COINTEGRATION IN FREQUENCY DOMAIN

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Abstract. Existence of a cointegration relationship between two time series in the time domain imposes restrictions on the series zero-frequency behaviour in terms of their squared coherence, phase and gain, in the frequency domain. I derive these restrictions by studying cross-spectral properties of a cointegrated bivariate system. Specifically, I demonstrate that if two difference stationary series, X_t and Y_t, are cointegrated with a cointegrating vector $[1 \quad b]$ and thus share a common stochastic trend, then at the zero frequency, the squared coherence of $(1 - L)X_t$ and $(1 - L)Y_t$ will equal one, their phase will equal zero, and their gain will equal $|b|$.

Keywords. Common stochastic trend; cointegration; frequency domain analysis; cross-spectrum; zero-frequency.

1. INTRODUCTION

Since the introduction of cointegration and common trend analysis in econometrics and statistics by Engle and Granger (1987) and Stock and Watson (1988), integration and cointegration tests have by now become an essential part of applied econometricians' and macroeconomists' standard tool kit. These tests are routinely applied to economic time series because the notion of cointegration has a natural economic interpretation: existence of a cointegration relationship between two variables indicates that the series 'move together' in the long run, and so they share common stochastic trend, although, in the short run, the series may diverge from each other. Since many economic theories make these kinds of long-run and short-run differential predictions about economic time series co-movements, many economic models (and particularly macroeconomic models) lend themselves naturally to co-integration testing (Engle and Granger, 1987).

The cointegration property is a long-run property and, therefore, in frequency domain, it refers to the zero-frequency relationship of the time series. Therefore, there is a frequency-domain equivalent of the time-domain cointegration property. Specifically, existence of a cointegration relationship between two time series in the time domain imposes restrictions on the series zero-frequency

behaviour in terms of their cross-spectral measures in the frequency domain. The purpose of this paper is to use a bivariate setting to derive these frequency-domain restrictions in terms of the time series' squared coherence, phase and gain–the measures practitioners typically consider when studying cross-spectral properties of time series.

Squared coherence is analogous to the square of the correlation coefficient and measures the degree to which one series can be represented as a linear function of the other. Phase measures the phase difference or the timing (i.e., lead or lag) between the frequency components of the two series. Gain indicates how much the spectrum of one series has been amplified to approximate the corresponding frequency component of the other. It is essentially the regression coefficient of one series on another at frequency ω. Thus, the squared coherence, phase, and gain are frequency-domain equivalents of correlation coefficient, time-delay (lag), and regression coefficient, respectively, and, therefore, they have a natural interpretation in terms of the standard time domain regression analysis.

The paper proceeds as follows. Cross-spectral properties of a cointegrated bivariate system are derived by beginning with two non-stationary time series that are cointegrated with a cointegration vector $[1 \quad b]$. Then using standard Fourier Transform methods and matrix algebra, frequency domain properties of the series' co-movement are derived in terms of their squared coherence, phase and gain. Specifically, it is shown that the squared coherence between such series, after differencing, will equal one, their phase will equal zero, while their gain will equal $|b|$. The paper ends with a brief conclusion in Section 3.

2. CROSS-SPECTRAL PROPERTIES OF A COINTEGRATED BIVARIATE SYSTEM

Let the time series of X_t and Y_t be difference stationary. Thus, let $X_t \sim I(1)$ and $Y_t \sim I(1)$, so that they can be written as

$$X_t = X_{t-1} + u_t \quad \text{and} \quad Y_t = Y_{t-1} + v_t$$

respectively, where $u_t \sim I(0)$, and $v_t \sim I(0)$. Moreover, let us assume that X_t and Y_t are cointegrated with the cointegration vector $[1 \quad b]$, so that they satisfy

$$Y_t = bX_t + \mu_t$$

where $\mu_t \sim I(0)$. Then X_t and Y_t processes share a common stochastic trend and, therefore, can be written in a matrix notation

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} T_t + x_t \\ bT_t + y_t \end{bmatrix} \tag{1}$$

where T_t is the common stochastic trend with the property $(1 - L)T_t = z_t, z_t \sim \text{iid}(0, \sigma_z^2)$ is a white noise process, $x_t \sim I(0)$, and $y_t \sim I(0)$. Applying difference operator $\Delta = (1 - L)$ to (1) yields a bivariate stationary process

$$\begin{bmatrix} (1-L)X_t \\ (1-L)Y_t \end{bmatrix} = \begin{bmatrix} z_t + (1-L)x_t \\ bz_t + (1-L)y_t \end{bmatrix} \tag{2}$$

with the spectral matrix

$$\mathbf{f}(\omega) = \begin{bmatrix} f_{\Delta X} & f_{\Delta X \Delta Y} \\ f_{\Delta Y \Delta X} & f_{\Delta Y} \end{bmatrix} \tag{3}$$

The diagonal elements of the $\mathbf{f}(\omega)$ matrix are the spectral density functions of $(1-L)X_t$ and $(1-L)Y_t$, defined by

$$f_{\Delta X} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma_{\Delta X}(\tau) e^{-i\tau\omega} d\tau \tag{4a}$$

and

$$f_{\Delta Y} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma_{\Delta Y}(\tau) e^{-i\tau\omega} d\tau \tag{4b}$$

where $\gamma_{\Delta X}(\tau)$ and $\gamma_{\Delta Y}(\tau)$ are the autocovariance functions of $(1-L)X_t$ and $(1-L)Y_t$, defined by

$$\gamma_{\Delta X}(\tau) = \mathrm{E}[(\Delta X_{t+\tau} - \mu_{\Delta X})(\Delta X_t - \mu_{\Delta X})] \tag{5a}$$

and

$$\gamma_{\Delta Y}(\tau) = \mathrm{E}[(\Delta Y_{t+\tau} - \mu_{\Delta Y})(\Delta Y_t - \mu_{\Delta Y})] \tag{5b}$$

respectively, where $\mu_{\Delta X}$ and $\mu_{\Delta Y}$ denote the means of $(1-L)X_t$ and $(1-L)Y_t$, respectively. The off-diagonal elements of the $\mathbf{f}(\omega)$ matrix are the cross spectral density functions of $(1-L)X_t$ and $(1-L)Y_t$, defined by

$$f_{\Delta X \Delta Y} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma_{\Delta X \Delta Y}(\tau) e^{-i\tau\omega} d\tau \tag{6a}$$

and

$$f_{\Delta Y \Delta X} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma_{\Delta Y \Delta X}(\tau) e^{-i\tau\omega} d\tau \tag{6b}$$

where $\gamma_{\Delta X \Delta Y}(\tau)$ and $\gamma_{\Delta Y \Delta X}(\tau)$ are the cross covariance functions of $(1-L)X_t$ and $(1-L)Y_t$, and $(1-L)Y_t$ and $(1-L)X_t$, defined by

$$\gamma_{\Delta X \Delta Y}(\tau) = \mathrm{E}[(\Delta X_{t+\tau} - \mu_{\Delta X})(\Delta Y_t - \mu_{\Delta Y})] \tag{7a}$$

and

$$\gamma_{\Delta Y \Delta X}(\tau) = \mathrm{E}[(\Delta Y_{t+\tau} - \mu_{\Delta Y})(\Delta X_t - \mu_{\Delta X})] \tag{7b}$$

respectively.

To compute the elements of the $\mathbf{f}(\omega)$ matrix, first the autocovariance-crosscovariance matrix of (2) has to be computed. It is given by

$$\boldsymbol{\gamma}(\tau) = \begin{bmatrix} \gamma_{\Delta X}(\tau) & \gamma_{\Delta X \Delta Y}(\tau) \\ \gamma_{\Delta Y \Delta X}(\tau) & \gamma_{\Delta Y}(\tau) \end{bmatrix}$$

$$= \mathrm{E} \begin{bmatrix} z_{t+\tau} + \Delta x_{t+\tau} \\ b z_{t+\tau} + \Delta y_{t+\tau} \end{bmatrix} [z_t + \Delta x_t \quad b z_t + \Delta y_t]$$

$$= \mathrm{E} \begin{bmatrix} (z_{t+\tau} + \Delta x_{t+\tau})(z_t + \Delta x_t) & (z_{t+\tau} + \Delta x_{t+\tau})(b z_t + \Delta y_t) \\ (b z_{t+\tau} + \Delta y_{t+\tau})(z_t + \Delta x_t) & (b z_{t+\tau} + \Delta y_{t+\tau})(b z_t + \Delta y_t) \end{bmatrix}$$

$$= \begin{bmatrix} \gamma_z(\tau) + \gamma_1(\tau) + \gamma_{z1}(\tau) + \gamma_{1z}(\tau) & b\gamma_z(\tau) + b\gamma_{1z}(\tau) + \gamma_{z2}(\tau) + \gamma_{12}(\tau) \\ b\gamma_z(\tau) + \gamma_{2z}(\tau) + b\gamma_{z1}(\tau) + \gamma_{21}(\tau) & b^2\gamma_z(\tau) + \gamma_2(\tau) + b\gamma_{z2}(\tau) + b\gamma_{2z}(\tau) \end{bmatrix}$$

$$(8)$$

where subscripts 1 and 2 denote Δx and Δy, respectively, for notational simplicity, the diagonal elements of the last matrix in (8) are the autocovaraiance functions, $\gamma_{\Delta X}(\tau)$ and $\gamma_{\Delta Y}(\tau)$, and the off-diagonal elements are the cross cova-riance functions, $\gamma_{\Delta X \Delta Y}(\tau)$ and $\gamma_{\Delta Y \Delta X}(\tau)$, respectively, as defined in (5a)–(5b) and (7a)–(7b).

Applying the Fourier transform to both sides of (8), multiplying through by $1/(2\pi)$, and using spectrum and cross-spectrum definitions provided by (4a)–(4b) and (6a)–(6b), gives the spectral matrix

$$\mathbf{f}(\omega)$$

$$= \begin{bmatrix} f_z(\omega) + f_1(\omega) + f_{z1}(\omega) + f_{1z}(\omega) & b f_z(\omega) + b f_{1z}(\omega) + f_{z2}(\omega) + f_{12}(\omega) \\ b f_z(\omega) + f_{2z}(\omega) + b f_{z1}(\omega) + f_{21}(\omega) & b^2 f_z(\omega) + f_2(\omega) + b f_{z2}(\omega) + b f_{2z}(\omega) \end{bmatrix}$$

$$(9)$$

which can be rewritten as

$$\mathbf{f}(\omega) = \begin{bmatrix} f_z(\omega) & b f_z(\omega) \\ b f_z(\omega) & b^2 f_z(\omega) \end{bmatrix}$$
$$+ \begin{bmatrix} f_1(\omega) + f_{z1}(\omega) + f_{1z}(\omega) & b f_{1z}(\omega) + f_{z2}(\omega) + f_{12}(\omega) \\ f_{2z}(\omega) + b f_{z1}(\omega) + f_{21}(\omega) & f_2(\omega) + b f_{z2}(\omega) + b f_{2z}(\omega) \end{bmatrix} \quad (10)$$

The cross-spectrum terms in (10) can be written in Cartesian form because the spectral matrix $\mathbf{f}(\omega)$ is, in general, a complex-valued function. Thus, for example, we can write

$$f_{mn}(\omega) = c_{mn}(\omega) - i q_{mn}(\omega) \tag{11}$$

where $c_{mn}(\omega)$ denotes the cospectral function of m and n, and $q_{mn}(\omega)$ denotes the quadrature spectral function of m and n. Therefore, using Priestley's (1981, p. 668, Equation 9.1.53) result that $f_{mn}(\omega) = \overline{f_{nm}(\omega)}$, (10) can be rewritten as

$$\mathbf{f}(\omega) = \begin{bmatrix} f_z(\omega) & bf_z(\omega) \\ bf_z(\omega) & b^2 f_z(\omega) \end{bmatrix}$$
$$+ \begin{bmatrix} f_1(\omega) + f_{z1}(\omega) + \overline{f_{z1}(\omega)} & bf_{1z}(\omega) + f_{z2}(\omega) + f_{12}(\omega) \\ f_{2z}(\omega) + bf_{z1}(\omega) + f_{21}(\omega) & f_2(\omega) + bf_{z2}(\omega) + b\overline{f_{z2}(\omega)} \end{bmatrix} \quad (12)$$

where bar denotes complex conjugate. Combining (12) with Cartesian representation of $f_{z1}(\omega)$ and $f_{z2}(\omega)$,

$$f_{z1}(\omega) = c_{z1}(\omega) - \mathrm{i} q_{z1}(\omega) \quad (13\text{a})$$

and

$$f_{z2}(\omega) = c_{z2}(\omega) - \mathrm{i} q_{z2}(\omega) \quad (13\text{b})$$

yields

$$\mathbf{f}(\omega) = \begin{bmatrix} f_z(\omega) & bf_z(\omega) \\ bf_z(\omega) & b^2 f_z(\omega) \end{bmatrix}$$
$$+ \begin{bmatrix} f_1(\omega) + 2c_{z1}(\omega) & bf_{1z}(\omega) + f_{z2}(\omega) + f_{12}(\omega) \\ f_{2z}(\omega) + bf_{z1}(\omega) + f_{21}(\omega) & f_2(\omega) + 2bc_{z2}(\omega) \end{bmatrix} \quad (14)$$

Now, consider the value of the spectral matrix $\mathbf{f}(\omega)$ at frequency $\omega = 0$, which using (3) and (14) can be written as

$$\mathbf{f}(0) = \begin{bmatrix} f_{\Delta X}(0) & f_{\Delta X \Delta Y}(0) \\ f_{\Delta Y \Delta X}(0) & f_{\Delta Y}(0) \end{bmatrix} = \begin{bmatrix} f_z(0) & bf_z(0) \\ bf_z(0) & b^2 f_z(0) \end{bmatrix}$$
$$+ \begin{bmatrix} f_1(0) + 2c_{z1}(0) & bf_{1z}(0) + f_{z2}(0) + f_{12}(0) \\ f_{2z}(0) + bf_{z1}(0) + f_{21}(0) & f_2(0) + 2bc_{z2}(0) \end{bmatrix} \quad (15)$$

Recall that z_t is a white noise process and, therefore, its theoretical spectrum is flat and equals $f_z(\omega) = \sigma_z^2/(2\pi)$ for all frequencies $-\pi \le \omega \le \pi$. In addition, Δx and Δy are I(-1); therefore, their zero-frequency spectral density, cross spectral density and cospectral density functions equal zero. Thus, every element of the second matrix of the right-hand side of (15) vanishes; therefore, the spectral matrix, evaluated at frequency $\omega = 0$, becomes

$$\mathbf{f}(0) = \begin{bmatrix} f_{\Delta X}(0) & f_{\Delta X \Delta Y}(0) \\ f_{\Delta Y \Delta X}(0) & f_{\Delta Y}(0) \end{bmatrix} = \begin{bmatrix} \dfrac{\sigma_z^2}{2\pi} & \dfrac{b\sigma_z^2}{2\pi} \\ \dfrac{b\sigma_z^2}{2\pi} & \dfrac{b^2 \sigma_z^2}{2\pi} \end{bmatrix} \quad (16)$$

To see the implications of this result for the behaviour of the theoretical squared coherence, phase and gain, recall from polar representation of $\mathbf{f}(\omega)$ that

$$\mathcal{K}_{\Delta X \Delta Y}^2(\omega) = |f_{\Delta X \Delta Y}(\omega)|^2 [f_{\Delta X}(\omega) f_{\Delta Y}(\omega)]^{-1} \quad (17)$$

$$\Phi_{\Delta X \Delta Y}(\omega) = \arctan\left\{ -\dfrac{\mathrm{Im}[f_{\Delta X \Delta Y}(\omega)]}{\mathrm{Re}[f_{\Delta X \Delta Y}(\omega)]} \right\} \quad (18)$$

and

$$\Gamma_{\Delta X \Delta Y}(\omega) = |f_{\Delta X \Delta Y}(\omega)|[f_{\Delta X}(\omega)]^{-1} \tag{19}$$

where $\text{Im}[f_{\Delta X \Delta Y}(\omega)]$ and $\text{Re}[f_{\Delta X \Delta Y}(\omega)]$ denote the imaginary and real parts of $f_{\Delta X \Delta Y}(\omega)$; and $\mathcal{K}^2_{\Delta X \Delta Y}(\omega)$, $\Phi_{\Delta X \Delta Y}(\omega)$ and $\Gamma_{\Delta X \Delta Y}(\omega)$ denote the squared coherence, phase and the gain of $(1-L)X_t$ and $(1-L)Y_t$, respectively (Jenkins and Watts, 1968). Then, using the matrix (16), along with the definitions of squared coherence, phase and gain provided in (17)–(19), at the zero frequency, the following equalities hold.

For the squared coherence of $(1-L)X_t$ and $(1-L)Y_t$, (16) and (17) imply that, at the frequency $\omega = 0$,

$$\mathcal{K}^2_{\Delta X \Delta Y}(0) = \left|\frac{b\sigma_z^2}{2\pi}\right|^2 \left(\frac{\sigma_z^2}{2\pi}\frac{b^2\sigma_z^2}{2\pi}\right)^{-1} = 1 \tag{20}$$

To determine the phase of $(1-L)X_t$ and $(1-L)Y_t$, note that from the Cartesian representation of $f_{\Delta X \Delta Y}(\omega)$

$$f_{\Delta X \Delta Y}(\omega) = c_{\Delta X \Delta Y}(\omega) - iq_{\Delta X \Delta Y}(\omega) \tag{21}$$

However, from (16) at the zero frequency

$$f_{\Delta X \Delta Y}(0) = \frac{b\sigma_z^2}{2\pi} \tag{22}$$

Rewrite (21) for $\omega = 0$,

$$f_{\Delta X \Delta Y}(0) = c_{\Delta X \Delta Y}(0) - iq_{\Delta X \Delta Y}(0) \tag{23}$$

and compare the resulting equation (23) to (22). The equality of the two equations requires that their right-hand sides be equal. However, for a complex number to equal a real number, it is necessary that the imaginary part of the complex number be zero. In other words, for the equality of (22) and (23), it is necessary that at the frequency $\omega = 0$, the cospectrum of $(1-L)X_t$ and $(1-L)Y_t$ satisfy

$$c_{\Delta X \Delta Y}(0) = \text{Re}[f_{\Delta X \Delta Y}(0)] = \frac{b\sigma_z^2}{2\pi} \tag{24}$$

and the quadrature spectrum of $(1-L)X_t$ and $(1-L)Y_t$ satisfy

$$q_{\Delta X \Delta Y}(0) = \text{Im}[f_{\Delta X \Delta Y}(0)] = 0 \tag{25}$$

Substituting (24) and (25) into the definition of phase (18) for the frequency $\omega = 0$ gives

$$\Phi_{\Delta X \Delta Y}(0) = \arctan\left\{-\frac{\text{Im}[f_{\Delta X \Delta Y}(0)]}{\text{Re}[f_{\Delta X \Delta Y}(0)]}\right\} = \arctan(0) = 0 \tag{26}$$

Finally, to determine the gain of $(1-L)X_t$ and $(1-L)Y_t$, combining (16) and (19) and evaluating the result for the frequency $\omega = 0$ yields

$$\Gamma_{\Delta X \Delta Y}(0) = \left|\frac{b\sigma_z^2}{2\pi}\right|\left[\frac{\sigma_z^2}{2\pi}\right]^{-1} = |b| \tag{27}$$

where b is the coefficient that measures the extent of the long-run relationship between X_t and Y_t. That is, b is the coefficient in the cointegration relationship, $Y_t = bX_t + \mu_t$, where $\mu_t \sim I(0)$.

Equations (20), (26) and (27) establish the main result of this paper: if two difference stationary series, X_t and Y_t, are cointegrated with the cointegrating vector $[1 \quad b]$, then the zero frequency squared coherence, phase and gain of $(1 - L)X_t$ and $(1 - L)Y_t$ will equal one, zero and $|b|$, respectively. This is a generalization of Levy (2000) who only focuses on the behaviour of squared coherence and gain, and only for the case $b = -1$.

3. CONCLUSION

The cointegration property is a long-run property and, therefore, in frequency domain, it refers to the zero-frequency relationship of the time series. Therefore, there is a frequency-domain equivalent of the time-domain cointegration property: existence of a cointegration relationship between two time series in the time domain imposes restrictions on the series zero-frequency behaviour in terms of their squared coherence, phase and gain in the frequency domain. This paper derives these frequency-domain restrictions in a bivariate setting. Specifically, it demonstrates that if two difference stationary series, X_t and Y_t, are cointegrated with the cointegrating vector $[1 \quad b]$, then the zero frequency squared coherence, phase and gain of $(1 - L)X_t$ and $(1 - L)Y_t$ will equal one, zero and $|b|$, respectively.

It is well known that the standard time series cointegration tests have a low power. The results derived in this paper suggest that it may be useful to test for cointegration in frequency domain. Further work should examine limiting null distributions and finite sample properties of such tests, so as to assess their practical usefulness.

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