# Comparing Local Risks by Acceptance and Rejection

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**Abstract**

It is said that risky asset $h$ *acceptance dominates* risky asset $k$ if any decision maker who rejects the investment in $h$ also rejects the investment in $k$. As Hart (2011) shows, acceptance dominance is an incomplete order on an ordinary set of gambles. We extend the definition of acceptance dominance order to risky assets whose values follow random processes. We refer to the risk that arises from investing in such assets with a short investment time horizon as *local* risk and show that for short investment time horizons, the acceptance dominance order is a complete order that can be represented by an index of local risk. Moreover, we show that the measures of riskiness proposed by Aumann and Serrano (2008), Foster and Hart (2009), and Schreiber (2011) all coincide with our index. We use differential calculus as an analytical tool to present our results.

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# 1 Introduction

The renewed interest in measures of riskiness which started with the seminal work of Aumann & Serrano (2008) focuses on the problem of accepting or rejecting risky assets. In this literature, risky assets are characterized by random variables whose values are interpreted either as absolute returns ("gambles") or relative returns ("securities"). Based on the simple decision problem of acceptance or rejection of gambles, Hart (2011) defines an order on the set of gambles, called *acceptance dominance*, and Schreiber (2011) extends its definition to securities. Given two risky assets $h$ and $k$, be they two gambles or two securities, it is said that $h$ acceptance dominates $k$ if any risk-averse decision maker who rejects the investment in $h$ also rejects the investment in $k$. On a regular set of risky assets—that is, those characterized by a random variable—acceptance dominance is an incomplete order. Indeed, there are many cases of pairs of risky assets where none of them acceptance dominates the other.

In this paper we extend the acceptance dominance order to financial assets whose price follows continuous-time random processes. We say that a decision maker accepts an asset if she is better off investing in such an asset; otherwise she rejects it. In principle, accepting or rejecting such assets should depend on the investment time horizon. We limit our discussion only to short investment time horizons. We show that on this set of assets, the acceptance dominance order is a complete order that can be represented by a *measure of riskiness.* Moreover, we show that several measures of riskiness of regular assets that are compatible with the acceptance dominance order on regular assets induce the acceptance dominance order in the continuous-time setup. For instance, the Aumann & Serrano (2008) Economic Index of Riskiness, the Foster & Hart (2009) Operational Measure of Riskiness, and the Schreiber (2011) Economic Index of Relative Riskiness all coincide with our measure of riskiness in the continuous-time setup.

The paper is organized as follows. In Section 2 we define the acceptance dominance order on a set of risky assets whose value follows a continuous-time random process. We also propose a measure of local riskiness of such assets that induces the acceptance dominance order. In Section 3 we present several measures of riskiness, defined on discrete-time assets, that relate to the acceptance and rejection problem. We show that these measures of riskiness coincide with our measure of local riskiness in the continuous-time environment. In Section 4 we discuss some properties of the measure of local riskiness. Section 5 concludes. The main proofs are relegated to the Appendix.

## 2 Acceptance Dominance and Local Riskiness

In this paper a utility function is a von Neumann–Morgenstern utility function for money which is strictly monotonic, strictly concave, and twice continuously differentiable. For simplicity we assume that utilities are defined over the entire real line.

It is quite common in the financial literature to model securities by continuous-time random processes. A (continuous-time) *security* $s$ is an asset whose value (price) follows a continuous-time random process. The value of $s$ at time zero, $s_0$, is given and can be any real value. For any other time $t$, $0 < t < T$, the value of $s$ at $t$ ($s_t$) is the unique strong solution of a stochastic differential equation (SDE) of the form

$$ds_t = \mu_t dt + \sigma_t^T dW_t, \tag{1}$$

where $W_t$ is a vector of $K$ independent standard Wiener processes. The superscript $T$ means transpose.[1] A more rigorous description of the continuous-time framework is relegated to the Appendix. We denote the collection of all those securities by $S$.

The question whether an agent benefits from buying one unit of a security depends, among other parameters, on the investment time horizon, which is the length of time that an investment is held before it is liquidated. If, for instance, the investment time horizon is $t$, buying one unit of security $s$ causes the wealth to be distributed as $w - s_0 + s_t$, where $w$ is the initial wealth. We focus on only short investment time horizons. We say that an agent with utility $u_i$ and initial wealth $w$ *accepts* $s$, if there exists $T^* > 0$ such that

$$E\big[u_i(w - s_0 + s_t)\big] > u_i(w),$$

for all $t$, $0 < t < T^*$. Otherwise, she *rejects* it.

Hart (2011) defines an incomplete order on a set of risky assets, which he calls *acceptance dominance*. In particular, Hart defines acceptance dom-

[1]The drift $\mu_t$ and the vector of diffusion $\sigma_t$ are both functions of $s_t$ and $t$, i.e., $\mu_t = \mu(s_t, t)$ and $\sigma_t = \sigma(s_t, t)$. In addition, we assume that the drift and the diffusion are both continuous functions and that $\sigma_t^T \sigma_t \neq 0$.

inance on the set of discrete-time gambles which are basically random variables.[2] In the present paper, we extend the definition of acceptance dominance to securities in the continuous-time framework. Let $s$ and $k$ be two securities in $S$.

**Definition 1.** *$s$ acceptance dominates $k$, denoted by $s \geq_A k$, if any decision maker who rejects $s$ at $w$ also rejects $k$ at $w$, for every $w$.*

The result presented here is that the acceptance dominance order in the continuous-time setup is a complete order and can be represented by a measure of riskiness. In general, a measure of riskiness is a real-valued function defined on risky assets. The riskiness of a discrete-time asset—that is, an asset characterized by a random variable—is simply a real number. By contrast, if assets are characterized by random processes as in our framework, the riskiness is defined locally, i.e., indexed by time, as it may change over time.

Let $s$ be a security whose value is described by the SDE

$$ds_t = \mu_t dt + \sigma_t^T dW_t.$$

Let $\sigma$ and $\mu$ denote the drift and the diffusion at time zero, i.e., $\sigma \equiv \sigma_0$ and $\mu \equiv \mu_0$. We define the local riskiness of $s$ at time zero as follows:

$$R_l(s) = \frac{\sigma^T \sigma}{2\mu}. \tag{2}$$

More generally, the local riskiness of security $s$ at $t$ is defined by $R_l(s,t) = \sigma_t^T \sigma_t/(2\mu_t)$. In what follows we will refer to the local riskiness only at time zero.

Obviously, $R_l$ induces a complete order on the set of securities $S$. Our claim is that the acceptance dominance order and the order that is induced by $R_l$ are equivalent. Formally, let $s$ and $k$ be two securities in $S$. Then,

**Theorem 2.1.** *$s \geq_A k \Leftrightarrow R_l(k) \geq R_l(s)$.*

Theorem 2.1 asserts that if security $k$ is riskier than security $s$, then any decision maker who benefits from buying one unit of $k$ will benefit also from buying one unit of $s$. Limiting investors to buying only one unit may seem a strong restriction. It turns out that this restriction is unneeded. In other

[2]Formal definition is given in Section 3.

words, $k$ is riskier than $s$ if and only if any decision maker who benefits from buying $x$ units of $k$ will benefit also from buying $x$ units of $s$, for every $x > 0$. This generalization of Theorem 2.1 is derived directly from the following lemma:

**Lemma 2.2.** *Given two securities $s$ and $k$, the following claims are equivalent:*

1. *Any decision maker who rejects $s$ at $w$ also rejects $k$ at $w$, for every $w$.*
2. *Any decision maker who rejects $x$ units of $s$ at $w$ also rejects $x$ units of $k$ at $w$, for every $x > 0$ and $w$.*

To see why the lemma is correct, note that rejecting $x$ units of security $s$ is equivalent to rejecting the security $xs$. Hence, Lemma 2.2 follows directly from the homogeneity of the local riskiness; see Section 4.

# 3 Riskiness and Local Riskiness

## 3.1 Discrete-Time Gambles and Securities

In the previous section we defined an order of riskiness on a set of continuous-time securities. Several recent papers deal with other types of risky assets, namely, discrete-time gambles and securities. A *gamble* $g$ is a real-valued random variable with positive expectation and some negative values. A discrete-time *security* $r$ is a real-valued random variable with a geometric mean greater than one and some values less than one.[3]

We say that a decision maker accepts gamble $g$ if she is better off investing in $g$, i.e., if $E[u(w+g)] > E(u(w))$; otherwise she rejects it. Similarly, we say that a decision maker accepts the investment in a discrete-time security $r$ (or simply accepts $r$), if she is better off investing all her initial wealth[4] $w$ in $r$, i.e., if $E[u(wr)] > u(w)$; otherwise she rejects it. Based on the problem of accepting or rejecting gambles, Hart (2011) defines an order on the set

[3]The definitions of a gamble and a security are taken from Aumann & Serrano (2008) and Schreiber (2011), respectively.

[4]In principle, a similar result can be derived if the definition of acceptance relates to only a certain fraction of the initial wealth. Following Schreiber (2011), we limit our discussion to investment of all the initial wealth.

of gambles, called *acceptance dominance*, and Schreiber (2011) extends its definition to the set of securities: if $h$ and $k$ are two risky assets (two gambles or two securities), we say that $h$ acceptance dominates $k$ ($h \geq_A k$), if any decision maker who rejects $h$ at $w$ also rejects $k$ at $w$.

A measure of riskiness is compatible with the acceptance dominance order if $h \geq_A k$ implies that $k$ is riskier than $h$. Since the acceptance dominance order is an incomplete order on the set of gambles and on the set of securities, there exist different measures of riskiness that are compatible with the acceptance dominance order. Two such measures of riskiness are the Aumann & Serrano (2008) index of riskiness, and the Foster & Hart (2009) measure of riskiness, both defined on gambles. Formally, given a gamble $g$, the Aumann–Serrano index of riskiness, $R^{AS}$, is defined implicitly by

$$E\left[\exp\left(-\frac{g}{R^{AS}(g)}\right)\right] = 1, \tag{3}$$

and the Foster–Hart measure of riskiness, $R^{FH}$, is defined implicitly by

$$E\left[\log\left(1+\frac{g}{R^{FH}(g)}\right)\right] = 0. \tag{4}$$

Originally the two measures were based on different considerations: $R^{AS}$ is based on the dual relationship between risk and (absolute) risk aversion, while $R^{FH}$ is based on considerations of avoiding bankruptcy. However, Hart (2011) shows that the two measures are related to the acceptance dominance order. In this work, Hart defines two orders that extend the acceptance dominance order, namely, "wealth-uniform dominance" and 'utility-uniform dominance". He shows that $R^{AS}$ induces the wealth-uniform dominance order, while $R^{FH}$ induces the utility-uniform dominance order.[5]

As we showed, locally, acceptance dominance is a complete order. Hence, locally, there is no difference between utility-uniform dominance and wealth-uniform dominance as both of them coincide with the acceptance dominance order. Formally, we show that the "local versions" of $R^{AS}$ and $R^{FH}$ coincide

[5] These orders are defined as follows: gamble $g$ *wealth-uniformly dominates* gamble $h$ if any risk-averse utility function that rejects $g$ at all wealth levels also rejects $h$ at all wealth levels; and $g$ *utility-uniformly dominates* $h$ if any wealth level at which all risk-averse utility functions reject $g$ is also a wealth level at which they all reject $h$. Note that Hart (2011) considers a specific set of utilities which he calls "regular utilities".

with our measure of local riskiness $R_l$.[6]

Another relevant index of riskiness that extends the acceptance dominance order is the Schreiber (2011) index of riskiness, which is defined on securities rather than on gambles. Following Aumann & Serrano (2008) who characterize their index of riskiness of gambles by the duality of risk and absolute risk aversion, Schreiber (2011) characterizes his index by the duality of risk and relative risk aversion. Formally, if $r$ is a discrete-time security, the Schreiber measure of riskiness, denoted by $R^S$, is defined implicitly by

$$E\left[r^{-1/R^S(r)}\right] = 1. \tag{5}$$

Schreiber (2011) defines the adjusted riskiness to be

$$\widetilde{R}^S(r) = R^S(r)/(1 + R^S(r)). \tag{6}$$

Note that $R^S$ and $\widetilde{R}^S$ are ordinally equivalent; i.e., given any two securities, the two measures will always agree on which one of the securities is riskier.

## 3.2 Local Riskiness as a Limit of Riskiness

Let $s$ be a continuous-time security. Recall that $s_t$—the value of $s$ at time $t$—is a random variable. Given a measure of riskiness of regular assets, $R^*$, we define the *local riskiness of $s$ at time zero*, based on $R^*$, as the limit of the riskiness of $s$ as $t$ goes to zero. The local riskiness of $s$ is not well defined for all measures of riskiness and all securities, as some measures of riskiness are not well defined on all random variables.[7] We denote the local riskiness

[6] The practical use of these two measures has been analyzed by Kadan & Liu (2011), who show how these two measures can be applied to address the problem of tail events and rare disasters.

[7] For instance, Aumann & Serrano (2008) and Foster & Hart (2009) relate only to gambles that have a positive expectation and take negative values with a positive probability. If we dispense with these constraints, equations (3) and (4) may have no solution. See Schulze (2010) who studies on which distributions the Aumann–Serrano index is well defined. The idea that a measure of riskiness is not well defined on the whole set of risky assets should come as no surprise, as almost any other familiar measure of riskiness is not well defined on the entire set of random variables. For instance, it is quite common in the financial literature (and in the industry) to measure the riskiness of a security by its variance. But there are random variables that have no variance, such as random variables that have the Cauchy distribution.

by $R_l^*(s)$, which can be formally defined as follows:

$$R_l^*(s) \equiv \lim_{t \to 0} R^*(s_t). \tag{7}$$

Note that even if $R^*$ is well defined on $s_t$ for all $t$, $0 < t < T^*$, it does not guarantee that the expression in (7) is well defined.

The following theorem shows the connection between the local riskiness based on the above-mentioned measures of riskiness and the local riskiness as defined in Section 2.

**Theorem 3.1.**

1. *Let $s \in S$ be a security where $s_0 = 0$. There exists $T^*$ such that $R^{AS}(s_t)$ is well defined for all $t$, $0 < t < T^*$ and*

$$R_l^{AS}(s) = R_l(s). \tag{8}$$

2. *Let $s \in S$ be a security where $s_0 = 0$. There exists $T^*$ such that $R^{FH}(s_t)$ is well defined for all $t$, $0 < t < T^*$ and*

$$R_l^{FH}(s) = R_l(s). \tag{9}$$

3. *Let $s \in S$ be a security where $s_0 = 1$ and assume that $2\mu^s/\sigma^{sT}\sigma^s > 1$. There exists $T^*$ such that $\tilde{R}^S(s_t)$ is well defined for all $t$, $0 < t < T^*$ and*

$$\tilde{R}_l^S(s) = R_l(s). \tag{10}$$

The decision problem underlying the measures of riskiness proposed by Aumann–Serrano and Foster–Hart concerns whether to accept or reject a gamble whose price is zero. Hence, the first two parts of Theorem 3.1 relate only to securities whose price at $t = 0$ is zero. In this case, $R_l^{AS}$ and $R_l^{FH}$ coincide with $R_l$. Similarly, the decision problem underlying the measure of riskiness of Schreiber (2011) concerns whether to accept or reject a security whose price at $t = 0$ is a positive number. Hence, the third part of Theorem 3.1 relates only to securities whose price at $t = 0$ is a positive number (normalized to one). The condition $2\mu^s/\sigma^{sT}\sigma^s > 1$ in the third part of the theorem is equivalent to the property of a discrete-time security asserting that its geometric mean is greater than one.

# 4 Properties

The concepts of *stochastic dominance* provide the most uncontroversial notions of riskiness (Hadar & Russell (1969), Levy & Hanoch (1969), Rothschild & Stiglitz (1970)). For two given discrete-time risky assets, $h$ and $k$, whether gambles or securities, it is said that $h$ (second-degree) stochastically dominates $k$ if and only if all decision makers prefer $h$ to $k$. As Hart (2011) shows, acceptance dominance is an extension of the stochastic dominance order, in the sense that if an asset $h$ stochastically dominates an asset $k$, then $h$ also acceptance dominates $k$ (the opposite does not hold). In order to study these relations in our setup of continuous time, we extend the definition of stochastic dominance to continuous-time processes as follows. If $s$ and $k$ are two securities, we say that $s$ stochastically dominates $k$ at $t = 0$ (denoted by $s >_{SD} k$) if there exist $T^* > 0$ such that for all $0 < t < T^*$, $s_t$ stochastic dominates $k_t$. The following theorem asserts that the relation between stochastic dominance and acceptance dominance is preserved in the continuous-time setup.

**Theorem 4.1.**

$$s >_{SD} k \Rightarrow R_l(k) > R_l(s).$$

Theorem 4.1 follows from the relationship between stochastic dominance and acceptance dominance of standard assets: if $s >_{SD} k$, then, by definition, there is a range of time in which $s_t >_{SD} k_t$ implies that in the same range of time $s_t >_{AD} k_t$ implies that $R_l(k) > R_l(s)$.

Other properties of the measure of local riskiness are summarized in the following theorem.

**Theorem 4.2.** *Let $s$ and $k$ be two continuous-time securities. The local riskiness has the following properties:*

1. *Homogeneity: $R_l(\lambda s) = \lambda R_l(s)$ for every $\lambda > 0$.*
2. *Subadditivity: $R_l(s + k) \leq R_l(s) + R_l(k)$.*
3. *Convexity: $R_l(\lambda s + (1-\lambda)k) \leq \lambda R_l(s) + (1-\lambda)R_l(k)$ for every $0 < \lambda < 1$.*

## Duality of Local Risk and Risk Aversion

Aumann & Serrano (2008) characterize their index of riskiness by two axioms: duality and homogeneity. The duality axiom reflects the idea that "risk is what risk averters hate" (Machina and Rothschild (2008)). We take a similar approach and characterize our measure of local riskiness by similar axioms.

We denote by $\rho_i$ the well-known Arrow-Pratt coefficient of absolute risk aversion (ARA) of a decision maker $i$ with a utility function $u_i$ and an initial wealth $w_i$, i.e., $\rho_i = -u_i''(w_i)/u_i'(w_i)$. Let $i$ and $j$ be two decision makers and let $s$ and $k$ be two securities. Let $Q_l$ be a measure of local riskiness; i.e., $Q_l$ is a real-valued function of continuous-time securities. The following axiom characterizes a kind of dual relationship between local risk ($Q_l$) and risk aversion ($\rho$).

**Axiom 1.** *If $\rho_i > \rho_j$, $i$ accepts $s$ at $w_i$, and $Q_l(s) > Q_l(k)$, then $j$ accepts $k$ at $w_k$.*

As Aumann & Serrano (2008) explain, the axiom says that if the more risk-averse of two decision makers accepts the riskier of two securities, then a fortiori the less risk-averse decision maker accepts the less risky security. The second axiom concerns homogeneity:

**Axiom 2.** *$Q_l(\lambda s) = \lambda Q_l(s)$ for all $\lambda > 0$.*

As Aumann & Serrano (2008) explain, positive homogeneity embodies the cardinal nature of riskiness. If $s$ is a security, it makes sense to say that $2s$ is "twice as" risky as $s$, not just "more" risky.

**Theorem 4.3.** *For each security $s \in S$, $Q_l = R_l$ satisfies Axioms 1 and 2, and any index that satisfies these axioms is a positive multiple of $R_l$.*

Note that Axiom 1 and its equivalent—the original Aumann–Serrano duality axiom—are not identical. While both of them describe the relationship between risk and risk aversion, the definition of risk aversion in the two axioms is different: Axiom 1 refers to the Arrow–Pratt absolute risk aversion, and the Aumann–Serrano duality axiom refers to an incomplete order of the set of decision makers, which they call the "uniform risk aversion" order. Note that this difference is not crucial for our characterization of local riskiness. In other words, Axiom 1 could be formulated in terms of uniform risk aversion (instead of Arrow–Pratt risk aversion) and Theorem 4.3 would still be valid. Yet, the opposite is not true; i.e., the Aumann–Serrano Index cannot be characterized by a duality axiom that refers to the risk aversion instead of the uniform risk aversion.

# 5 Conclusion

We extended the acceptance dominance order, originally defined by Hart (2011) on gambles, to include risky assets whose values follow random processes in a continuous-time setup. We presented a measure of local riskiness of such assets that induces the acceptance dominance order. This shows that on this set of assets, acceptance dominance is a complete order. In addition, we showed that several measures of riskiness defined on random variables coincide locally with our measure of local riskiness.

The focus on risky assets whose random returns evolve continuously over time enables us to analyze cases where the risks are in some sense small, by focusing only on short investment time horizons. Such situations have been studied already by different methods. For example, Pratt (1964) showed that if the distribution of the returns is sufficiently concentrated, which means that the third absolute central moment is sufficiently small compared with the variance, then for any decision maker, the magnitude of the so-called *risk premium* is correlated with the level of the decision maker's risk aversion. Another similar interpretation of risk-aversion measures was developed independently by Arrow (1965). In addition, Samuelson (1970) showed that the classic mean-variance analysis, initiated by Markowitz (1959), applies approximately to all utility functions, in situations that involve what he calls "compact" distribution.

Analysis of decision problems in these methods is interesting since the decision of a decision maker regarding investments with "concentrated" or "small" returns depends on her utility function (and its derivatives) only at the initial wealth. This enables us to find an order on risks in a way that is relevant to all decision makers. In our analysis, this order is the acceptance dominance order. By contrast, if changes of wealth are not infinitesimally small, the utility at any possible future wealth level should be taken into account, and an order that is relevant for all decision makers does not exist.

Since our measure of local riskiness is relevant to all decision makers, we would expect any measure of riskiness in the discrete-time environment, which relates to the acceptance or rejection decision problem, to coincide with our measure of local riskiness in the continuous-time environment. In this sense, our measure of local riskiness can be used to identify proper measures of risks.

# A The Securities Model

The uncertainty in this model is generated by $K$ standard Wiener processes $W^1, \ldots, W^K$ defined on a filtered probability space $(\Omega, F_T, F, P)$ that satisfies the so-called usual conditions. The filtration $F = (F_t)_{t \in [0,T]}$ is the augmentation of the natural filtration $F^W$, generated by the vector $W = \{W(t) = W^1(t) \ldots W^K(t), t \in [0,T]\}$ of standard independent Wiener processes; see Karatzas & Shreve (1998).

Let $S$ be the set of securities whose prices follow continuous-time random processes and whose properties are described as follows. Let $s \in S$ be a security. The value of $s$ at time zero is given, denoted by $s_0$. For any other value of time $t$, $0 < t < T$, $s_t$ is the unique strong solution of a stochastic differential equation, described by

$$ds_t = \mu_t dt + \sigma_t^T dW_t, \tag{11}$$

where $\mu_t \equiv \mu(s_t, t)$ is a continuous function ($\forall t,\ \mu_t > 0$) and $\sigma_t \equiv \sigma(s_t, t)$ is a (column) vector of continuous functions. The superscript $T$ means "transpose". We assume also that $\sigma_t \neq 0$ $a.s.$

# B Proofs

Throughout the proofs we shall use Ito's lemma several times. It is worthwhile to recall a simple version of this lemma. If $s$ is a random process described by

$$ds = \mu dt + \sigma^T dW, \tag{12}$$

and $f(s,t)$ is a twice differentiable function of two variables, then

$$df_{s,t} = [\mu_t f_s + 0.5\sigma^T \sigma f_{ss} + f_t]dt + f_s \sigma^T dW, \tag{13}$$

where $f_s$ and $f_{ss}$ are the first and second derivatives of $f$ in relation to $s$, and $f_t$ is the first derivative of $f$ in relation to $t$.

***Proof of Theorem 2.1***.
Given a security $s$, an agent benefits from buying a unit of $s$ with investment time horizon $t$ if and only if $Eu(w - s_0 + s_t) - u(w) > 0$. According to Ito's lemma,

$$\begin{aligned} & Eu(w - s_0 + xs_t) - u(w) \\ = \ & E\left[\int_0^t xu'(w - s_0 + s_t)\mu_k + \frac{1}{2}u''(w - s_0 + s_t)\sigma_k^T \sigma_k dk\right]. \end{aligned} \tag{14}$$

Since the expression $Eu(w - s_0 + s_t) - u(w)$ is continuous over time, the agent accepts $s$ if and only if the limit of $Eu(w - s_0 + s_t) - u(w)$, as $t$ goes to zero, is positive. Following (14), this condition can be written as

$$\lim_{t\to 0} E\left[\int_0^t \left(u'(w - s_0 + s_t)\mu_s + \tfrac{1}{2}u''(w - s_0 + s_t)\sigma_s^T \sigma_s\right)ds \Big/ t\right] > 0$$

$$u'(w)\mu_0 + \tfrac{1}{2}u''(w)\sigma_0^T \sigma_0 > 0$$

$$-\frac{u'(w)}{u''(w)} > \frac{\sigma_0^T \sigma_0}{2\mu_0}. \tag{15}$$

The left-hand side of the equation is the reciprocal of the ARA of the decision maker and the right-hand side is $R_l(s)$. It follows from (15) that the question whether a decision maker accepts or rejects a security depends on two parameters only: the riskiness of the security $R_l$, and her Arrow-Pratt absolute risk aversion at the initial wealth. Hence, any decision maker who rejects a security $s$ at $w$ also rejects a security $k$ at $w$ if and only if $R_l(k) \geq R_l(s)$. □

The following lemma will be useful in the proof of Theorem 3.1.

**Lemma B.1.** *Let $f_t(x)$, $t > 0$, be a set of continuous real-valued functions, defined on the set of real numbers. Let $f(x)$ be a function and let $x_0 > 0$ be a number, such that:*

1. $\lim_{t\to 0} f_t(x) = f(x)$ *for all* $x > 0$.
2. $f(x_0) = 0$
3. *There exists* $\delta^* > 0$ *such that* $f$ *is strictly monotonic on* $(x_0 - \delta^*, x_0 + \delta^*)$.

*Then,*

1. $\exists\epsilon$ s.t. $\forall t$ $0 < t < \epsilon$ $\exists x_t$ s.t. $f_t(x_t) = 0$.

2. $\lim_{t\to 0} x_t = x_0$.

*Proof.* Without loss of generality we can assume that $f(x_0 - \delta^*) < 0$ and $f(x_0 + \delta^*) > 0$. For any $\delta$, $0 < \delta \leq \delta^*$, we define

$$l(\delta) = \min\{|f(x_0 - \delta)|, f(x_0 + \delta)\}.$$

It follows from the first condition that for all $\delta$, $0 < \delta \leq \delta^*$ there exists $\epsilon_\delta$ s.t. $\forall t < \epsilon_\delta$ :

$$|f(x_0 + \delta) - f_t(x_0 + \delta)| < l(\delta)$$
$$|f(x_0 - \delta) - f_t(x_0 - \delta)| < l(\delta).$$

Hence, $f_t(x_0 + \delta) > 0$ and $f_t(x_0 - \delta) < 0$ and since $f_t$ are continuous, $\forall t$ $0 < t < \epsilon_\delta$, there exists $x_t \in (x_0 - \delta, x_0 + \delta)$ s.t. $f_t(x_t) = 0$. Define a series $\delta_n = \delta^*/n$, the limit of the appropriate series $x_t$ is $x_0$.

□

Let $f$ be a real-valued function defined on the real numbers. In addition assume that its first derivative is positive and its second derivative is negative. Define the measure of riskiness $R^f$ on gambles implicitly by the equation

$$Ef(g/R^f(g)) = f(0), \tag{16}$$

the following lemma will also be useful for the proof of Theorem 3.1.

**Lemma B.2.** *Let $s$ be a security in the continuous-time environment, $s_0 = 0$, and $ds_t = \mu_t dt + \sigma_t dW$. There exists $T^* > 0$ such that $R^f$ is well defined on $s_t$ for all $t$, $0 < t < T^*$, and*

$$R^f_l(s) = -\frac{f''}{f'}\frac{\sigma^T\sigma}{2\mu},$$

*where $f'$ and $f''$ are the first and second derivatives of $f(x)$ where $x = 0$. $R^f_l$ is the local riskiness defined by $R^f$ by Equation 7.*

*Proof.* Given a security $s$, we look at the stochastic process $f(s_k/x)$, where $0 < k < T$. Using Ito's lemma, $f$ is characterized by the SDE

$$\begin{aligned} df_k &= \left(\frac{1}{x}\mu_k f'_k + \frac{1}{2}\frac{1}{x^2}\sigma_k^T\sigma_k f''_k\right)dk \\ &+ \frac{1}{x}\sigma_k^T f'_k dW, \end{aligned} \tag{17}$$

where $f'_k$ denotes the first derivative of $f$ at the point $s_k/x$ and $f''_k$ denotes the second derivative of $f$ at the point $s_k/x$. Taking the expectation of $f(s_t/x)$, we get

$$E_0\Big[f(s_t/x)\Big] = f(0) + E_0\left[\int_0^t \left(\frac{1}{x}\mu_k f'_k + \frac{1}{2}\frac{1}{x^2}\sigma_k^T\sigma_k f''_k\right)dk\right]. \tag{18}$$

It remains to show that for all $t$, $0 < t < T$ (for some $T > 0$), there exists $x_t$ such that

$$E_0\Big[f(s_t/x_t)\Big] = f(0), \tag{19}$$

or

$$E_0\left[\int_0^t \left(\frac{1}{x_t}\mu_k f'_k + \frac{1}{2}\frac{1}{x_t^2}\sigma_k^T\sigma_k f''_k\right)dk\right] = 0, \tag{20}$$

and that the limit of $x_t$, as $t$ goes to zero, is $R_l^f(s)$.

Let $h_t$ be a set of functions defined by

$$h_t(x) = E\left[\int_0^t \left(\frac{1}{x}\mu_k f'_k + \frac{1}{2}\frac{1}{x^2}\sigma_k^T\sigma_k f''_k\right)dk\right]\Big/ t, \tag{21}$$

and let $h(x)$ be defined as the limit of $h_t(x)$ as $t$ goes to zero:

$$h(x) = \frac{1}{x}\mu_0 f'_0 + \frac{1}{2}\frac{1}{x^2}\sigma_0^T\sigma_0 f''_0. \tag{22}$$

Now, let $x_0$ be s.t.

$$h(x_0) = 0 \Rightarrow x_0 = \frac{-f''_0}{f'_0}\frac{\sigma_0^T\sigma_0}{2\mu_0} \equiv \frac{-f''}{f'}\frac{\sigma^T\sigma}{2\mu}.$$

Since $\lim_{t\to 0} h_t(x) = h(x)$ and $h(x_0) = 0$, the first and the second conditions of Lemma B.1 are satisfied. To see that the third condition of the lemma is also satisfied, note that $h'(x_0) = 4(f'\mu)^3/(f''\sigma^T\sigma)^2 > 0$. So it follows from Lemma B.1 that there exists $\epsilon > 0$ s.t. for all $t$, $0 < t < \epsilon$, there exists $x_t$ s.t. $h_t(x_t) = 0$ and $\lim_{t\to 0} x_t = x_0$. Defining $T^* = \epsilon$, $R^f(s_t) = x_t$, and $R^f_l(s) = x_0$ completes the proof.

□

***Proof of Theorem 3.1***. We define $f^{AS}(g/x) = 1 - e^{-g/x}$ and $f^{FH}(g/x) = \log(1 + g/x)$. The proof of the first two parts of the theorem follows from Lemma (B.2). The proof of the third part of the theorem is as follows.

It follows from (5) and (6) that if $r$ is a discrete-time security, $\widetilde{R}^S(r)$ can be defined implicitly by the equation

$$E\left[r^{1-\frac{1}{\widetilde{R}^S(r)}}\right] = 1, \tag{23}$$

where $\widetilde{R}^S(r) \neq 1$. Now let $s \in S$ be a security and assume that $s_0 = 1$. We look at the stochastic process $f(s_k) = s_k^{1-1/x}$, where $0 < k < T$. Using Ito's lemma, $f(s_k)$ can be described by the SDE as

$$\begin{aligned} df_k &= \left((1-\frac{1}{x})\mu_k s_k^{-\frac{1}{x}} - \frac{1}{2}\frac{1}{x}(1-\frac{1}{x})\sigma_k^T\sigma_k s_k^{-(\frac{1}{x}+1)}\right)dk \\ &+ \frac{1}{x}s_k^{-\frac{1}{x}}\sigma_k^T dW, \end{aligned} \tag{24}$$

where by the assumption of the theorem, $f(s_0) \equiv s_0^{1-\frac{1}{x}}$=1.

Taking the expectation in (24), we get

$$\begin{aligned} &E_0\left[s_t^{1-\frac{1}{x}}\right] = s_0^{1-\frac{1}{x}} \\ &+E_0\left[\int_0^t \left((1-\frac{1}{x})\mu_k s_k^{-\frac{1}{x}} - \frac{1}{2}\frac{1}{x}(1-\frac{1}{x})\sigma_k^T\sigma_k s_k^{-(\frac{1}{x}+1)}\right)dk\right]. \end{aligned} \tag{25}$$

It remains to show that for $t$, $0 < t < T^*$ (for some $T^* > 0$), there exists $x_t$ such that

$$E_0\left[s_t^{1-\frac{1}{x_t}}\right] = 1 \tag{26}$$

or

$$E_0\left[\int_0^t \left((1-\frac{1}{x_t})\mu_k s_k^{-\frac{1}{x_t}} - \frac{1}{2}\frac{1}{x_t}(1-\frac{1}{x_t})\sigma_k^T\sigma_k s_k^{-(\frac{1}{x_t}+1)}\right)dk\right] = 0, \qquad (27)$$

and that the limit of $x_t$, as $t$ goes to zero, is $\tilde{R}_l^S(s)$.

Let $h_t$ be a set of real-valued functions, defined by

$$h_t(x) = E_0\left[\int_0^t \left((1-\frac{1}{x})\mu_k s_k^{-\frac{1}{x}} - \frac{1}{2}\frac{1}{x}(1-\frac{1}{x})\sigma_k^T\sigma_k s_k^{-(\frac{1}{x}+1)}\right)dk\right]\Big/ t, \qquad (28)$$

and let $h(x)$ be defined as the limit of $h_t(x)$ as $t$ goes to zero:

$$h(x) = (1-\frac{1}{x})\mu_0 - \frac{1}{2}\frac{1}{x}(1-\frac{1}{x})\sigma_0^T\sigma_0. \qquad (29)$$

Now, let $x_0$ be s.t. $h(x_0) = 0$. Recall that $\tilde{R}_l \neq 1$ so we are not interested in the solution $x_0 = 1$. Hence,

$$x_0 = \frac{\sigma_0^T\sigma_0}{2\mu_0}.$$

Since $\lim_{t\to 0} h_t(x) = h(x)$ and $h(x_0) = 0$, the first and the second conditions of Lemma B.1 are satisfied. To see that the third condition of the lemma is also satisfied, note that $h'(x_0) = 2\mu^2/\sigma^T\sigma - 4\mu^3/(\sigma^T\sigma)^2 < 0$. So it follows from Lemma B.1 that there exists $\epsilon > 0$ s.t. for all $t$, $0 < t < \epsilon$, there exists $x_t$ s.t. $h_t(x_t) = 0$ and $\lim_{t\to 0} x_t = x_0$. Defining $T^* = \epsilon$, $\tilde{R}^S(s_t) = x_t$, and $\tilde{R}_l^S(s) = x_0$ completes the proof. □

***Proof of Theorem 4.2.***

**Homogeneity.** If $ds = \mu dt + \sigma^T dW$, the security $\lambda s$ is described by the SDE $d(\lambda s) = \lambda\mu dt + \lambda\sigma^T dW$. Hence, $R_l(\lambda s) = \lambda\frac{\sigma^T\sigma}{2\mu} = \lambda R_l(s)$.

**Subadditivity.** Let $ds = \mu^s dt + \sigma^{s\ T} dW$ and $dk = \mu^k dt + \sigma^{k\ T} dW$ be two securities. The sum $s+k$ is characterized by the SDE $d(s+k) = (\mu^s + \mu^k)dt + (\sigma^{s\ T} + \sigma^{s\ T})dW$. By definition, the riskiness of $s+k$ is

$$R_l(s+k) = \frac{(\sigma^{s\ T} + \sigma^{k\ T})(\sigma^s + \sigma^k)}{\mu^s + \mu^k}.$$

We have to show that

$$\frac{(\sigma^{s\ T} + \sigma^{k\ T})(\sigma^s + \sigma^k)}{\mu^s + \mu^k} \leq \frac{\sigma^{s\ T}\sigma^s}{\mu^s} + \frac{\sigma^{k\ T}\sigma^k}{\mu^k}. \tag{30}$$

Note that

1. $\sigma^{s\ T}\sigma^s = \Sigma_{i=1}^K (\sigma_i^s)^2$.
2. $\sigma^{k\ T}\sigma^k = \Sigma_{i=1}^K (\sigma_i^k)^2$.
3. $\sigma^{s\ T}\sigma^k = \sigma^{k\ T}\sigma^s = \Sigma_{i=1}^K \sigma_i^s \sigma_i^k$.

Hence, (30) can be rewritten as

$$\begin{aligned}
\frac{\Sigma_{i=1}^K (\sigma_i^s)^2 + \Sigma_{i=1}^K (\sigma_i^k)^2 + 2\Sigma_{i=1}^K \sigma_i^s \sigma_i^k}{\mu^s + \mu^k} &\leq \frac{\Sigma_{i=1}^K (\sigma_i^s)^2}{\mu^s} + \frac{\Sigma_{i=1}^K (\sigma_i^k)^2}{\mu^k} \\
(\Sigma_{i=1}^K (\sigma_i^s)^2 + \Sigma_{i=1}^K (\sigma_i^k)^2 + 2\Sigma_{i=1}^K \sigma_i^s \sigma_i^k)\mu^s \mu^k &\leq \Sigma_{i=1}^K (\sigma_i^s)^2 \mu^k (\mu^s + \mu^k) \\
&\quad + \Sigma_{i=1}^K (\sigma_i^k)^2 \mu^s (\mu^s + \mu^k) \\
\mu^s \mu^k 2\Sigma_{i=1}^K \sigma_i^s \sigma_i^k &\leq (\mu^k)^2 \Sigma_{i=1}^K (\sigma_i^s)^2 \\
&\quad + (\mu^s)^2 \Sigma_{i=1}^K (\sigma_i^k)^2 \\
0 &\leq \Sigma_{i=1}^K (\mu^k \sigma_i^s - \mu^s \sigma_i^k)^2. \quad (31)
\end{aligned}$$

Since the last line is always satisfied, it completes the proof of subadditivity.

**Convexity.** Follows directly from homogeneity and subadditivity. □

***Proof of Theorem 4.3*** .
To see that $R_l$ satisfies the first axiom, recall that according to Equation 15, a decision maker $dm = (u, w)$ accepts a security $s$ if and only if

$$-\frac{u'(w)}{u''(w)} > \frac{\sigma^{sT}\sigma^s}{2\mu^s}.$$

The left-hand side of the equation is the reciprocal of the ARA of the decision maker and the right-hand side is $R_l(s)$. So if a decision maker $i$ is less averse to risk than decision maker $j$ and if $j$ accepts a security, $i$ would accept this security and any riskier security. The second axiom (homogeneity) is proved above.

To see that any measure of local riskiness that satisfies the axioms is a positive multiple of $R_l$, assume that $H_l$ is a measure of riskiness that satisfies the two axioms. The first axiom implies that $R_l$ and $H_l$ are ordinally equivalent, i.e., for any two securities $s$ and $r$, if $s$ is riskier according to $R_l$ then $s$ is also riskier according to $H_l$. Indeed, it follows from (15) that if a measure of riskiness $H_l$ satisfies the first axiom, then $H_l(s) > H_l(r)$ if and only if

$$\frac{\sigma^{sT}\sigma^s}{2\mu^s} > \frac{\sigma^{rT}\sigma^r}{2\mu^r}.$$

Now let $r$ be an arbitrary but fixed security and set $\lambda = R_l(r)/H_l(r)$. If $s$ is any security and $t = H_l(s)/H_l(r)$, then $H_l(tr) = tH_l(r) = H_l(s)$, so $R_l(tr) = tR_l(r) = R_l(s)$ by the ordinal equivalence between $H_l$ and $R_l$, so $R_l(s)/R_l(r) = t = H_l(s)/H_l(r)$, so $H_l(s)/R_l(s) = R_l(r)/H_l(r) = \lambda$, so $H_l(s) = \lambda R_l(s)$. This completes the proof of the theorem. □

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