Online Supplementary Appendix

(Not for Publication)

Interpolation and Shock Persistence of Prewar U.S. Macroeconomic Time Series: A Reconsideration

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To obtain the variance ratio for the interpolated series (section 3 in the paper), we need to find $\sigma_{1,x}^2 = \operatorname{var}\left(x_{t,i} - x_{t,i-1}\right)$, the variance of the series' 1-period growth or the "short variance," and $\sigma_{s,x}^2 = \operatorname{var}\left(x_{t,i} - x_{t-1,i}\right)$, the variance of the series' s-period growth or the "long variance."

Short Variance

The short variance, i.e., the variance of the "short difference" can be obtained by taking the variance of the short difference in the interpolated series, as shown below.

$$
\begin{aligned}
\sigma_{1,x}^2 &= \operatorname{var}\left(x_{t,i} - x_{t,i-1}\right) \\
&= \operatorname{var}\left[(1/s)\left(y_{t,s} - y_{t-1,s}\right) + \left(\varepsilon_{t,i} - \theta\varepsilon_{t,i-1}\right)\psi\left(i \neq s\right) - \left(\varepsilon_{t,i-1} - \theta\varepsilon_{t,i-2}\right)\psi\left(i \neq 1\right)\right] \\
&= \operatorname{var}\left\{(1/s)\left[\left(\varepsilon_{t,1} - \theta\varepsilon_{t-1,s}\right) + \left(\varepsilon_{t,2} - \theta\varepsilon_{t,1}\right) + \left(\varepsilon_{t,3} - \theta\varepsilon_{t,2}\right) + \cdots + \left(\varepsilon_{t,s-1} - \theta\varepsilon_{t,s-2}\right) + \left(\varepsilon_{t,s} - \theta\varepsilon_{t,s-1}\right)\right]\right\} \\
&+ \operatorname{var}\left[\left(\varepsilon_{t,i} - \theta\varepsilon_{t,i-1}\right)\psi\left(i \neq s\right)\right] \\
&+ \operatorname{var}\left[\left(\varepsilon_{t,i-1} - \theta\varepsilon_{t,i-2}\right)\psi\left(i \neq 1\right)\right] \\
&+ 2\operatorname{cov}\left\{\left[(1/s)\left[\left(\varepsilon_{t,1} - \theta\varepsilon_{t-1,s}\right) + \left(\varepsilon_{t,2} - \theta\varepsilon_{t,1}\right) + \left(\varepsilon_{t,3} - \theta\varepsilon_{t,2}\right) + \cdots + \left(\varepsilon_{t,s-1} - \theta\varepsilon_{t,s-2}\right) + \left(\varepsilon_{t,s} - \theta\varepsilon_{t,s-1}\right)\right]\right],\right. \\
&\left.\left[\left(\varepsilon_{t,i} - \theta\varepsilon_{t,i-1}\right)\psi\left(i \neq s\right)\right]\right\} \\
&- 2\operatorname{cov}\left\{\left[(1/s)\left[\left(\varepsilon_{t,1} - \theta\varepsilon_{t-1,s}\right) + \left(\varepsilon_{t,2} - \theta\varepsilon_{t,1}\right) + \left(\varepsilon_{t,3} - \theta\varepsilon_{t,2}\right) + \cdots + \left(\varepsilon_{t,s-1} - \theta\varepsilon_{t,s-2}\right) + \left(\varepsilon_{t,s} - \theta\varepsilon_{t,s-1}\right)\right]\right],\right. \\
&\left.\left[\left(\varepsilon_{t,i-1} - \theta\varepsilon_{t,i-2}\right)\psi\left(i \neq 1\right)\right]\right\} \\
&+ 2\operatorname{cov}\left\{\left[\left(\varepsilon_{t,i} - \theta\varepsilon_{t,i-1}\right)\psi\left(i \neq s\right)\right],\left[\left(\varepsilon_{t,i-1} - \theta\varepsilon_{t,i-2}\right)\psi\left(i \neq 1\right)\right]\right\}
\end{aligned}
$$

Applying the definitions of variances and covariances, we obtain

$$
\begin{aligned}
\sigma_{1,x}^2 &= (1/s)^2\left\{\operatorname{var}\left(\varepsilon_{t,1} - \theta\varepsilon_{t-1,s}\right) + \cdots + \operatorname{var}\left(\varepsilon_{t,s} - \theta\varepsilon_{t,s-1}\right) + 2E\left[\left(\varepsilon_{t,1} - \theta\varepsilon_{t-1,s}\right)\left(\varepsilon_{t,2} - \theta\varepsilon_{t,1}\right)\right] + \cdots\right. \\
&\left. + 2E\left[\left(\varepsilon_{t,s-1} - \theta\varepsilon_{t,s-2}\right)\left(\varepsilon_{t,s} - \theta\varepsilon_{t,s-1}\right)\right]\right\} \\
&+ \operatorname{var}\left[\left(\varepsilon_{t,i} - \theta\varepsilon_{t,i-1}\right)\psi\left(i \neq s\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\operatorname{var}\left[\left(\varepsilon_{t,i-1}-\theta\varepsilon_{t,i-2}\right)\psi\left(i\neq 1\right)\right]\\
&+2\operatorname{cov}\left\{\left[\left(1/s\right)\left[\left(\varepsilon_{t,1}-\theta\varepsilon_{t-1,s}\right)+\left(\varepsilon_{t,2}-\theta\varepsilon_{t,1}\right)+\left(\varepsilon_{t,3}-\theta\varepsilon_{t,2}\right)+\cdots+\left(\varepsilon_{t,s-1}-\theta\varepsilon_{t,s-2}\right)+\left(\varepsilon_{t,s}-\theta\varepsilon_{t,s-1}\right)\right]\right],\right.\\
&\left.\left[\left(\varepsilon_{t,i}-\theta\varepsilon_{t,i-1}\right)\psi\left(i\neq s\right)\right]\right\}\\
&-2\operatorname{cov}\left\{\left[\left(1/s\right)\left[\left(\varepsilon_{t,1}-\theta\varepsilon_{t-1,s}\right)+\left(\varepsilon_{t,2}-\theta\varepsilon_{t,1}\right)+\left(\varepsilon_{t,3}-\theta\varepsilon_{t,2}\right)+\cdots+\left(\varepsilon_{t,s-1}-\theta\varepsilon_{t,s-2}\right)+\left(\varepsilon_{t,s}-\theta\varepsilon_{t,s-1}\right)\right]\right],\right.\\
&\left.\left[\left(\varepsilon_{t,i-1}-\theta\varepsilon_{t,i-2}\right)\psi\left(i\neq 1\right)\right]\right\}\\
&+2\operatorname{cov}\left\{\left[\left(\varepsilon_{t,i}-\theta\varepsilon_{t,i-1}\right)\psi\left(i\neq s\right)\right],\left[\left(\varepsilon_{t,i-1}-\theta\varepsilon_{t,i-2}\right)\psi\left(i\neq 1\right)\right]\right\}
\end{aligned}
$$

which can be simplified

$$
\begin{aligned}
\sigma_{1,x}^{2}\ &=\left(1/s\right)^{2}\left[s\left(\sigma_{\varepsilon}^{2}+\theta^{2}\sigma_{\varepsilon}^{2}\right)+\left(s-1\right)\left(-2\theta\sigma_{\varepsilon}^{2}\right)\right]+\left[\left(\sigma_{\varepsilon}^{2}+\theta^{2}\sigma_{\varepsilon}^{2}\right)\psi\left(i\neq s\right)\right]\\
&+\left[\left(\sigma_{\varepsilon}^{2}+\theta^{2}\sigma_{\varepsilon}^{2}\right)\psi\left(i\neq 1\right)\right]+\left[\left(2\theta\sigma_{\varepsilon}^{2}\right)\psi\left(i\neq 1,i\neq s\right)\right]\\
&+\left[\left(\frac{2\sigma_{\varepsilon}^{2}}{s}-\frac{2\theta\sigma_{\varepsilon}^{2}}{s}-\frac{2\theta\sigma_{\varepsilon}^{2}}{s}+\frac{2\theta^{2}\sigma_{\varepsilon}^{2}}{s}\right)\psi\left(i\neq 1,i\neq s\right)\right]\\
&+\left[\left(\frac{2\sigma_{\varepsilon}^{2}}{s}-\frac{2\theta\sigma_{\varepsilon}^{2}}{s}+\frac{2\theta^{2}\sigma_{\varepsilon}^{2}}{s}\right)\psi\left(i=1\right)\right]\\
&+\left[\left(-\frac{2\sigma_{\varepsilon}^{2}}{s}+\frac{2\theta\sigma_{\varepsilon}^{2}}{s}-\frac{2\theta^{2}\sigma_{\varepsilon}^{2}}{s}\right)\psi\left(i=2\right)\right]\\
&+\left[\left(-\frac{2\sigma_{\varepsilon}^{2}}{s}+\frac{2\theta\sigma_{\varepsilon}^{2}}{s}+\frac{2\theta\sigma_{\varepsilon}^{2}}{s}-\frac{2\theta^{2}\sigma_{\varepsilon}^{2}}{s}\right)\psi\left(s\geq i>2\right)\right]
\end{aligned}
$$

Collecting terms, we obtain:

$$
\begin{aligned}
\sigma_{1,x}^{2}\left(i\right)\ &=\left(\frac{\sigma_{\varepsilon}^{2}}{s^{2}}\right)\left[s\left(1+\theta^{2}\right)-2\left(s-1\right)\theta\right]+\left[\sigma_{\varepsilon}^{2}\left(1+\theta^{2}\right)\psi\left(i\neq s\right)\right]\\
&+\left[\sigma_{\varepsilon}^{2}\left(1+\theta^{2}\right)\psi\left(i\neq 1\right)\right]+\left[\sigma_{\varepsilon}^{2}\left(2\theta\right)\psi\left(i\neq 1,i\neq s\right)\right]\\
&+\left(\frac{\sigma_{\varepsilon}^{2}}{s}\right)\left[\left(2-4\theta+2\theta^{2}\right)\psi\left(i\neq 1,i\neq s\right)\right]+\left(\frac{\sigma_{\varepsilon}^{2}}{s}\right)\left[\left(2-2\theta+2\theta^{2}\right)\psi\left(i=1\right)\right]\\
&+\left(\frac{\sigma_{\varepsilon}^{2}}{s}\right)\left[\left(-2+2\theta-2\theta^{2}\right)\psi\left(i=2\right)\right]+\left(\frac{\sigma_{\varepsilon}^{2}}{s}\right)\left[\left(-2+4\theta-2\theta^{2}\right)\psi\left(s\geq i>2\right)\right]
\end{aligned}
$$

Note that the variance depends on *i* which is the index within the interpolation segment. We use the notation $\sigma_{1,x}^2(i)$ to emphasize the dependence of the short variance of the interpolated series on *i*. This interpolation-caused dependency is referred to as *periodic nonstationarity*.[1] To remove this conditionality, we integrate *i* out of this equation using the fact that *i* follows a uniform distribution with support [1, 2, …*s*].

Using the summation rules,

$$\frac{1}{s}\sum_{i=1}^{s}\psi(i \neq 1) = \frac{1}{s}\sum_{i=1}^{s}\psi(i \neq s) = \frac{s-1}{s}$$

$$\frac{1}{s}\sum_{i=1}^{s}\psi(i = 1) = \frac{1}{s}\sum_{i=1}^{s}\psi(i = 2) = \frac{1}{s}$$

$$\frac{1}{s}\sum_{i=1}^{s}\psi(i \neq 1, i \neq s) = \frac{1}{s}\sum_{i=1}^{s}\psi(s \geq i > 2) = \frac{s-2}{s}$$

we obtain

$$\begin{aligned}\sigma_{1,x}^2 = &\left(\frac{\sigma_\varepsilon^2}{s^2}\right)\left[s\left(1+\theta^2\right)-2(s-1)\theta\right] \\ &+\sigma_\varepsilon^2\left[\left(1+\theta^2\right)\left(\frac{s-1}{s}\right)+\left(1+\theta^2\right)\left(\frac{s-1}{s}\right)+2\theta\left(\frac{s-2}{s}\right)\right] \\ &+\left(\frac{\sigma_\varepsilon^2}{s}\right)\left[\left(2-4\theta+2\theta^2\right)\left(\frac{s-2}{s}\right)+\left(2-2\theta+2\theta^2\right)\left(\frac{1}{s}\right)\right. \\ &\left.+\left(-2+2\theta-2\theta^2\right)\left(\frac{1}{s}\right)+\left(-2+4\theta-2\theta^2\right)\left(\frac{s-2}{s}\right)\right]\end{aligned}$$

which is the unconditional short variance of the interpolated series. Simplifying the equation, after collecting terms, we obtain

[1] Dezhbakhsh and Levy (1994) derive the variance, the covariance, and the autocorrelation functions of linearly interpolated trend-stationary series, and find that they all vary with *i*, which they term “periodic variation.”

$$\sigma_{1,x}^2 = \left(\frac{\sigma_\varepsilon^2}{s^2}\right)\left[s\left(1+\theta^2\right)\left(2s-1\right)+2\theta\left(s^2-3s+1\right)\right]$$

This is the expression for the short variance that is given in equation (6) in the paper.

Long Variance

The long variance, i.e., the variance of the "long difference" can be obtained by taking the variance of the long difference in the interpolated series, as shown below.

$$\begin{aligned}
\sigma_{s,x}^2 &= \operatorname{var}\left(x_{t,i}-x_{t-1,i}\right)\\
&= \operatorname{var}\left[\left(\frac{s-i}{s}\right)\sum_{j=1}^{s}u_{t-1,j}+\left(\frac{i}{s}\right)\sum_{j=1}^{s}u_{t,j}+\left(\varepsilon_{t,i}-\theta\varepsilon_{t,i-1}-\varepsilon_{t-1,i}+\theta\varepsilon_{t-1,i-1}\right)\psi\left(i\neq s\right)\right]\\
&= \left(\frac{s-i}{s}\right)^2\operatorname{var}\left[\left(\varepsilon_{t-1,1}-\theta\varepsilon_{t-2,s}\right)+\left(\varepsilon_{t-1,2}-\theta\varepsilon_{t-1,1}\right)+\cdots+\left(\varepsilon_{t-1,s}-\theta\varepsilon_{t-1,s-1}\right)\right]\\
&+\left(\frac{i}{s}\right)^2\operatorname{var}\left[\left(\varepsilon_{t,1}-\theta\varepsilon_{t-1,s}\right)+\left(\varepsilon_{t,2}-\theta\varepsilon_{t,1}\right)+\cdots+\left(\varepsilon_{t,s}-\theta\varepsilon_{t,s-1}\right)\right]\\
&+\operatorname{var}\left[\left(\varepsilon_{t,i}-\theta\varepsilon_{t,i-1}-\varepsilon_{t-1,i}+\theta\varepsilon_{t-1,i-1}\right)\psi\left(i\neq s\right)\right]\\
&+2\operatorname{cov}\left\{\left(\frac{s-i}{s}\right)\left[\left(\varepsilon_{t-1,1}-\theta\varepsilon_{t-2,s}\right)+\left(\varepsilon_{t-1,2}-\theta\varepsilon_{t-1,1}\right)+\cdots+\left(\varepsilon_{t-1,s}-\theta\varepsilon_{t-1,s-1}\right)\right],\right.\\
&\left.\left(\frac{i}{s}\right)\left[\left(\varepsilon_{t,1}-\theta\varepsilon_{t-1,s}\right)+\left(\varepsilon_{t,2}-\theta\varepsilon_{t,1}\right)+\cdots+\left(\varepsilon_{t,s}-\theta\varepsilon_{t,s-1}\right)\right]\right\}\\
&+2\operatorname{cov}\left\{\left[\left(s-i\right)/s\right]\left[\left(\varepsilon_{t-1,1}-\theta\varepsilon_{t-2,s}\right)+\left(\varepsilon_{t-1,2}-\theta\varepsilon_{t-1,1}\right)+\cdots+\left(\varepsilon_{t-1,s}-\theta\varepsilon_{t-1,s-1}\right)\right],\right.\\
&\left.\left[\left(\varepsilon_{t,i}-\theta\varepsilon_{t,i-1}-\varepsilon_{t-1,i}+\theta\varepsilon_{t-1,i-1}\right)\psi\left(i\neq s\right)\right]\right\}\\
&+2\operatorname{cov}\left\{\left(i/s\right)\left[\left(\varepsilon_{t,1}-\theta\varepsilon_{t-1,s}\right)+\left(\varepsilon_{t,2}-\theta\varepsilon_{t,1}\right)+\cdots+\left(\varepsilon_{t,s}-\theta\varepsilon_{t,s-1}\right)\right],\right.\\
&\left.\left[\left(\varepsilon_{t,i}-\theta\varepsilon_{t,i-1}-\varepsilon_{t-1,i}+\theta\varepsilon_{t-1,i-1}\right)\psi\left(i\neq s\right)\right]\right\}
\end{aligned}$$

which yields

$$
\begin{aligned}
\sigma_{s,x}^2 =& \left(\frac{s-i}{s}\right)^2 \left[s\left(\sigma_\varepsilon^2 + \theta^2\sigma_\varepsilon^2\right) + (s-1)\left(-2\theta\sigma_\varepsilon^2\right)\right] \\
&+ \left(\frac{i}{s}\right)^2 \left[s\left(\sigma_\varepsilon^2 + \theta^2\sigma_\varepsilon^2\right) + (s-1)\left(-2\theta\sigma_\varepsilon^2\right)\right] \\
&+ \left[\left(\sigma_\varepsilon^2 + \theta^2\sigma_\varepsilon^2 + \sigma_\varepsilon^2 + \theta^2\sigma_\varepsilon^2\right)\psi(i \neq s)\right] \\
&+ \left(\frac{s-i}{s}\right)\left(\frac{i}{s}\right)\left(-2\theta\sigma_\varepsilon^2\right) \\
&+ \left(\frac{s-i}{s}\right)\left[2\left(-\theta\sigma_\varepsilon^2 - \sigma_\varepsilon^2 + \theta\sigma_\varepsilon^2 - \theta^2\sigma_\varepsilon^2\right)\psi(i=1) + 2\left(-\sigma_\varepsilon^2 + 2\theta\sigma_\varepsilon^2 - \theta^2\sigma_\varepsilon^2\right)\psi(i \neq 1, i \neq s)\right] \\
&+ \left(\frac{i}{s}\right)\left[2\left(\sigma_\varepsilon^2 - \theta\sigma_\varepsilon^2 + \theta^2\sigma_\varepsilon^2\right)\psi(i=1) + 2\left(\sigma_\varepsilon^2 - 2\theta\sigma_\varepsilon^2 + \theta^2\sigma_\varepsilon^2\right)\psi(i \neq 1, i \neq s)\right]
\end{aligned}
$$

which after further simplification becomes

$$
\begin{aligned}
\sigma_{s,x}^2(i) =& \sigma_\varepsilon^2 \left[\left(\frac{s-i}{s}\right)^2 + \left(\frac{i}{s}\right)^2\right]\left[s\left(1+\theta^2\right) - 2(s-1)\theta\right] \\
&+ \sigma_\varepsilon^2\left(2+2\theta^2\right)\left[\psi(i \neq s)\right] - 2\sigma_\varepsilon^2\theta i\left(\frac{s-i}{s^2}\right) \\
&+ \sigma_\varepsilon^2\left(\frac{s-i}{s}\right)\left[\left(-2-2\theta^2\right)\psi(i=1) + \left(-2+4\theta-2\theta^2\right)\psi(i \neq 1, i \neq s)\right] \\
&+ \sigma_\varepsilon^2\left(\frac{i}{s}\right)\left[\left(2-2\theta+2\theta^2\right)\psi(i=1) + \left(2-4\theta+2\theta^2\right)\psi(i \neq 1, i \neq s)\right]
\end{aligned}
$$

where we use the notation $\sigma_{s,x}^2(i)$ to emphasize that the long variance of the interpolated series also depends on *i*. Following the same steps as above to remove the conditionality of the long variance on *i*, and using the summation for the indicator functions as we did above in this Appendix, we obtain

$$
\begin{aligned}
\sigma_{s,x}^2 =& \sigma_\varepsilon^2\left(\frac{s^2 - 2s\bar{i} + 2\bar{i}^2}{s^2}\right)\left[s\left(1+\theta^2\right) - 2(s-1)\theta\right] \\
&+ \sigma_\varepsilon^2\left(2+2\theta^2\right)\left[\left(\frac{1}{s}\right)\sum_{i=1}^{s}\psi(i \neq s)\right] - 2\sigma_\varepsilon^2\theta\left(\frac{s\bar{i} - \bar{i}^2}{s^2}\right) \\
&+ \sigma_\varepsilon^2\left(\frac{s-\bar{i}}{s}\right)\left\{\left(-2-2\theta^2\right)\left[\left(\frac{1}{s}\right)\sum_{i=1}^{s}\psi(i=1)\right] + \left(-2+4\theta-2\theta^2\right)\left[\left(\frac{1}{s}\right)\sum_{i=1}^{s}\psi(i \neq 1, i \neq s)\right]\right\} \\
&+ \sigma_\varepsilon^2\left(\frac{\bar{i}}{s}\right)\left\{\left(2-2\theta+2\theta^2\right)\left[\left(\frac{1}{s}\right)\sum_{i=1}^{s}\psi(i=1)\right] + \left(2-4\theta+2\theta^2\right)\left[\left(\frac{1}{s}\right)\sum_{i=1}^{s}\psi(i \neq 1, i \neq s)\right]\right\}
\end{aligned}
$$

where

$$\bar{i} = \left(\frac{1}{s}\right)\sum_{i=1}^{s} i = \frac{s+1}{2} \qquad \text{and} \qquad \overline{i^2} = \left(\frac{1}{s}\right)\sum_{i=1}^{s} i^2 = \frac{(s+1)(2s+1)}{6}$$

After the substitution, we have

$$\sigma_{s,x}^2 = \sigma_\varepsilon^2 \left\{ \frac{s^2 - 2\left[s(s+1)/2\right] + 2\left[(s+1)(2s+1)/6\right]}{s^2} \right\} \left[s\left(1+\theta^2\right) - 2\left(s-1\right)\theta \right]$$
$$+ \sigma_\varepsilon^2 \left(2 + 2\theta^2\right)\left(\frac{s-1}{s}\right) - 2\sigma_\varepsilon^2 \theta \left\{ \frac{s\left[(s+1)/2\right] - \left[(s+1)(2s+1)/6\right]}{s^2} \right\}$$
$$+ \sigma_\varepsilon^2 \left\{ \frac{s - \left[(s+1)/2\right]}{s} \right\} \left\{ \left(-2 - 2\theta^2\right)\left(\frac{1}{s}\right) + \left(-2 + 4\theta - 2\theta^2\right)\left(\frac{s-2}{s}\right) \right\}$$
$$+ \sigma_\varepsilon^2 \left[\frac{(s+1)/2}{s} \right] \left\{ \left(2 - 2\theta + 2\theta^2\right)\left(\frac{1}{s}\right) + \left(2 - 4\theta + 2\theta^2\right)\left(\frac{s-2}{s}\right) \right\}$$

which, after simplification yields

$$\sigma_{s,x}^2 = \left(\frac{\sigma_\varepsilon^2}{s^2}\right)\left(\frac{1+\theta^2}{3}\right)\left(2s^3 + 6s^2 + s - 6\right) - \left(\frac{\sigma_\varepsilon^2}{s^2}\right)\left(\frac{\theta}{3}\right)\left(4s^3 - 3s^2 + 17s - 24\right)$$

This is the expression for the long variance that is given in equation (10) in the paper.

Mathematica commands for plotting Figures 1, 2, and 3

Figure 1

Plot[{($\frac{s(1+(-0.99)^2)+\left(2(1-s)(-0.99)\right)}{s(1+(-0.99)^2)}$),

$\left(\frac{s(1+(-0.75)^2)+\left(2(1-s)(-0.75)\right)}{s(1+(-0.75)^2)}\right)$,

$\left(\frac{s(1+(-0.5)^2)+\left(2(1-s)(-0.5)\right)}{s(1+(-0.5)^2)}\right)$,

$\left(\frac{s(1+(-0.25)^2)+\left(2(1-s)(-0.25)\right)}{s(1+(-0.25)^2)}\right)$,

$\left(\frac{s(1+(0)^2)+\left(2(1-s)(0)\right)}{s(1+(0)^2)}\right)$,

$\left(\frac{s(1+(0.25)^2)+\left(2(1-s)(0.25)\right)}{s(1+(0.25)^2)}\right)$,

$\left(\frac{s(1+(0.5)^2)+\left(2(1-s)(0.5)\right)}{s(1+(0.5)^2)}\right)$,

$\left(\frac{s(1+(0.75)^2)+\left(2(1-s)(0.75)\right)}{s(1+(0.75)^2)}\right)$,

($\frac{s(1+(0.99)^2)+\left(2(1-s)(0.99)\right)}{s(1+(0.99)^2)}$)},

{s, 2,30},

PlotLabels → {"Teta = −0.99", "Teta = −0.75", "Teta = −0.50", "Teta = −0.25", "Teta = 0.00", "Teta = 0.25", "Teta = 0.50", "Teta = 0.75", "Teta = 0.99"}, PlotTheme−> "Monochrome"]

Figure 2

Plot[{$\frac{(1+(-0.99)^2)(2s^3+6s^2+s-6)-(-0.99)(4s^3-3s^2+17s-24)}{3s^2(1+(-0.99)^2)(2s-1)+6s(-0.99)(s^2-3s+1)}$,

$\frac{(1+(-0.75)^2)(2s^3+6s^2+s-6)-(-0.75)(4s^3-3s^2+17s-24)}{3s^2(1+(-0.75)^2)(2s-1)+6s(-0.75)(s^2-3s+1)}$,

$\frac{(1+(-0.5)^2)(2s^3+6s^2+s-6)-(-0.5)(4s^3-3s^2+17s-24)}{3s^2(1+(-0.5)^2)(2s-1)+6s(-0.5)(s^2-3s+1)}$,

$\frac{(1+(-0.25)^2)(2s^3+6s^2+s-6)-(-0.25)(4s^3-3s^2+17s-24)}{3s^2(1+(-0.25)^2)(2s-1)+6s(-0.25)(s^2-3s+1)}$,

$\frac{(1+(0)^2)(2s^3+6s^2+s-6)-(0)(4s^3-3s^2+17s-24)}{3s^2(1+(0)^2)(2s-1)+6s(0)(s^2-3s+1)}$,

$\frac{(1+(0.25)^2)(2s^3+6s^2+s-6)-(0.25)(4s^3-3s^2+17s-24)}{3s^2(1+(0.25)^2)(2s-1)+6s(0.25)(s^2-3s+1)}$,

$\frac{(1+(0.5)^2)(2s^3+6s^2+s-6)-(0.5)(4s^3-3s^2+17s-24)}{3s^2(1+(0.5)^2)(2s-1)+6s(0.5)(s^2-3s+1)}$,

$\frac{(1+(0.75)^2)(2s^3+6s^2+s-6)-(0.75)(4s^3-3s^2+17s-24)}{3s^2(1+(0.75)^2)(2s-1)+6s(0.75)(s^2-3s+1)}$,

$\frac{(1+(0.99)^2)(2s^3+6s^2+s-6)-(0.99)(4s^3-3s^2+17s-24)}{3s^2(1+(0.99)^2)(2s-1)+6s(0.99)(s^2-3s+1)}$},

{s, 2,30},

PlotLabels → {"Teta = −0.99", "Teta = −0.75", "Teta = −0.50", "Teta = −0.25", "Teta = 0.00", "Teta = 0.25", "Teta = 0.50", "Teta = 0.75", "Teta = 0.99"}, PlotTheme− > "Monochrome"]

Figure 3

Plot[{($\frac{s(1+(-0.99)^2)+\big(2(1-s)(-0.99)\big)}{s(1+(-0.99)^2)}$

$$-\frac{(1+(-0.99)^2)(2s^3+6s^2+s-6)-(-0.99)(4s^3-3s^2+17s-24)}{3s^2(1+(-0.99)^2)(2s-1)+6s(-0.99)(s^2-3s+1)}),$$

$$\left(\frac{s(1+(-0.75)^2)+\big(2(1-s)(-0.75)\big)}{s(1+(-0.75)^2)}\right.$$

$$\left.-\frac{(1+(-0.75)^2)(2s^3+6s^2+s-6)-(-0.75)(4s^3-3s^2+17s-24)}{3s^2(1+(-0.75)^2)(2s-1)+6s(-0.75)(s^2-3s+1)}\right),$$

$$\left(\frac{s(1+(-0.5)^2)+\big(2(1-s)(-0.5)\big)}{s(1+(-0.5)^2)}\right.$$

$$\left.-\frac{(1+(-0.5)^2)(2s^3+6s^2+s-6)-(-0.5)(4s^3-3s^2+17s-24)}{3s^2(1+(-0.5)^2)(2s-1)+6s(-0.5)(s^2-3s+1)}\right),$$

$$\left(\frac{s(1+(-0.25)^2)+\big(2(1-s)(-0.25)\big)}{s(1+(-0.25)^2)}\right.$$

$$\left.-\frac{(1+(-0.25)^2)(2s^3+6s^2+s-6)-(-0.25)(4s^3-3s^2+17s-24)}{3s^2(1+(-0.25)^2)(2s-1)+6s(-0.25)(s^2-3s+1)}\right),$$

$$\left(\frac{s(1+(0)^2)+\big(2(1-s)(0)\big)}{s(1+(0)^2)}\right.$$

$$\left.-\frac{(1+(0)^2)(2s^3+6s^2+s-6)-(0)(4s^3-3s^2+17s-24)}{3s^2(1+(0)^2)(2s-1)+6s(0)(s^2-3s+1)}\right),$$

$$\left(\frac{s(1+(0.25)^2)+\big(2(1-s)(0.25)\big)}{s(1+(0.25)^2)}\right.$$

$$\left.-\frac{(1+(0.25)^2)(2s^3+6s^2+s-6)-(0.25)(4s^3-3s^2+17s-24)}{3s^2(1+(0.25)^2)(2s-1)+6s(0.25)(s^2-3s+1)}\right),$$

$$\left(\frac{s(1+(0.5)^2)+\big(2(1-s)(0.5)\big)}{s(1+(0.5)^2)}\right.$$

$$\left.-\frac{(1+(0.5)^2)(2s^3+6s^2+s-6)-(0.5)(4s^3-3s^2+17s-24)}{3s^2(1+(0.5)^2)(2s-1)+6s(0.5)(s^2-3s+1)}\right),$$

$$\left(\frac{s(1+(0.75)^2)+\big(2(1-s)(0.75)\big)}{s(1+(0.75)^2)}\right.$$

$$\left.-\frac{(1+(0.75)^2)(2s^3+6s^2+s-6)-(0.75)(4s^3-3s^2+17s-24)}{3s^2(1+(0.75)^2)(2s-1)+6s(0.75)(s^2-3s+1)}\right),$$

$$
(\frac{s(1+(0.99)^2)+\left(2(1-s)(0.99)\right)}{s(1+(0.99)^2)} - \frac{(1+(0.99)^2)(2s^3+6s^2+s-6)-(0.99)(4s^3-3s^2+17s-24)}{3s^2(1+(0.99)^2)(2s-1)+6s(0.99)(s^2-3s+1)})\},
$$

$\{s, 2{,}30\}$,

PlotLabels → {"Teta = −0.99", "Teta = −0.75", "Teta = −0.50", "Teta = −0.25", "Teta = 0.00", "Teta = 0.25", "Teta = 0.50", "Teta = 0.75", "Teta = 0.99"}, PlotTheme−> "Monochrome"]