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January 1, 2015

RESEARCH INSTITUTE FOR ECONOMETRICS

DISCUSSION PAPER NO. 1-15

DEPARTMENT OF ECONOMICS
BAR-ILAN UNIVERSITY
RAMAT-GAN 5290002, ISRAEL

http://econ.biu.ac.il/en/node/2473

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Ruth Ben-Yashar* and Leif Danziger**

Abstract

We consider a superior decision rule for making collective choices. In our framework the optimal decision rule depends simply on the decision makers' posterior probabilities of a particular state of nature. In contrast to voting schemes, there is no need to force the decision makers to provide dichotomous information nor to estimate the different abilities of the various decision makers. An important insight is that voting is generally not an efficient way to make collective choices. The purpose of the paper is to shed light on the relationship between the optimal decision rule and voting mechanisms. We derive the conditions under which the optimal decision rule is equivalent to a well-known voting procedure (weighted supermajority, weighted majority, and simple majority) and show that these conditions are very stringent. More general voting procedures that, for example, allow for abstentions, are also considered, and we show that the conditions for reaching the optimal collective choice are still very stringent.

JEL Classifications: D70, D71
Keywords: Voting rule; common goal; collective choice; posterior probability

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Financial support from the Research Institute for Econometrics at Bar-Ilan University is greatly appreciated.

1 Introduction

Group decision making is a common practice and has been extensively studied, especially for the case in which the decision makers share a common goal but may differ about what action should be implemented. In particular, there is a large theoretical literature about the use of voting to aggregate the opinions of different decision makers in order to choose the most desirable action. The standard assumption is that each decision maker must cast his vote either for or against a proposal and perhaps be permitted to abstain. However, even though some voting procedures weigh the votes according to the decision makers' abilities, using voting to aggregate the decision makers' opinions is generally an inefficient way of making collective choices.

If different proposals are preferred in different states of nature, the degree of certainty with which a decision maker can identify the state of nature is critical. A fundamental weakness of all common-goal voting mechanisms that we are familiar with is the presumption that each decision maker has a known and invariant probability of correctly identifying the state of nature and must vote in favor of one proposal or another.[1] Therefore, a decision maker may vote in favor of a proposal both if he believes it to be the correct collective choice with probability 0.99 and if he believes it to be the correct collective choice with probability 0.51. It is clear that his vote should be given considerably more weight in the former case, and the optimal collective choice may well be to accept the proposal if the probability is 0.99, but to reject it if the probability is 0.51. However, within a voting framework the decision maker is unable to signal his precise assessment of the probability that he correctly identifies the state of nature. Essentially, it is assumed that in all cases where the optimal collective choice

[1] Recent contributions include Ben-Yashar and Kraus (2002), Baharad and Nitzan (2007a, 2007b), Berend and Sapir (2007), Visser and Swank (2007), Christensen and Knudsen (2010), Dietrich (2010a), Dietrich and List (2010), and Csaszar and Eggers (2013). These authors all assume that each decision maker has a known decision-making ability (as perhaps derived from the decision maker's track record) represented by a given probability of correctly identifying the state of nature.

is to accept a proposal, a particular decision maker has the same probability to correctly identify when acceptance is the preferred action. In reality, though, the probability that the decision maker correctly identifies the preferred action is not uniquely given and depends on the decision maker's idiosyncratic circumstances for the particular case.

To the best of our knowledge, this paper is the first to consider a rule for group decision making that is generally superior to any other rule for the case in which the decision makers share a common goal but may differ about what action should be implemented. The major innovation is that each decision maker provides his posterior probability of a particular state of nature. Importantly, the posterior probability is not limited to taking just two possible values as is implicitly or explicitly assumed in most of the voting literature. Within this framework, the optimal decision rule determining the collective choice depends simply on the decision makers' posterior probabilities of a particular state of nature. Nevertheless, the optimal decision rule cannot, in general, be formulated as a voting rule where the individual decision makers vote for or against a proposal.

The purpose of this paper is to shed light on the relationship between the optimal decision rule and voting mechanisms, a topic which, as far as we know, has hitherto been ignored by the literature. Toward this purpose, we derive the conditions under which the optimal decision rule is equivalent to a voting rule and show that the conditions for equivalence are very stringent. To wit, the optimal decision rule can be represented by a weighted supermajority (also called a qualified majority) voting rule[2] if and only if each decision maker's posterior probability of a particular state of nature can take at most two values. If the environment is symmetric (as defined in section 2), the optimal decision rule can be represented by a weighted majority voting rule if and only if these two posterior probabilities are complements (i.e., sum to unity), and by a simple majority voting rule if and only if the

[2] See Ben-Yashar and Nitzan (1997) and Fey (2003).

two values are complements and identical for all the decision makers.[3]

The model also allows us to consider more general voting schemes. In particular, if decision makers are allowed to abstain, then the optimal decision rule can be represented by a weighted supermajority voting rule with abstentions if and only if each decision maker's posterior probability of a particular state of nature can take at most three values.

In many voting models of collective choice, a tacit assumption is that a decision maker's vote is based on a single private signal which is what limits a decision maker's posterior probability of a state of nature to taking just two values.[4] However, in this paper we consider a more general framework where a decision maker's posterior probability of a state need not be limited to having only two possible values. The warranted interpretation is that a decision maker may receive numerous private signals whose informational contents he combines to obtain his posterior probability of a particular state of nature. Indeed, we illustrate the inferiority of the simple majority voting rule in two cases where decision makers may receive more than a single private signal.

2 The Model

We consider a group of $I \in \mathbb{N}^+$ decision makers who collectively need to decide between two mutually exclusive proposals A and B. There are two possible states of nature, s_A and s_B. The true state of nature is unknown at the time the collective choice must be made but the decision makers do know that the prior probability of state s_A is $p_A \in (0, 1)$ and of state s_B is $1 - p_A$.

Each decision maker $i \in \{1, 2, ..., I\}$ provides his posterior probability $p_{Ai} \in (0, 1)$ which

[3] See Nitzan and Paroush (1982), Shapley and Grofman (1984), Harstad (2005), and Barberà and Jackson (2006).

[4] See Ladha (1995), Young (1995), Nageeb et al. (2008), and Ben-Yashar and Danziger (2011, forthcoming). An exception is Austen-Smith and Banks (1996) who also consider the possibility that a decision maker receives two signals.

is based on all his available information z_i that s_A is the state of nature. Specifically, decision maker i's posterior probability p_{Ai} is given by Bayes' theorem

$$p_{Ai} = \frac{p_A \Pr(z_i \mid s_A)}{p_A \Pr(z_i \mid s_A) + (1 - p_A) \Pr(z_i \mid s_B)}. \tag{1}$$

Therefore, the odds ratio that the ith decision maker attaches to s_A being the true state is

$$\begin{aligned} \psi_{Ai} &\equiv \frac{p_{Ai}}{(1 - p_{Ai})\gamma} \\ &= \frac{\Pr(z_i \mid s_A)}{\Pr(z_i \mid s_B)}, \end{aligned}$$

where $\gamma \equiv p_A/(1 - p_A)$ is the common prior odds of state s_A.

Let

$$\pi_A \equiv \frac{p_A \Pr(z_1, z_2, ..., z_I \mid s_A)}{p_A \Pr(z_1, z_2, ..., z_I \mid s_A) + (1 - p_A) \Pr(z_1, z_2, ..., z_I \mid s_B)}$$

denote the collective probability that the I decision makers attach to s_A being the true state. Assuming that the z_i's are conditionally independent of the state of nature, we have that

$$\begin{aligned} \pi_A &\equiv \frac{p_A \prod_i \Pr(z_i \mid s_A)}{p_A \prod_i \Pr(z_i \mid s_A) + (1 - p_A) \prod_i \Pr(z_i \mid s_B)} \\ &= \frac{1}{1 + (1/\gamma) \prod_i [\Pr(z_i \mid s_B)/ \Pr(z_i \mid s_A)]}. \end{aligned}$$

Furthermore, let

$$\Psi_A \equiv \frac{\pi_A}{(1 - \pi_A)\gamma}$$

denote the collective odds ratio that the decision makers attach to s_A being the true state. It follows that[5]

$$\begin{aligned} \Psi_A &= \prod_i \frac{\Pr(z_i \mid A)}{\Pr(z_i \mid B)} \\ &= \prod_i \psi_{Ai}. \end{aligned} \tag{2}$$

[5] See Bordley (1982), Barrett and Pattanaik (1987), Dietrich (2010b), and Allard et al. (2012).

The gain from implementing proposal A in state s_A (s_B) is $G(A, s_A)$ ($G(A, s_B)$), and the gain from implementing proposal B in state s_A (s_B) is $G(B, s_A)$ ($G(B, s_B)$). We assume that $G(A, s_A) > G(B, s_A)$ and $G(B, s_B) > G(A, s_B)$ so that the largest gain in state s_A is obtained by choosing proposal A and the largest gain in state s_B is obtained by choosing proposal B.

The expected gain from choosing A given the collective probability π_A that the decision makers collectively attach to s_A being the correct state of nature is

$$\pi_A G(A, s_A) + (1 - \pi_A) G(A, s_B).$$

Similarly, the expected gain from choosing B given the collective probability π_A is

$$(1 - \pi_A) G(B, s_B) + \pi_A G(B, s_A).$$

The common objective of the decision makers is to maximize the expected gain from choosing either A or B. Let $\mathbf{G} \equiv [G(A, s_A), G(B, s_A), G(A, s_B), G(B, s_B)]$ denote the profile of the gains, and let $\mathbf{p}_A \equiv (p_{A1}, p_{A2}, ..., p_{AI})$ denote the profile of the decision makers' posterior probabilities that s_A is the true state of nature. The optimal decision rule is a function $f(\mathbf{p}_A, p_A, \mathbf{G})$ that selects A or B for any combination of the posterior probabilities and the environmental parameters, namely, the prior probability and the gains from choosing correctly and incorrectly in each state in order to maximize the expected gain from the collective choice.

The optimal decision rule takes the following simple form:

$$f(\mathbf{p}_A, p_A, \mathbf{G}) = \begin{cases} A & \text{if } \Psi_A > \Psi_A^*, \\ B & \text{if } \Psi_A < \Psi_A^*, \\ A \text{ or } B & \text{if } \Psi_A = \Psi_A^*, \end{cases}$$

where

$$\Psi_A^* \equiv \frac{G(B, s_B) - G(A, s_B)}{[G(A, s_A) - G(B, s_A)]\gamma}$$

is the critical value of the collective odds ratio. Any decision rule that chooses A for some $\Psi_A < \Psi_A^*$ or B for some $\Psi_A > \Psi_A^*$ is inferior. This includes all rules entailing that the decision makers' available information is not used efficiently (for example, voting rules constraining the decision makers to vote in favor of either one or the other of the proposals) or attaching importance to irrelevant information (for example, the order in which the decision makers provide their p_{Ai} or a hierarchical structure of the decision makers).

The collective choice between alternatives A and B is determined by comparing the collective odds ratio Ψ_A with its critical value Ψ_A^*. Hence, the larger the expected gain from choosing A rather than B if the state is s_A, the smaller is the critical odds ratio for which A will be chosen. In particular, if the environment is symmetric (i.e., $\gamma = 1$ and $G(A, s_A) - G(B, s_A) = G(B, s_B) - G(A, s_B)$ so that the expected gain from choosing A rather than B if the state is s_A equals the expected gain from choosing B rather than A if the state is s_B), then $\Psi_A^* = 1$. In that case, A is chosen if $\Psi_A > 1$, B is chosen if $\Psi_A < 1$, while A or B is chosen if $\Psi_A = 1$.

The optimal decision rule presumes that the decision makers are truthful when providing their posterior probabilities that s_A is the true state. It is clear, however, that it is a Nash equilibrium that each decision maker i is truthful when providing his p_{Ai}.[6]

3 When Is Voting Optimal?

In our model, there is no constraint on the distributions of the possible values of the decision makers' posterior probabilities p_{Ai}'s, and, hence, on the possible values of their odds ratios ψ_{Ai}'s. Therefore, our framework is more general than the voting models where the decision makers are constrained to only providing dichotomous information by voting for proposal

[6] With a nonoptimal decision rule, strategic considerations may play an important role. See Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1996, 1998), Dekel and Piccione (2000), Persico (2004), Gerardi and Yariv (2007), Ben-Yashar and Milchtaich (2007), and Oliveros (2013).

A or proposal B. Accordingly, if at least one decision maker's available information p_{Ai} can take more than two different values, the decision maker will be unable to provide exact information about his p_{Ai} as he is forced to make the same vote for at least two different p_{Ai} values. Hence, his available information cannot be precisely represented by a vote for proposal A or proposal B.

Consider the special case that each decision maker's p_{Ai} and hence ψ_{Ai} can take at most two different values. We now prove that only in this special case is the optimal decision rule equivalent to the optimally weighted supermajority voting rule. A weighted supermajority voting rule, which is the most refined voting mechanism, gives the different decision makers' votes for A and B different individualized weights, and A is chosen if the value of the sum of the weighted votes for A less the sum of the weighted votes for B exceeds a certain threshold, B is chosen if this value is less than the threshold, and A or B is chosen if the value equals the threshold.

Theorem 1: The optimal decision rule $f(\mathbf{p}_A, p_A, \mathbf{G})$ can be represented by a weighted supermajority voting rule if and only if each decision maker's odds ratio can take at most two different values, i.e., $\psi_{Ai} \in \{\xi_i, \xi_i'\}\ \forall i$.

Proof: (1) We first prove that if each decision maker's odds ratio can take at most two different values, the optimal decision rule can be represented by a weighted supermajority voting rule. Suppose that the ith decision maker votes for A if $\psi_{Ai} = \xi_i$, for B if $\psi_{Ai} = \xi_i'$ and $\xi_i \neq \xi_i'$, and for A or B if $\xi_i = \xi_i'$. Furthermore, suppose that his vote is weighted by $\ln \psi_{Ai}$ if he votes for A and by $-\ln \psi_{Ai}$ if he votes for B.[7] If $I_A \subseteq I$ denotes the set of decision makers who vote for A, then the weighted votes for A less the weighted votes for B

[7] If $\xi_i = \xi_i'$, then $\psi_{Ai} = 1$. In this case, $\ln \psi_{Ai} = 0$ without it mattering whether the decision maker votes for A or B. Similarly, in the proofs of Corollary 1 and 2 and Theorem 2, if $\psi_{Ai} = 1$, then $\ln \psi_{Ai} = 0$ without it mattering whether the decision maker votes for A or B (or, in the case of Theorem 2, abstains).

is

$$
\begin{aligned}
& \sum_{i \in I_A} \ln \xi_i - \sum_{i \notin I_A} (-\ln \xi_i') \\
= & \sum_{i \in I_A} \ln \xi_i + \sum_{i \notin I_A} \ln \xi_i' \\
= & \ln \Psi_A .
\end{aligned}
$$

With $\ln \Psi_A^*$ being the threshold, A is chosen if $\Psi_A > \Psi_A^*$, B is chosen if $\Psi_A < \Psi_A^*$, and A or B is chosen if $\Psi_A = \Psi_A^*$. Thus, there exists a weighted supermajority voting rule that leads to the optimal decision.

(2) We next prove that if at least one decision maker's odds ratio can take three or more different values, the optimal decision rule cannot be represented by a weighted supermajority voting rule. The logic is that at least one of the decision makers is forced to cast the same vote for two or more different odds ratios. Specifically, we will show that there will be at least two cases where one decision maker votes for A and the other for B, with the optimal collective choice being A in one of the cases and B in the other. Hence, the collective choice may not be optimal.

Since the optimal decision rule is defined for any given combination of the profile of posterior probabilities $\mathbf{p}_A$ and the environmental parameters p_A and G, it suffices to show this impossibility for particular values of the ψ_{Ai}'s in a symmetric environment. Suppose that there are two decision makers. Further, suppose that decision maker 1 has two possible odds ratios, $\psi_{A1} \in \{v^{-1}, v\}$, $v > 1$, and that decision maker 2 has at least three possible odds ratios, $\psi_{A2} \in \{v^{-2}, 1, v^2, v^3, ...v^n\}$, $n \geq 2$, and that $\Psi_A^* = 1$. The optimal collective choice is then A if $\psi_{A2} \in \{v^2, v^3, ...v^n\}$, or if $\psi_{A1} = v$ and $\psi_{A2} = 1$. The optimal collective choice is B if $\psi_{A2} = v^{-2}$, or if $\psi_{A1} = v^{-1}$ and $\psi_{A2} = 1$.

If decision maker 2 votes the same whether he has $\psi_{A2} \in \{v^2, v^3, ...v^n\}$ or $\psi_{A2} = 1$, and the collective choice is A if $\psi_{A2} \in \{v^2, v^3, ...v^n\}$ (which is optimal), then the collective choice is also A (which may not be optimal) if $\psi_{A2} = 1$, independently of ψ_{A1}. That is, if $\psi_{A2} = 1$ and $\psi_{A1} = v^{-1}$, then B will not be chosen even though it is optimal. On the other hand, if decision maker 2 votes the same whether he has $\psi_{A2} = v^{-2}$ or $\psi_{A2} = 1$, and the collective choice is B if $\psi_{A2} = v^{-2}$ (which is optimal), then the collective choice is also B (which may not be optimal) if $\psi_{A2} = 1$, independently of ψ_{A1}. That is, if $\psi_{A2} = 1$ and $\psi_{A1} = v$, then A will not be chosen even though it is optimal. Finally, if decision maker 2 votes the same whether he has $\psi_{A2} \in \{v^2, v^3, ...v^n\}$ or $\psi_{A2} = v^{-2}$, then the collective choice does not depend on decision maker 1's odds ratio. Hence, the collective choice is not optimal. It follows that the optimal decision rule cannot be represented by a weighted supermajority voting rule if at least one of the decision maker's odds ratio can take three or more different values. □

Accordingly, if the optimal decision can be reached by a weighted supermajority voting rule, knowledge of the decision makers' odds ratios is equivalent to both knowing how each of the decision makers vote and possessing the information needed to calculate the optimal weight for each decision maker's vote.

To obtain an important implication of Theorem 1, suppose that each decision maker's p_{Ai} and hence their odds ratio can take at most two different values, and also that the p_{Ai}'s are complements and hence the odds ratio are reciprocals, i.e., $\xi_i' = 1/\xi_i$, $\forall i$. Furthermore, suppose that the environment is symmetric so that $\Psi_A^* = 1$. Theorem 1 then implies that the optimal decision rule $f(\mathbf{p}_A, p_A, \mathbf{G})$ that chooses A or B for any combination of the posterior probabilities and the environmental parameters is equivalent to a weighted majority voting rule. That is, the different decision makers' votes are given different individualized weights that are identical for A and B, and A is chosen if the value of the sum of the weighted votes for A less the sum of the weighted votes for B is positive, B is chosen if the value is negative, and A or B is chosen if the value equals zero. Thus, Theorem 1 implies the following corollary:

Corollary 1: Suppose that the environment is symmetric. Then the optimal decision rule $f(\mathbf{p}_A, p_A, \mathbf{G})$ can be represented by a weighted majority voting rule if and only if each decision maker's odds ratio can take at most two values that are reciprocals, i.e., $\psi_{Ai} \in \{\xi_i, 1/\xi_i\}\ \forall i$.

Proof: (1) We first prove that the optimal decision rule can be represented by a weighted majority voting rule. Assume, without loss of generality, that $\xi_i \geq 1$, and suppose that the ith decision maker votes for A if $\psi_{Ai} > 1$, for B if $\psi_{Ai} < 1$, and for A or B if $\psi_{Ai} = 1$. Furthermore, suppose that his vote is weighted by $\ln \psi_{Ai}$ if he votes for A, and by $-\ln \psi_{Ai}$ if he votes for B. The weighted votes for A less the weighted votes for B is

$$\begin{aligned} & \sum_{i \in I_A} \ln \xi_i - \sum_{i \notin I_A} - \ln \left(\frac{1}{\xi_i} \right) \\ = & \sum_{i \in I_A} \ln \xi_i + \sum_{i \notin I_A} \ln \left(\frac{1}{\xi_i} \right) \\ = & \sum_{i=1}^{I} \ln \psi_{Ai} \\ = & \ln \Psi_A . \end{aligned}$$

In a symmetric environment $\Psi_A^* = 1 \Leftrightarrow \ln \Psi_A^* = 0$. It follows then that A is chosen if $\Psi_A > 1$, B is chosen if $\Psi_A < 1$, and A or B is chosen if $\Psi_A = 1$. Thus, there exists a weighted majority voting rule that leads to the optimal collective choice.

(2) We next prove that if at least one decision maker's odds ratios can take more than two values or if the decision maker's odds ratios are not reciprocals, the optimal decision rule cannot be represented by a weighted majority voting rule. Since a weighted majority voting rule is a special case of a weighted supermajority voting rule, we only need to show that if the decision makers' odds ratio can take only two values, the optimal decision rule cannot be represented by a weighted majority voting rule unless the odds ratios are reciprocals. Since the optimal decision rule is defined for any given profile $\mathbf{p}_A$, it suffices to show this impossibility for particular values of the ψ_{Ai}'s. Suppose, therefore, that there are two decision

makers and that $\psi_{A1} \in \{v^{-1}, v^2\}$ and $\psi_{A2} \in \{v^{-1}, v^2\}$, $v > 1$. The optimal decision is then A if $\psi_{A1} = v^2$ or if $\psi_{A2} = v^2$, and B if $\psi_{A1} = \psi_{A2} = v^{-1}$. Thus, if the weight of decision maker 1 is w_1 and of decision maker 2 is w_2, then in order for A to always be chosen if $\psi_{A1} = v^2$ it must be the case that $w_1 - w_2 > 0$, while in order for A to always be chosen if $\psi_{A2} = v^2$ it must be the case that $w_2 - w_1 > 0$, which is impossible. $\square$

Suppose, as above, that the decision makers' p_{Ai}'s can take at most two values which are complements, and hence their odds ratios can take at most two values which are reciprocals, and in addition that the decision makers are homogeneous. Then $\{\xi_i, \xi_i'\} = \{\xi, 1/\xi\}$, $\forall i$. It then follows that in a symmetric environment, the optimal decision rule is equivalent to a simple majority voting rule. That is, A is chosen if a majority votes for A, B is chosen if a majority votes for B, and A or B is chosen if the same number of decision makers vote for A and B. More precisely, Theorem 1 also implies the following corollary:

Corollary 2: Suppose that the environment is symmetric. Then the optimal decision rule $f(\mathbf{p}_A, p_A, \mathbf{G})$ can be represented by a simple majority rule if and only if the decision makers are homogeneous and each decision maker's odds ratio can take at most two values that are reciprocals, i.e., $\psi_{Ai} \in \{\xi, 1/\xi\}$ $\forall i$.

Proof: (1) We first prove that the optimal decision rule can be represented by a simple majority voting rule. Assume, without loss of generality, that $\xi_i \geq 1$, and suppose that the ith decision maker votes for A if $\psi_{Ai} > 1$, for B if $\psi_{Ai} < 1$, and for A or B if $\psi_{Ai} = 1$. Suppose also that the decision maker's vote is weighted by $\ln \psi_{Ai}$ if he votes for A, and by $-\ln \psi_{Ai}$ if he votes for B. The weighted votes for A less the weighted votes for B is

$$\begin{aligned} & \sum_{i \in I_A} \ln \xi - \sum_{i \notin I_A} - \ln \left(\frac{1}{\xi} \right) \\ = & \sum_{i=1}^{I} \ln \psi_{Ai} \\ = & \ln \Psi_A. \end{aligned}$$

Furthermore,

$$\Psi_A \gtreqless 1$$
$$\Leftrightarrow \xi^{I_A-(I-I_A)} \gtreqless 1$$
$$\Leftrightarrow I_A \gtreqless \tfrac{1}{2}I.$$

It follows that A is chosen if $\Psi_A > 1 \Leftrightarrow I_A > \frac{1}{2}A$, B is chosen if $\Psi_A < 1 \Leftrightarrow I_A < \frac{1}{2}A$, and A or B is chosen if $\Psi_A = 1 \Leftrightarrow I_A = \frac{1}{2}A$. Consequently, a simple majority voting rule leads to the optimal decision.

(2) We next prove that if at least one decision maker's odds ratios can take more than two values, or if the decision makers are not homogenous, or if the decision maker's odds ratios are not reciprocals, the optimal decision rule cannot be represented by a simple majority voting rule. Since a simple majority voting rule is a special case of a weighted majority voting rule, we need only to show that if the decision makers' odds ratio can take only two values that are reciprocals, the optimal decision rule cannot be represented by a simple majority voting rule unless the decision makers are homogenous. Since the optimal decision rule is defined for any given profile $\mathbf{p}_A$, it suffices to show this impossibility for particular values of the ψ_{Ai}'s. Suppose, therefore, that there are three decision makers and that $\psi_{A1} \in \{v^{-3}, v^3\}$, $\psi_{A2} \in \{v^{-1}, v\}$, and $\psi_{A3} \in \{v^{-1}, v\}$, $v > 1$. The optimal decision is then A if $\psi_{A1} = v^3$ and B if $\psi_{A1} = v^{-3}$, independently of ψ_{A2} and ψ_{A3}. However, if $\psi_{A1} = v^3$, $\psi_{A2} = v^{-1}$, and $\psi_{A3} = v^{-1}$, and the optimal decision is A, a simple majority voting rule would choose B. □

The standard voting model forces decision makers to vote for one of two options and does not permit abstentions. However, it is simple to generalize that model to include abstentions. In particular, if the decision makers have at most three different p_{Ai}'s and hence odds ratios, the optimal decision is equivalent to a weighted supermajority voting rule with abstentions. That is, suppose that the different decision makers' votes for A and B as well as their abstentions are given different individualized weights. Then A is chosen if the value of the sum of the different decision makers' weighted votes for A less the sum of the weighted votes for B and the weighted abstentions exceeds a certain threshold, B is chosen if this value is less than the threshold, and A or B is chosen if this value is equal to the threshold. Thus,

Theorem 2: The optimal decision rule can be represented by a weighted supermajority voting rule with abstentions if and only if each decision maker's odds ratio can take at most three different values, i.e., $\psi_{Ai} \in \{\xi_i, \xi_i', \xi_i''\}$ $\forall i$.

Proof: (1) We first prove that if each decision maker's odds ratio can take at most three different values, the optimal decision rule can be represented by a weighted supermajority voting rule with abstentions. Suppose, therefore, that the ith decision maker votes for A if $\psi_{Ai} = \xi_i$, for B if $\psi_{Ai} = \xi_i'$, and abstains if $\psi_{Ai} = \xi_i''$. Furthermore, suppose that if he votes for A then his weight is $\ln \psi_{Ai}$, and if he votes for B or abstains then his weight is $-\ln \psi_{Ai}$.

If $I_B \subseteq I$ denotes the set of decision makers who vote for B, then the weighted votes for A less the weighted votes for B and the weighted abstentions are

$$\begin{aligned} & \sum_{i \in I_A} \ln \xi_i - \sum_{i \in I_B} (-\ln \xi_i') - \sum_{i \notin (I_A \cup I_B)} (-\ln \xi_i'') \\ = \; & \ln \Psi_A. \end{aligned}$$

With $\ln \Psi_A^*$ being the threshold, A is chosen if $\Psi_A > \Psi_A^*$, B is chosen if $\Psi_A < \Psi_A^*$, and A or B is chosen if $\Psi_A = \Psi_A^*$. Thus, there exists a weighted supermajority voting rule with abstentions that leads to the optimal decision.

(2) We next prove that if at least one decision maker's odds ratio can take four or more different values, the optimal decision rule cannot be represented by a weighted supermajority voting rule with abstentions. Since the optimal decision rule is defined for any given profile $\mathbf{p}_A$ and thus for all possible values of ψ_{Ai} $\forall i$, it suffices to show this impossibility for particular values of the ψ_{Ai}'s. Therefore, suppose that there are two decision makers. Decision maker 1 has three possible odds ratios, $\psi_{A1} \in \{v^{-1}, 1, v\}$, $v > 1$, and decision maker 2 has at

least four possible odds ratios, $\psi_{A2} \in \{v^{-2}, v^{-1/2}, v^{1/2}, v^2, v^3, ...v^n\}$, $n \geq 2$, and that $\Psi_A^* = 1$. The optimal collective choice is then A if $\psi_{A2} \in \{v^2, v^3, ...v^n\}$, or if $\psi_{A1} = v$ and $\psi_{A2} \in \{v^{-1/2}, v^{1/2}\}$, or if $\psi_{A1} = 1$ and $\psi_{A2} = v^{1/2}$. The optimal collective choice is B if $\psi_{A2} = v^{-2}$, or if $\psi_{A1} = v^{-1}$ and $\psi_{A2} \in \{v^{-1/2}, v^{1/2}\}$, or if $\psi_{A1} = 1$ and $\psi_{A2} = v^{-1/2}$.

By the same logic as in part (2) of the proof of Theorem 1, there are at least two cases where one decision maker votes for A and the other for B, with the optimal collective choice being A in one of the cases and B in the other. The collective choice may therefore not be optimal. As a consequence, the optimal decision rule cannot be represented by a weighted supermajority voting rule with abstentions if at least one of the decision maker's odds ratio can take four or more different values. □

The crucial assumption in Theorem 2, as in Theorem 1, is that the number of different voting options is at least equal to the number of a decision maker's possible p_{Ai}'s and odds ratios. As a result, a decision maker's vote is able to convey exact information about his odds ratio. This would not be possible if the number of voting options is less than the number of a decision maker's possible p_{Ai}'s and odds ratios. Thus, if decision makers are only allowed to vote for A or B with no possibility of abstention, but have three odds ratios, then a decision maker's vote cannot uniquely identify the underlying odds ratio.

Consider the case where decision makers, in addition to being able to vote for A, B, and to abstain, can also choose not to vote at all. If each decision maker's odds ratio can take at most four different values, it is straightforward to show that the optimal decision rule is equivalent to a weighted supermajority voting rule with abstentions and non-voters, where each of these four possibilities corresponds to one of a decision maker's odds ratios.

More generally, suppose that the number of a decision maker's different voting options is at least equal to the number of his possible odds ratios (for example, he can "vote strongly for A", "vote weakly for A", etc.). The optimal collective choice can then be reached by a voting rule which gives different individualized weights to each of a decision maker's voting possibilities.

4 Many Private Signals

Until now we have not discussed the nature of the available information z_i that underlies decision maker ith's posterior probability p_{Ai} that s_A is the state of nature. However, we wish to point out that z_i need not consist of a single private signal, but may be based on many conditionally independent private signals. In this section we show how to express the collective odds ratio as a function of all the decision makers' many private signals. An important implication of such an aggregation is that the optimal decision rule can be formulated in terms of a critical number of private signals. We use this formulation to illustrate the inferiority of the simple majority voting rule in two cases where decision makers may receive more than a single private signal.

Suppose that decision maker i receives $n_i \geq 1$ random and conditionally independent private signals about the true state of nature. A signal can be either "A" or "B". The signals are drawn from a state-dependent distribution, with each signal correctly reflecting the true state of nature with probability $q \in (\frac{1}{2}, 1)$. In other words, if the state is s_A, then q is the probability that a signal is "A" and $1 - q$ that a signal is "B", while if the state is s_B, then q is the probability that a signal is "B" and $1 - q$ that a signal is "A". Hence, if k_i of the n_i signals are "A", decision maker i's privately available information is given by $z_i = (k_i, n_i)$. Therefore, if the unknown state is s_A, the probability that k_i of the n_i signals are "A" is $\binom{n_i}{k_i} q^{k_i}(1-q)^{n_i-k_i}$, while if the unknown state is s_B, the probability that k_i of the n_i signals are "A" is $\binom{n_i}{k_i}(1-q)^{k_i} q^{n_i-k_i}$.

Given decision maker i's information, his posterior probability p_{Ai} is

$$p_{Ai} = \frac{p_A \binom{n_i}{k_i} q^{k_i}(1-q)^{n_i-k_i}}{p_A \binom{n_i}{k_i} q^{k_i}(1-q)^{n_i-k_i} + p_B \binom{n_i}{k_i}(1-q)^{k_i} q^{n_i-k_i}}$$

$$= \frac{1}{1 + (p_B/p_A)\,[q/(1-q)]^{n_i - 2k_i}}$$

$$= \frac{1}{1 + (1/\gamma)\alpha^{n_i - 2k_i}}, \tag{3}$$

where $\alpha \equiv q/(1-q)$ is the odds that a signal correctly reflects the true state. Hence, the odds ratio that the ith decision maker attaches to s_A being the true state is

$$\psi_{Ai} = \frac{p_{Ai}}{(1-p_{Ai})\gamma} = \alpha^{2k_i - n_i}.$$

Since p_{Ai} depends on only n_i and k_i (for a given α and γ), ψ_{Ai} depends on only n_i and k_i (for a given α). Let $N \equiv \sum_{i=1}^{I} n_i$ denote the total number of private signals that the decision makers receive, and $K \equiv \sum_{i=1}^{I} k_i$ the total number of "A" signals among them. Then, eq. (2) implies that the collective odds ratio is

$$\Psi_A = \alpha^{2K-N},$$

which depends on only N and K (for a given α).

The optimal decision rule can be formulated as a function of K, N, p_A, and $\mathbf{G}$ instead of as a function of $\mathbf{p}_A$, p_A, and $\mathbf{G}$. That is, A is chosen if $K > K^*$, B is chosen if $K < K^*$, and A or B is chosen if $K = K^*$, where $K^* \equiv \frac{1}{2}\left(\ln \Psi_A^* / \ln \alpha + N\right)$ is the critical number of "A" signals.

The collective choice is the same for every combination of k_i's and n_i's that lead to the same K and N. The only relevant information for determining the collective odds ratio Ψ_A that the state of nature is s_A is the total numbers K of "A" signals that the decision makers actually receive and the maximum number N of "A" signals that the decision makers could possibly receive.[8] An important consequence is that the distribution of the possible

[8] It is only for simplicity that we have assumed that $q > \frac{1}{2}$. If instead $q < \frac{1}{2}$, then A is chosen if $K < K^*$, B if $K > K^*$, and A or B if $K = K^*$. If $q = \frac{1}{2}$, then K has no effect on the collective odds ratio Ψ_A, which equals unity. Therefore, A is chosen if $\Psi_A^* < 1$, B if $\Psi_A^* > 1$, and A or B if $\Psi_A^* = 1$.

Ψ_A's is the same for a group consisting of N decision makers each of whom receives a single signal as for a single decision maker who receives N signals. Interestingly, in the case that $K = \frac{1}{2}N$, the collective probability equals the prior probability. The signals then do not add any information about the state of nature and we have that $\Psi_A = 1$.

Consider now the special case where each decision maker receives only one signal, i.e., $n_i = 1$ for all $i \in I$ (and hence $N = I$), and votes A if he receives an "A" signal and B if he receives a "B" signal. The total number of votes for A is then equal to the total number of "A" signals received by the decision makers. Since all decision makers have the same probability q of receiving an "A" signal, it follows that the odds ratio can only take two values that are reciprocals, namely α and $1/\alpha$. Therefore, if the environment is symmetric, Corollary 2 shows that a simple majority voting rule where each decision maker's vote is given the same positive weight is optimal.

Consider next the special case where each decision maker receives one or two signals with at least one decision maker receiving two signals, i.e., $n_i \in \{1, 2\}$ for all $i \in I$ and $n_i = 2$ for at least one i (and hence $I + 1 \leq N \leq 2I$). At least one decision maker's odds ratio can then take three different values in which case his vote for A cannot accurately reflect the number of "A" signals he received. Consequently, Theorem 1 shows that a weighted supermajority is not generally optimal. However, according to Theorem 2, the optimal collective choice can be reached by a weighted supermajority voting rule with abstentions.

4.1 Two Numerical Illustrations

In a framework where decision makers may receive many private signals, what matters in our model is the total number of private "A" signals that the decision makers actually receive and the maximum number N of "A" signals that the decision makers could possibly receive. Consequently, with our approach the informational content of the signals is used as efficiently as possible. In contrast, in voting models each decision maker combines the informational contents of his private signals in order to decide whether to vote for proposal A or proposal B, with a voting rule employed to make the collective choice. Since a decision maker may vote the same for many different combinations of signals he receives, the voting process inevitably destroys some of the informational content of the signals. An essential difference between our approach and voting models is therefore that only in our approach the informational content of the signals is always used efficiently.

We now present two examples that highlight the inferiority of the simple majority voting rule. In both examples the environment is symmetric and the decision makers are homogeneous, each receiving three private signals. In the first example we illustrate the advantage of having an additional decision maker as it always increases the probability of making the correct collective choice.[9] This is in contrast to the simple majority voting rule where, if the number of decision makers is odd, having an additional decision maker does not increase the probability of making the correct collective choice.[10] In the second example we illustrate that due to its greater efficiency, the optimal decision rule might have a higher probability of making the correct collective choice even with a smaller number of decision makers than the simple majority voting rule. As such, the optimal decision rule may facility cost savings by reducing the number of decision makers needed to reach a desired minimum level of the probability of making the correct collective choice.

Example 1: Consider the case in which we start with one decision maker and then add a second decision maker. With one decision maker there are three signals and with two decision makers there are six signals. The relative increase in the probability of the correct collective choice is

$$\frac{1}{q}\left[\sum_{m=4}^{6}\binom{6}{m}q^m(1-q)^{6-m}+\tfrac{1}{2}\binom{6}{3}q^3(1-q)^3-\sum_{m=2}^{3}\binom{3}{m}q^m(1-q)^{3-m}\right]$$

[9] Increasing the expected gain from choosing either A or B is equivalent to increasing the probability of making the correct collective choice.

[10] Under our assumptions, the commonly used simple majority voting rule is the best possible voting rule.

$$= \quad 3q(2q^3 - 5q^2 + 4q - 1),$$

which is always positive (it would equal zero for $q = \frac{1}{2}$ and $q = 1$). It reaches its maximum of 7.4% for $q = \frac{2}{3}$.

On the other hand, with a simple majority voting rule, adding a second decision maker does not change the probability of making the correct collective choice. More precisely, the probability that two decision makers make the correct collective choice is equal to the probability that both decision makers get at least two correct signals plus half of the probability that only one decision maker gets at least two correct signals. Thus, the probability is

$$(3q^2 - 2q^3)^2 + (3q^2 - 2q^3)(1 - 3q^2 + 2q^3) = 3q^2 - 2q^3,$$

which is the same as the probability that one decision maker makes the correct choice.

Example 2: Consider the case with six decision makers using the optimal decision rule vs. seven decision makers using the simple majority voting rule. The difference in the probability of making the correct collective choice is

$$\sum_{x=10}^{18} \binom{18}{x} q^x (1-q)^{18-x} + \tfrac{1}{2}\binom{18}{9} q^9 (1-q)^9$$
$$-\sum_{x=4}^{7} \binom{7}{x} \left[q^3 + 3q^2(1-q)\right]^x \left\{1 - \left[q^3 + 3q^2(1-q)\right]\right\}^{7-x},$$

which is always positive (shown by simulations). That is, six decision makers using the optimal decision rule have a higher probability of making the correct collective choice than seven decision makers using the simple majority voting rule.

5 Conclusion

In this paper we have considered a superior decision rule for making collective choices. The framework is very general and only requires that each decision maker provides his posterior probability that a particular state of nature is the true one. In particular, the decision makers are not asked to vote in favor of one or another proposal. Given the prior probability of a particular state, the decision makers' posterior probabilities can be transformed into their odds ratios and then into the collective odds ratio. We have showed that a proposal should be accepted if the collective odds ratio exceeds a critical level and rejected if the collective odds ratio is less.

In contrast to voting schemes, with our optimal decision rule there is no need to force the decision makers to provide dichotomous information nor to estimate the different abilities of the various decision makers. Indeed, we have showed that the conditions under which voting procedures (weighted supermajority, weighted majority, and simple majority) would lead to the optimal collective choice are very stringent. For example, in order for the simple majority voting rule to yield the optimal collective choice, the environment must be symmetric and each decision maker's posterior probability of a particular state of nature take at most two values that are complements and identical for all the decision makers. The framework allows us to also consider more general voting procedures, as for example allowing for abstentions, but we show that the conditions under which these more general voting procedures would lead to the optimal collective choice are still very stringent.

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