Problem Set 3 – Special Functions

Linear Functions and the Equation of the Line

1.

a. Describe the family of linear functions satisfying f(4) = –3

$$\boldsymbol{f(x) = ax - (4a + 3)}$$

b. Which of the functions you found above satisfy the equation f(2) = 7?

Since we must satisfy $\boldsymbol{a = -5}$, the function is $\boldsymbol{f(x) = -5x + 17}$

c. Which one has a slope of 3/4?

Since we must satisfy $\boldsymbol{a = \frac{3}{4}}$, the function is $\boldsymbol{f(x) = \frac{3}{4}x - 6}$

2. Find the equation of the line parallel to the line x+2y = 32 forming a triangle of area 1 with the two axes.

$$\boldsymbol{x + 2y = 2 \text{ or } x + 2y = -2}$$

3. Fine the equations of the lines containing the sides of the triangle whose vertice are (-4, 5) ,(0, 3) ,(2, -1) .

Side 1: $\boldsymbol{y = -2x + 3}$. Side 2: $\boldsymbol{y = -x + 1}$. Side 3: $\boldsymbol{x + 2y - 6 = 0}$

4. Given the family of lines defined by $(k^2 - 36)x + (k^2 - 15k + 36)y + (k-1)^2 = 0$, where k is a parameter, find the values of k such that the associated line satisfies the property that:

a. It is parallel to the x-axis

$$\boldsymbol{k = 6 \text{ or } k = -6}$$

b. It passes through the origin

$$\boldsymbol{k = 1}$$

c. It cuts through the angle between the axes

Never

d. It is parallel to the line 2y + 5x = 3

$$\boldsymbol{k = 4 \text{ or } k = 21}$$

5. Given the functions

   $f(x) = x^2 - 2x - 3$

   $g(x) = -2x^2 + 13x - 15$

   a. Describe the domains and ranges of f and g.

      The domain of both is the set of all real numbers. The range of f is $[-4, \infty)$. The range of g is $(-\infty, 6.125]$.

   b. Calculate the roots of f and of f – g.

      Roots of f: 3 and –1. Roots of f – g: 4 and 1.

6. Given the function $f(x) = (m-4)x^2 + 10x + m$,

   for what value of m does the range of this function include only numbers that are less than 4?

   $$\{m | m < -1\}$$

7. Given the function $x^2 - mx + m + 3 = 0$,

   for what value of m does this function have two solutions, one less than 3 and the other greater than 3?

   $$\{m | m > 6\}$$

8. For what values of m is the expression $(m^2 - 1)x^2 + 2(m-1)x + 2$ positive for all x?

   $$\{m | m > 1 \text{ or } m < -3\}$$

9. Production costs for producing Q items at a certain firm depends on the number of machines being used. If 5 machines are run, the total production cost is

$TC_1(Q) = 2Q^2 + 8Q + 10$. If 15 machines are in use, the total production cost is $TC_2(Q) = Q^2 + 4Q + 231$.

For any given production amount Q, the firm will choose the number of machines (either 5 or 15) that reduce production costs to a minimum. For which values of Q will the firm choose to use only 5 machines, and for which values of Q will the firm choose to use 15 machines?

$$\{\boldsymbol{Q} | -\mathbf{15} < \boldsymbol{Q} < \mathbf{13}\}$$

10. Suppose the following game is given: the player chooses a number x such that $0 \leq x \leq 3$, and then receives a payoff the lesser of $2x+3$ and x^2-x+5.
    a. Find the function that gives the payoff that the player receives as a function of x.

$$\boldsymbol{f(x)} = \begin{cases} \mathbf{2x+3} & \boldsymbol{if}\ \mathbf{0 \leq x \leq 1}\ \boldsymbol{or}\ \mathbf{2 \leq x \leq 3} \\ \mathbf{x^2 - x + 5} & \boldsymbol{if}\ \mathbf{1 < x < 2} \end{cases}$$

    b. Which x should the player choose in order to receive the maximal payoff?

x=3

Sequences

11. Determine whether the sequence $a_n = \frac{n-1}{3n+5}$ is monotonic and whether it is bounded.

Monotonically increasing and bounded from above by 1/3 and from below by 0

12. Given the sequence

$$a_n = \begin{cases} \frac{2n}{n+3} & if \quad n \quad even \\ 2 & otherwise \end{cases}$$

write down the first five elements of the sequence. Determine whether this sequence is monotonic. Is it bounded? Is it injective?

First five elements: 2, 4/5, 2, 8/7, 2. Not monotonic. Bounded from above by 3 and from below by 0. Not injective.

13. Given the sequence 1, 1.1, 1.11, 1.111 ...
    a. Write this sequence as a function of n.

$$f(n) = \frac{\sum_{j=1}^{n-1} 10^j}{10^{n-1}}$$

    b. Determine whether this sequence is monotonic. Is it bounded?

Monotonically increasing and bounded.

14. Given the sequence $\frac{(-1)^{n+1}}{n+1}$
    a. Write down the first five elements of this sequence.

First five elements: 1/2, -1/3, 1/4, -1/5, 1/6.

    b. Determine whether this sequence is monotonic. Is it bounded? If it is bounded, find its upper bound and its lower bound.

Not monotonic. Bounded by 1 and –1 .

15. Given the function $f(n) = \frac{4-2n}{3n+2}$ $\quad f: N \to R$
    a. Determine if this is a surjective function. **No**
    b. Is it injective? **Yes**
    c. Is it monotonic? If yes, is it monotonically increasing or monotonically decreasing? **Monotonically decreasing.**

Is the function bounded? If yes, find its bounds. **Bounded by 1 and –1 .**