rfloor}(\beta)\mu + Y_{\lfloor nr \rfloor}^* \sim n\mu e^{a'B_u(r)}\int_0^r e^{-a'B_u(p)}dp + \sqrt{n}G_a(r). \tag{14}$$

The first term in (14) contributes an additional drift $(n\mu dr)$ to the differential equation (13), leading to the following approximate continuous time law of motion for $Y_n(r)$

$$dY_n(r) = n\mu dr + \sqrt{n}\left(G_a(r)\,a'dB_u(r) + dB_\varepsilon(r) + \left[\frac{a'\Sigma_u a}{2}G_a(r) - a'\Sigma_{u\varepsilon}\right]dr\right). \tag{15}$$

In this system the nonlinear diffusion process G_a affects both the martingale component and the drift. When $a = 0$, the system reduces to $dY_n(r) = n\mu dr + \sqrt{n}dB_\varepsilon(r)$, which corresponds in form to the classic equation (5).

3 A STUR Extension of the BS Model

3.1 The Price Process

A fundamental building block in the BS option price formula involves a random walk with a drift, which in the discrete case amounts to equation (1) with $\beta = 1$, i.e., the STUR process with $a = 0$. We use the results of the last section to suggest a suitable generalization of the BS formula. To fix ideas, it is convenient to set the parameters as $\mu_T = n\mu$, and $\sigma_{\varepsilon,T} = \sqrt{n}\sigma_\varepsilon$, so that when $\beta = 1$

$$Y_n(r) \sim \mu_T r + \sigma_{\varepsilon,T}W(r),$$

which corresponds to (4) in annual terms. The subscripts A and T will be used in what follows to distinguish between annualized and T-year period quantities, respectively.

A key assumption in the BS option price model is that stock prices, $S(t)$, follow a geometric Brownian motion, viz.,

$$\frac{dS(r)}{S(r)} = T\mu_A dr + \sqrt{T}\sigma_{\varepsilon,A}dW(r) = \mu_T dr + \sigma_{\varepsilon,T}dW(r). \tag{16}$$

The right side specializes (15) as above, thereby suggesting the latter as a suitable extension giving a geometric nonlinear diffusion. We will use this extension to obtain derivative pricing formulae under weaker conditions than BS, including the price of a European call option.

Define $B^* := (B_u^*, B_\varepsilon^*)' = \Sigma^{-1/2}B$, so that B^* is a vector standard Brownian motion. Write the lower triangular square root of Σ as

$$\begin{aligned}
\Sigma^{1/2} \quad : &= \begin{pmatrix} \left[\Sigma^{1/2}\right]_{1,1} & 0 \\ \left[\Sigma^{1/2}\right]_{2,1} & \left[\Sigma^{1/2}\right]_{2,2} \end{pmatrix} := \begin{pmatrix} \Sigma_u^{1/2} & 0 \\ \Sigma_{u\varepsilon}' \Sigma_u^{-1/2} & \left(\sigma_\varepsilon^2 - \Sigma_{u\varepsilon}' \Sigma_u^{-1} \Sigma_{u\varepsilon}\right)^{1/2} \end{pmatrix} \\
&=: \begin{pmatrix} \left[\Sigma^{1/2}\right]_1 \\ \left[\Sigma^{1/2}\right]_2 \end{pmatrix},
\end{aligned}$$

where $\Sigma_u^{1/2}$ is the positive definite square root of Σ_u. We can write (14) as

$$\begin{aligned}
Y_n(r) &= \mu_T e^{a' B_u(r)} \int_0^r e^{-a' B_u(p)} dp \\
&\quad + \sqrt{n} e^{a' B_u(r)} \left(\left[\Sigma^{1/2}\right]_2 \int_0^r e^{-a' B_u(p)} dB^*(p) - \frac{\left(\frac{a}{\sqrt{n}}\right)' (n\Sigma_{u\varepsilon}) \int_0^r e^{-a' B_u(p)} dp}{\sqrt{n}} \right) \\
&= \mu_T e^{a' B_u(r)} \int_0^r e^{-a' B_u(p)} dp \\
&\quad + e^{a' B_u(r)} \left(\left[\Sigma_T^{1/2}\right]_2 \int_0^r e^{-a' B_u(p)} dB^*(p) - a_n' \Sigma_{u\varepsilon,T} \int_0^r e^{-a' B_u(p)} dp \right), \qquad (17)
\end{aligned}$$

where $a_n = \frac{a}{\sqrt{n}}, \Sigma_{u\varepsilon,T} = n\Sigma_{u\varepsilon}$, and $\Sigma_T = n\Sigma$. Further, $\frac{a'}{\sqrt{n}} \sum_{j=1}^t u_j \sim a' B_u(r) = a' \Sigma_u^{1/2} B_u^*(r) = a_n' \left[\Sigma_T^{1/2}\right]_{1,1} B_u^*(r)$. Hence, (17) becomes

$$\begin{aligned}
Y_n(r) &= \mu_T e^{a_n' \left[\Sigma_T^{1/2}\right]_{1,1} B_u^*(r)} \int_0^r e^{-a_n' \left[\Sigma_T^{1/2}\right]_{1,1} B_u^*(p)} dp \\
&\quad + e^{a_n' \left[\Sigma_T^{1/2}\right]_{1,1} B_u^*(r)} \left(\left[\Sigma_T^{1/2}\right]_2 \int_0^r e^{-a_n' \left[\Sigma_T^{1/2}\right]_{1,1} B_u^*(p)} dB_\varepsilon^*(p) \right. \qquad (18) \\
&\quad \left. - a_n' \Sigma_{u\varepsilon,T} \int_0^r e^{-a_n' \left[\Sigma_T^{1/2}\right]_{1,1} B_u^*(p)} dp \right) \\
&= \mu_T e^{a_n' \left[\Sigma_T^{1/2}\right]_{1,1} B_u^*(r)} \int_0^r e^{-a_n' \left[\Sigma_T^{1/2}\right]_{1,1} B_u^*(p)} dp + G_{a_n,T}(r), \text{ say,}
\end{aligned}$$

where by a simple calculation

$$G_{a_n,T}(r) = e^{a_n'\left[\Sigma_T^{1/2}\right]_{1,1}B_u^*(r)}\left\{\left[\Sigma_T^{1/2}\right]_2\int_0^r e^{-a_n'\left[\Sigma_T^{1/2}\right]_{1,1}B_u^*(p)}dB_\varepsilon^*(p)\right.$$
$$\left.-a_n'\Sigma_{u\varepsilon,T}\int_0^r e^{-a_n'\left[\Sigma_T^{1/2}\right]_{1,1}B_u^*(p)}dp\right\} = \sqrt{n}G_a(r).$$

Equation (18) is simply equation (11), but the former is in period-T terms whereas the latter is given in instantaneous terms. The distinction is useful in matching the notation of the BS formula, as well as for the empirical work, where care is needed in determining the precise interpretation of the means and covariances.

Let $b(r) = (G_a(r)a', 1)'$ and $b_T(r) = (G_{a_n,T}(r)a_n', 1)'$. Using (13) and changing notation, we rewrite (15) as

$$\begin{aligned}
dY_n(r) &= n\mu dr + \sqrt{n}dG_a(r) = \mu_T dr + dG_{a_n,T}(r) \\
&= \mu_T dr + \left[\frac{a_n'\Sigma_{u,T}a_n}{2}G_{a_n,T}(r) - a_n'\Sigma_{u\varepsilon,T}\right]dr + b_T(r)'\,dB(r) \\
&= \left[\mu_T + \frac{a_n'\Sigma_{u,T}a_n}{2}G_{a_n,T}(r) - a_n'\Sigma_{u\varepsilon,T}\right]dr + b_T(r)'\left[\Sigma_T^{1/2}\right]dB^*(r).
\end{aligned}$$

In the light of this equation we replace the standard formula (16) with the above, which leads to the following geometric nonlinear diffusion that is based on the similarity STUR model

$$\frac{dS(r)}{S(r)} = \left[\mu_T + \frac{a_n'\Sigma_{u,T}a_n}{2}G_{a_n,T}(r) - a_n'\Sigma_{u\varepsilon,T}\right]dr + b_T(r)'\left[\Sigma_T^{1/2}\right]dB^*(r). \tag{19}$$

Whereas (16) is geometric Brownian motion, the system (19) is a geometric price process that involves the nonlinear diffusion $G_{a_n,T}(r)$ and Brownian motion driver process B^*. Observe that this system collapses to (16) when $a = 0$. Hence, we may regard (19) as a process that is parametrically local to geometric Brownian motion.

Next consider the process defined by $G(r) = \log(S(r))$ and let $[S]_r$ be the quadratic variation of $S(r)$. By stochastic differentiation we have

$$\begin{aligned}
dG(r) &= \frac{dS(r)}{S(r)} - \frac{1}{2}\frac{1}{S^2(r)} d[S]_r \\
&= \frac{1}{S(r)}\left\{\mu_T + \frac{a_n'\Sigma_{u,T}a_n}{2}G_{a_n,T}(r) - a_n'\Sigma_{u\varepsilon,T}\right\} S(r)\,dr \\
&\quad + \frac{1}{S(r)} S(r)\, b_T(r)' \left[\Sigma_T^{1/2}\right] dB^*(r) \\
&\quad - \frac{1}{2S^2(r)}\left\{b_T(r)' \left[\Sigma_T^{1/2}\right] S^2(r) \left[\Sigma_T^{1/2}\right]' b_T(r)\right\} dr \\
&= \left\{\mu_T + \frac{a_n'\Sigma_{u,T}a_n}{2}G_{a_n,T}(r) - a_n'\Sigma_{u\varepsilon,T} - \frac{1}{2} b_T(r)'\Sigma_T b_T(r)\right\} dr \\
&\quad + b_T(r)'\Sigma_T dB^*(r).
\end{aligned}$$

Now

$$b_T(r)'\Sigma_T b_T(r) = G^2_{a_n,T}(r)\, a_n'\Sigma_{u,T}a_n + 2G_{a_n,T}(r)\, a_n'\Sigma_{u\varepsilon,T} + \sigma^2_{\varepsilon,T} \tag{20}$$

and thus,

$$\begin{aligned}
dG(r) &= \Bigg(\mu_T + \frac{a_n'\Sigma_{u,T}a_n}{2}G_{a_n,T}(r) - a_n'\Sigma_{u\varepsilon,T} \\
&\quad - \frac{1}{2}\left(G^2_{a_n,T}(r)\, a_n'\Sigma_{u,T}a_n + 2G_{a_n,T}(r)\, a_n'\Sigma_{u\varepsilon,T} + \sigma^2_{\varepsilon,T}\right)\Bigg) dr \\
&\quad + b_T(r)'\Sigma_T dB^*(r) \\
&= \left(\mu_T - a_n'\Sigma_{u\varepsilon,T} - \frac{\sigma^2_{\varepsilon,T}}{2}\right) dr \\
&\quad + \left(\left(\frac{a_n'\Sigma_{u,T}a_n}{2} - a_n'\Sigma_{u\varepsilon,T}\right) G_{a_n,T}(r) - \frac{a_n'\Sigma_{u,T}a_n}{2} G^2_{a_n,T}(r)\right) dr \\
&\quad + b_T(r)'\, dB(r).
\end{aligned}$$

When $a = 0$, we retain the classic formula

$$\begin{aligned} dG(r) &= \left(\mu_T - \frac{\sigma_{\varepsilon,T}^2}{2}\right) dr + dB_\varepsilon(r) = \left(\mu_T - \frac{\sigma_{\varepsilon,T}^2}{2}\right) dr + \sigma_{\varepsilon,T} dB_\varepsilon^*(r) \\ &= T\left(\mu_A - \frac{\sigma_{\varepsilon,A}^2}{2}\right) dr + \sqrt{T}\sigma_{\varepsilon,A} dB_\varepsilon^*(r). \end{aligned} \tag{21}$$

In this case, (21) implies that $\log(S(r)) - \log(S(0)) \sim N\left(\left(\mu_T - \frac{\sigma_{\varepsilon,T}^2}{2}\right) r, \sigma_{\varepsilon,T}^2 r\right)$ so that

$$S(r) = S(0)\exp\left\{N\left(\left(\mu_T - \frac{\sigma_{\varepsilon,T}^2}{2}\right) r, \sigma_{\varepsilon,T}^2 r\right)\right\}. \tag{22}$$

When $a \neq 0$, the price process $S(r)$ satisfies

$$\begin{aligned} & G(r) - G(0) = \log(S(r)) - \log(S(0)) \\ = & \int_0^r \left(\mu_T - a_n' \Sigma_{u\varepsilon,T} - \frac{\sigma_{\varepsilon,T}^2}{2}\right) dr + \int_0^r \left(\left(\frac{a_n' \Sigma_{u,T} a_n}{2} - a_n' \Sigma_{u\varepsilon,T}\right) G_{a_n,T}(s) \right. \\ & \left. - \frac{a_n' \Sigma_{u,T} a_n}{2} G_{a_n,T}^2(s)\right) ds + \int_0^r b_T(s)' dB(s) \\ = & \left(\mu_T - a_n' \Sigma_{u\varepsilon,T} - \frac{\sigma_{\varepsilon,T}^2}{2}\right) r + \int_0^r \left\{\left(\frac{a_n' \Sigma_{u,T} a_n}{2} - a_n' \Sigma_{u\varepsilon,T}\right) G_{a_n,T}(s) - \frac{a_n' \Sigma_{u,T} a_n}{2} G_{a_n,T}^2(s)\right\} ds \\ & + a_n' \int_0^r G_{a_n,T}(s) dB_u(s) + B_\varepsilon(r). \end{aligned}$$

It follows that

$$\begin{aligned} S(r) = & S(0)\exp\left\{\left(\mu_T - \frac{\sigma_{\varepsilon,T}^2}{2} - a_n' \Sigma_{u\varepsilon,T}\right) r \right. \\ & + \int_0^r \left(\left(\frac{a_n' \Sigma_{u,T} a_n}{2} - a_n' \Sigma_{u\varepsilon,T}\right) G_{a_n,T}(s) - \frac{a_n' \Sigma_{u,T} a_n}{2} G_{a_n,T}^2(s)\right) ds \\ & \left. + a_n' \int_0^r G_{a_n,T}(s) dB_u(s) + B_\varepsilon(r)\right\}. \end{aligned} \tag{23}$$

Equation (23) can then be used in place of (22) when calculating BS option prices.

3.2 BS European Option Pricing

Following Hull (2009), let $f(r,x)$ be the price at r of a European-style derivative of a stock, such as a European call option, where $x \equiv S(r)$ is the stock price and $A(r)$ is the price of the riskless asset. Let $(\alpha_S(r), \alpha_A(r))$ be the associated self-financing portfolio at time r. The value of the portfolio is

$$V(r) = \alpha_S(r) S(r) + \alpha_A(r) \gamma(r),$$

where $\gamma(r) = e^{r_{f,T} r}$ and $r_{f,T}$ is the period-T risk-free rate of interest. The sde corresponding to the portfolio $V(r)$ is

$$dV(r) = \alpha_S(r) dS(r) + \alpha_A(r) r_{f,T} \gamma(r) dr, \tag{24}$$

which is the self-financing condition. We must have $df(r,x) = dV(r)$ and by direct calculation

$$\begin{aligned} dV(r) &= \alpha_S(r) \left\{ \left(\mu_T + \frac{a_n' \Sigma_{u,T} a_n}{2} G_{a_n,T}(r) - a_n' \Sigma_{u\varepsilon,T} \right) S(r) dr \right. \\ &\quad \left. + S(r) b_T(r)' dB(r) \right\} + r_{f,T} \alpha_A(r) \gamma(r) dr \\ &= \left\{ \alpha_S(r) \left(\mu_T + \frac{a_n' \Sigma_{u,T} a_n}{2} G_{a_n,T}(r) - a_n' \Sigma_{u\varepsilon,T} \right) S(r) \right. \\ &\quad \left. + r_{f,T} \alpha_A(r) \gamma(r) \right\} dr + \alpha_S(r) S(r) b_T(r)' dB(r). \end{aligned} \tag{25}$$

Now, in view of (19), we have

$$(d(S(r)))^2 = S^2(r) b_T(r)' \Sigma_T b_T(r) dr, \tag{26}$$

and since $df(r,x) = f_r dr + f_x dS(r) + \frac{1}{2} f_{xx} (d(S(r)))^2$, we deduce that

$$\begin{aligned} df(r, S(r)) &= f_r dr + f_x \left\{ \mu_T + \frac{a_n' \Sigma_{u,T} a_n}{2} G_{a_n,T}(r) - a_n' \Sigma_{u\varepsilon,T} \right\} S(r) dr \\ &\quad + f_x S(r) b_T(r)' dB(r) + \frac{1}{2} f_{xx} S^2(r) b_T(r)' \Sigma_T b_T(r) dr \\ &= \left\{ f_r + f_x \left\{ \mu_T + \frac{a_n' \Sigma_{u,T} a_n}{2} G_{a_n,T}(r) - a_n' \Sigma_{u\varepsilon,T} \right\} S(r) \right. \\ &\quad \left. + \frac{1}{2} f_{xx} S^2(r) b_T(r)' \Sigma_T b_T(r) \right\} dr + f_x S(r) b_T(r)' dB(r). \end{aligned} \tag{27}$$

Equating the coefficients of dr and of the stochastic component in (25) and (27) gives

$$\begin{aligned}
&\alpha_S(r)\left\{\mu_T+\frac{a_n'\Sigma_{u,T}a_n}{2}G_{a_n,T}(r)-a_n'\Sigma_{u\varepsilon,T}\right\}S(r)+r_{f,T}\alpha_A(r)\gamma(r) \\
= \; & f_r+f_x\left\{\mu_T+\frac{a_n'\Sigma_{u,T}a_n}{2}G_{a_n,T}(r)-a_n'\Sigma_{u\varepsilon,T}\right\}S(r) \\
&+\frac{1}{2}f_{xx}S^2(r)\,b_T(r)'\,\Sigma_T b_T(r)
\end{aligned} \tag{28}$$

and

$$\alpha_S(r)\,S(r)\,b_T(r)'\,dB(r)=f_xS(r)\,b_T(r)'\,dB(r)\,. \tag{29}$$

The latter yields

$$\alpha_S(r)=f_x \tag{30}$$

which is the condition in the classic case. Using this condition in (28) we have

$$\begin{aligned}
&\alpha_S(r)\left\{\mu_T+\frac{a_n'\Sigma_{u,T}a_n}{2}G_{a_n,T}(r)-a_n'\Sigma_{u\varepsilon,T}\right\}S(r)+r_{f,T}\alpha_A(r)\gamma(r) \\
= \; & f_r+\alpha_S(r)\left\{\mu_T+\frac{a_n'\Sigma_{u,T}a_n}{2}G_{a_n,T}(r)-a_n'\Sigma_{u\varepsilon,T}\right\}S(r) \\
&+\frac{1}{2}f_{xx}S^2(r)\,b_T(r)'\,\Sigma_T b_T(r)\,.
\end{aligned}$$

which implies $r_{f,T}\alpha_A(r)\gamma(r)=f_r+\frac{1}{2}f_{xx}S^2(r)\,b_T(r)'\,\Sigma_T b_T(r)$ and therefore,

$$\alpha_A(r)=\frac{1}{r_{f,T}\gamma(r)}\left\{f_r+\frac{1}{2}f_{xx}S^2(r)\,b_T(r)'\,\Sigma_T b_T(r)\right\}. \tag{31}$$

In view of (20), when $a=0$ the last condition collapses to the well known condition

$$\alpha_A(r)=\frac{1}{r_{f,T}\gamma(r)}\left\{f_r+\frac{1}{2}f_{xx}S^2(r)\,\sigma^2_{\varepsilon,T}\right\}.$$

Since $V(r) = f(r, x)$, it follows that $\alpha_S(r) S(r) + \alpha_A(r)\gamma(r) = f(r, x)$, so that using (30) and (31) we obtain

$$f_x S(r) + \frac{1}{r_{f,T}\gamma(r)} \left\{ f_r + \frac{1}{2} f_{xx} S^2(r) b_T(r)' \Sigma_T b_T(r) \right\} \gamma(r) = f,$$

and finally,

$$r_{f,T} f_x S(r) + f_r + \frac{1}{2} f_{xx} S^2(r) b_T(r)' \Sigma_T b_T(r) = r_{f,T} f. \tag{32}$$

Equation (32) is the generalized BS sde for a European style derivative of a stock. When $a = 0$ the formula reduces to the well-known relationship

$$r_{f,T} f_x S(r) + f_r + \frac{1}{2} f_{xx} S^2(r) \sigma^2_{\varepsilon,T} = r_{f,T} f.$$

When $K = 1$ and $a \neq 0$, (32) becomes

$$r_{f,T} f_x S(r) + f_r + \frac{1}{2} f_{xx} S^2(r) \left\{ G^2_{a_n,T}(r) a_n^2 \sigma^2_{u,T} + 2a\sigma_{u\varepsilon} G_{a_n,T}(r) + \sigma^2_{\varepsilon,T} \right\} = r_{f,T} f.$$

4 The Value of a European Call Option

At time 0, the value of a European call option maturing in T years is

$$C = e^{-r_{f,T}} E^Q \{S_T - K, 0\},$$

where Q is the risk neutral measure and K is the strike price. In our notation, $S_T \equiv S(1)$. From (23),

$$\begin{aligned} S(r) &= S(0) \exp \left\{ \left(\mu_T - \frac{\sigma^2_{\varepsilon,T}}{2} - a_n' \Sigma_{u\varepsilon,T} \right) r \right. \\ &\quad + \int_0^r \left(\left(\frac{a_n' \Sigma_{u,T} a_n}{2} - a_n' \Sigma_{u\varepsilon,T} \right) G_{a_n,T}(s) - \frac{a_n' \Sigma_{u,T} a_n}{2} G^2_{a_n,T}(s) \right) ds \\ &\quad \left. + a_n' \int_0^r G_{a_n,T}(s)\, dB_u(s) + B_\varepsilon(r) \right\} \end{aligned}$$

and when $K=1$ this reduces to

$$\begin{aligned} S(r) &= S(0)\exp\left\{\left(\mu_T-\frac{\sigma^2_{\varepsilon,T}}{2}-a_n\sigma_{u\varepsilon,T}\right)r\right. \\ &\quad +\int_0^r\left(\left(\frac{a_n^2\sigma^2_{u,T}}{2}-a_n\sigma_{u\varepsilon,T}\right)G_{a_n,T}(s)-\frac{a_n^2\sigma^2_{u,T}}{2}G^2_{a_n,T}(s)\right)ds \\ &\quad \left.+a_n\int_0^r G_{a_n,T}(s)\,dB_u(s)+B_\varepsilon(r)\right\}. \qquad (33) \end{aligned}$$

In order to simulate the value of a European call option maturing in T years, we calculate $G_{a_n,T}(r)$ over the grid $\{r=0,1/n,2/n,...,1\}$, after simulating a vector Brownian motion driver process and noting that T corresponds to $r=1$. We then calulate the sample mean of $\{S(1)-K,0\}$, denoted by $\overline{\{S(1)-K,0\}}$, using a large number of replications. The estimated European call option price is

$$\hat{C}=e^{-r_{f,T}}\overline{\{S(1)-K,0\}}.$$

If instead we choose to convert (33) to annual terms, we can rewrite it, with $r=1$, as

$$\begin{aligned} S(1) &= S(0)\exp\left\{\left(r_{f,A}-\frac{\sigma^2_{\varepsilon,A}}{2}-a_n\sigma_{u\varepsilon,A}\right)T\right. \\ &\quad +T\int_0^1\left(\left(\frac{a_n^2\sigma^2_{u,A}}{2}-a_n\sigma_{u\varepsilon,A}\right)G_{a_n,A}(s)-\frac{a_n^2\sigma^2_{A}}{2}G^2_{a_n,A}(s)\right)ds \\ &\quad \left.+\sqrt{T}\left(a_n\int_0^1 G_{a_n,A}(s)\,dB_u(s)+B_\varepsilon(r)\right)\right\}, \qquad (34) \end{aligned}$$

where $G_{a_n,A}(r)$ is defined in the following equation

$$\begin{aligned}
& G_{a_n,T}(r) = e^{a_n'\left[\Sigma_T^{1/2}\right]_{1,1} B_u^*(r)} \left(\left[\Sigma_T^{1/2}\right]_2 \int_0^r e^{-a_n'\left[\Sigma_T^{1/2}\right]_{1,1} B_u^*(p)} dB^*(p) \right. \\
& \left. -a_n' \Sigma_{u\varepsilon,T} \int_0^r e^{-a_n'\left[\Sigma_T^{1/2}\right]_{1,1} B_u^*(p)} dp \right) \\
= \; & e^{\sqrt{T} a_n'\left[\Sigma_A^{1/2}\right]_{1,1} B_u^*(r)} \left(\sqrt{T} \left[\Sigma_T^{1/2}\right]_2 \int_0^r e^{-\sqrt{T} a_n'\left[\Sigma_A^{1/2}\right]_{1,1} B_u^*(p)} dB^*(p) \right. \\
& \left. -T a_n' \Sigma_{u\varepsilon,A} \int_0^r e^{-\sqrt{T} a_n'\left[\Sigma_A^{1/2}\right]_{1,1} B_u^*(p)} dp \right) \qquad (35) \\
\equiv \; & G_{a_n,A}(r).
\end{aligned}$$

The estimated call option price is $\hat{C} = e^{-r_{f,A}T} \overline{\{S(1) - K, 0\}}$, where $S(1)$ is computed from (34) and (35).

5 An Empirical Application

We downloaded from Yahoo Finance daily data on the closing prices of Google and Nasdaq composite indexes (tickers GOOG and ^IXIC), over the period 1-2-2009 through to 11-20-2013, giving a total of 1231 observations for each series.

With an obvious notation, we fitted the following empirical STUR model

$$\log(\text{Google})_t = \mu + \exp\left(\frac{a}{\sqrt{n}} \Delta \log(\text{Nasdaq})_t\right) \log(\text{Google})_{t-1} + \varepsilon_t$$

obtaining an estimate for $\widehat{(a/\sqrt{n})}$. The Akaike (AIC), Schwarz (SC) and sum of squared errors (SSE) values for the model are -5.976, -5.968, and 0.182, respectively. On the other hand, for the model (1) the figures are -5.327, -5.319 and 0.349, respectively, and for the model (1) with $\beta = 1$ the corresponding values are -5.327, -5.323 and 0.349. Thus, in terms of selection criteria, the STUR model provides a clear improvement over the basic model.

We remark that the estimates of $a/\sqrt{n}$ seem to change very little with subsamples of n and in our data the value turned out to be $\widehat{(a/\sqrt{n})} = 0.14118$. We have also obtained the standard deviation of the return on Nasdaq in annual terms, $\hat{\sigma}_{u,A}$ (Nasdaq) $= 0.21037$ and the sample correlation between Google returns and Nasdaq returns, $\hat{\rho} = 0.692347$, noting that it is invariant to the time period (i.e., whether it is in annual terms or otherwise). Furthermore, for the annual risk-free rate we used the Federal Funds Rate from the Bloomberg website quoted as $r_{f,A} = 0.0007$ (0.07% per annum). The 3-month treasury yield was identical (in annual terms). Other choices of the risk-free rate, such as the 12 month treasury yield (0.11%) or the 2-year yield (0.29%), which may be better suited to use for options with longer expiration periods might also be considered. But for this empirical illustration we have used the Federal Funds Rate.

Google is a non-dividend paying stock and thus, is suited to this application. From the Nasdaq website we have downloaded 11-27-2013 call option prices on Google, with expiration on 1-3-2014 (36 days), 1-17-2015 (415 days) and 1-15-2016 (778 days). For the former, we considered strike prices $K =$1035, 1065, 1100 whereas for the other two, we considered $K =$1030, 1060, 1100. The closing price for Goggle stock on 11-27-2013 was 1063.11. Based on this data, we calculated in Mathematica the implied volatility of Google and this value was used for $\hat{\sigma}_A$ (Google) and in the construction of $\hat{\Sigma}_A$.

For each scenario, we have verified the call option prices quoted on the Nasdaq website in Mathematica by calculating the Black and Scholes classic formula (see, e.g., equation (13.20) of Hull (2009)). Using the implied volatility for Google in place of historical volatility thus enabled us to properly compare our formula with Nasdaq's quote serving as a benchmark. If the historical standard deviation were used instead, Nasdaq's quote would not have been an appropriate benchmark for comparison.

To simulate $G_{a_n,A}(r)$, we generated a vector of two standard Brownian motions scaled by $\hat{\Sigma}_A^{1/2}$. For a_n, we used the estimate $\widehat{(a/\sqrt{n})} = 0.14118$ and for T, we substituted the number of days to expiration divided by 365. For the calculation of $\hat{\sigma}_{u,A}$ (Nasdaq), we multiplied the standard deviation of the daily return by $\sqrt{252}$, as per standard practice. The call price $S(1)$ was calculated by calculating (34) with $n = 500$ integral points and 500 replications over simulated Brownian motions.

The results are provided in Table 1. For short T, such as $T = 36/365$, there is not much difference between our Model and the BS figures. The reason is that in this case $Ta_n = (36/365) \times 0.14118 \cong 0.0014$, which makes the exponent in $G_a(r)$ small for most realizations of $B_u(r)$ and hence, the results are inevitably close to those of BS. On the other hand, for longer expiration periods such as that where $Ta_n = (778/365) \times 0.14118 = 0.30093$, we have

$$Ee^{0.30093\times\hat{\sigma}_{u,A}(\text{Nasdaq})\times B_u(r)} = e^{\frac{1}{2}r(0.30093\times0.21037)^2} \cong 1.001.$$

Due to the embedded volatility in the TVC model, the difference in prices range from +21.7%, for the TVC over BS, when the option is deep in the money (i.e., $K = 1030$), to -9.7%, when the option is out of the money (i.e., $K = 1100$). To explore these findings further, we ran 1000 replications with $n = 1000$ integral points of option prices corresponding to the rhs sub-block of Table 1, viz., $T = 2.132 (= 778/365)$, $\rho = \hat{\rho}$ and $S(0) = 1063.1$, the results of which are given in Table 2 and Figure 1. Clearly, the STUR data has larger skewness and kurtosis coefficient estimates, with values 0.999 and 4.916, respectively, compared with the BS-based data generated from the lognormal distribution, with values 0.660 and 3.367, respectively. The kernel density estimates in Figure 1 emphasize that the STUR-based kernel density is more peaked relative to BS. These results are consistent with a volatility smile for equities – see for instance Ch 18 of Hull (2009).

The results are also sensitive to ρ, but more so when T is large and the option is out of the money or in the money. Overall, it appears that the TVC feature of the model results in a superior performance over the basic model in terms of the usual AIC, SC and SSE criteria and that our model is consistent with a volatility smile which is absent in the standard model.

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7 Appendix

Proof Lemma 1. For $t = \lfloor ns \rfloor$ for any $s > 0$, we have $n^{-1/2} \sum_{j=1}^{t} \eta_j = B(t/n) + o_p(1)$, so that

$$
\begin{aligned}
n^{-1/2} Y_t^* &= n^{-1/2} \sum_{s=1}^{t-1} e^{\frac{a'}{\sqrt{n}} \sum_{j=s+1}^{t} u_j} \varepsilon_s + O_p\left(n^{-1/2}\right) \\
&= n^{-1/2} e^{\frac{a'}{\sqrt{n}} \sum_{j=1}^{t} u_j} \sum_{s=1}^{t-1} e^{-\frac{a'}{\sqrt{n}} \sum_{j=1}^{s} u_j} \varepsilon_s + O_p\left(n^{-1/2}\right) \\
&= n^{-1/2} e^{\{a' B_u(t/n) + o_p(1)\}} \sum_{s=1}^{t-1} e^{\left\{-\frac{a'}{\sqrt{n}} \sum_{j=0}^{s-1} u_j - \frac{a'}{\sqrt{n}} u_s\right\}} \varepsilon_s + O_p\left(n^{-1/2}\right) \\
&= n^{-1/2} e^{a' B_u(t/n)} \\
&\quad \times \sum_{s=1}^{t-1} e^{-\{a' B_u((s-1)/n) + o_p(1)\}} \left(1 - \frac{a' u_s}{\sqrt{n}} + O_p\left(n^{-1}\right)\right) \varepsilon_s + o_p(1) \\
&= e^{a' B_u(t/n)} \sum_{s=1}^{t-1} e^{-a' B_u((s-1)/n)} \left(\frac{\sum_{j=1}^{s} \varepsilon_j}{\sqrt{n}} - \frac{\sum_{j=1}^{s-1} \varepsilon_j}{\sqrt{n}}\right) \\
&\quad - e^{a' B_u(t/n)} \sum_{s=1}^{t-1} e^{-a' B_u((s-1)/n)} \left(\frac{a' u_s \varepsilon_s}{n}\right) + o_p(1). \qquad (36)
\end{aligned}
$$

Setting $t = \lfloor nr \rfloor$ and noting that $\mathbb{E}\left(e^{-a' B_u(p)}\right)^2 < \infty$, the first term on the right side of (36) has limit

$$
e^{a' B_u\left(\frac{t}{n}\right)} \sum_{s=1}^{t-1} e^{-a' B_u\left(\frac{s-1}{n}\right)} dB_\varepsilon\left(\frac{s}{n}\right) \to_p e^{a' B_u(r)} \int_0^r e^{-a' B_u(p)} dB_\varepsilon(p) := G_a^*(r), \qquad (37)
$$

and the second term is

$$
\begin{aligned}
& -e^{a' B_u(t/n)} \sum_{s=1}^{t-1} e^{-a' B_u\left(\frac{s-1}{n}\right)} \left(\frac{a' u_s \varepsilon_s}{n}\right) = -a' e^{a' B_u(t/n)} \sum_{s=1}^{t-1} e^{-a' B_u\left(\frac{s-1}{n}\right)} \left(\frac{u_s \varepsilon_s - \Sigma_{u\varepsilon}}{n} + \frac{\Sigma_{u\varepsilon}}{n}\right) \\
&= -a' \Sigma_{u\varepsilon} e^{a' B_u(t/n)} \frac{1}{n} \sum_{s=1}^{t-1} e^{-a' B_u((s-1)/n)} + O_p\left(n^{-1/2}\right) \to_p -a' \Sigma_{u\varepsilon} e^{a' B_u(r)} \int_0^r e^{-a' B_u(p)} dp.
\end{aligned}
$$

Hence,

$$\begin{aligned} n^{-1/2} Y^{*}_{\lfloor nr \rfloor} &\rightarrow_p G^{*}_{a}(r) - a' \Sigma_{u\varepsilon} e^{a' B_u(r)} \int_0^r e^{-a' B_u(p)} dp \\ &= e^{a' B_u(r)} \left(\int_0^r e^{-a' B_u(p)} dB_\varepsilon(p) - a' \Sigma_{u\varepsilon} \int_0^r e^{-a' B_u(p)} dp \right) \\ &=: G_a(r), \end{aligned} \tag{38}$$

as required for (11). Next,

$$\frac{1}{n}\left\{ \sum_{s=1}^{t-1} \left(\prod_{j=s+1}^{t} \beta_j \right) + 1 \right\} \mu = \frac{1}{n} \left(\sum_{s=1}^{t-1} e^{\frac{a'}{\sqrt{n}} \sum_{j=s+1}^{t} u_j} + 1 \right) \mu \rightarrow_p \mu e^{a' B_u(r)} \int_0^r e^{-a' B_u(p)} dp,$$

giving (12).

Table 1. Google Call Option Prices

No. of Days	36	36	36	415	415	415	778	778	778
K	1035	1065	1100	1030	1060	1100	1030	1060	1100
BS	40.5	23.8	8.6	126.5	112.4	95.2	127.32	151.1	132.28
TVC($\rho = 0$)	39.95	24.03	8.92	126.93	111.69	89.1	163.01	133.08	121.4
TVC($\hat{\rho}$)	43.98	24.10	8.65	115.43	109.61	94.33	154.95	133.32	120.58
TVC($\rho = 0.95$)	42.23	25.35	9.52	114.96	104.62	75.87	150.77	131.42	135.34
$\sigma_{imp,A}$	.181	.185	.165	.245	.246	.245	.23	.242	.248

Note: 'No. of days' are the number of days to expiration as of 11-27-2013; $S(0) = 1063.11$; 'BS' is the price based on Black and Scholes's classic formula; 'TVC' is the option price based on our TVC model; $\sigma_{imp,A}$ is Google's volatility; $\hat{\rho}$ is based on Nasdaq's historical volatility and $\sigma_{imp,A}$.

Table 2. Summary Statistics for BS- and Similarity STUR Based Simulated Data

Statistic	Mean	Median	Maximum	Minimum	SD	Skewness	Kurtosis
BS	1059.694	1021.982	2131.101	488.988	279.072	.660	3.367
Similarity STUR	1058.555	1016.176	2578.852	435.438	283.003	.999	4.916

Note: The simulation is based on $T = 2.132\,(= 778/365)$ and $S\,(0) = 1063.11$

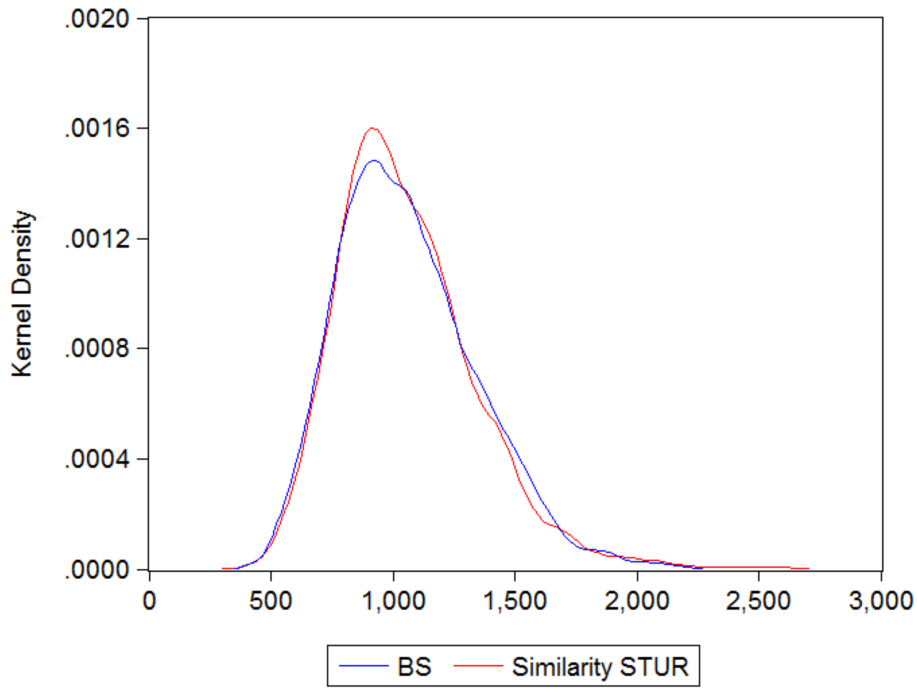

Figure 1: BS- and Similarity STUR based kernel density estimates for Google stock option prices with $T = 2.132$ and $S(0) = 1063.11$.