Online Supplement to: 'A Multivariate Stochastic Unit Root Model with an Application to Derivative Pricing'

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1 Summary

This Supplement provides: (i) details of the data, model and estimation of Heston's (1993) stochastic volatility (SV) model used in the empirical application of the paper; (ii) a complete set of proofs for the lemmas, theorems and corollaries given in the paper; (iii) figures relating to Sections 6-7 of the paper.

2 Empirics of the Heston (1993) SV Model

2.1 Data, Model and Estimation

Our data set covers the period from January 2, 2009 through to November 20, 2013 and its sources are described in Table 1. The stochastic volatility (SV) model is estimated using the approximate maximum likelihood technique developed by Aït-Sahalia and Kimmel (2007). This method requires a data on the joint time-series of the underlying stock price, a proxy for the unobserved volatility process, on the risk free interest rate and on the dividend yield. For each trading day we collected data on Google stock closing price, Google call options closing prices and the 3-months U.S. Treasury Bill rate. Black-Scholes (BS) implied volatility is used as a proxy for the unobserved volatility process and was calculated from short-maturity and at the money options, as was suggested by Aït-Sahalia and Kimmel (2007, p. 419).

The risk free interest rate is given by the 3-month U.S. Treasury Bill rate. The time to maturity is measured as the number of calendar days from the trading date to the Friday preceding the Saturday on which the option was due to expire.

Table 1. Data.

Variable	Description
r_t	3-month U.S. treasury Bill rate, secondary market
S_t	Google (GOOG) closing price at time t
s_t	Log of the Google (GOOG) closing price at time t
Y_t	Squared BS implied implied volatility
n_i	Time to maturity

The end of day options data and closing stock prices were obtained from Historical Option Data database (www.historicaloptiondata.com). For the estimation at each trading day we took the implied volatility from the short-maturity option with the smallest moneyness, defined as S_t/K_t, with K_t being the strike price on date t. The average maturity and moneyness are 40 days 1.019, respectively. The 3-month U.S. treasury Bill rates were obtained from the Federal Reserve Bank of St. Louis database (https://research.stlouisfed.org).

Under the Q-measure, the logarithmic stock prices $\{s_t\}_{t\geq 0}$ and their volatilities $\{Y_t\}_{t\geq 0}$ follow the dynamics

$$d\begin{bmatrix} s_t \\ Y_t \end{bmatrix} = \begin{bmatrix} r - d - \frac{1}{2}Y_t \\ \kappa'(\gamma' - Y_t) \end{bmatrix} dt + \begin{bmatrix} \sqrt{(1-\rho^2)Y_t} & \rho\sqrt{Y_t} \\ 0 & \sigma\sqrt{Y_t} \end{bmatrix} d\begin{bmatrix} W_1^Q(t) \\ W_2^Q(t) \end{bmatrix},$$

where $\{W_1^Q(t)\}_{t\geq 0}$ and $\{W_2^Q(t)\}_{t\geq 0}$ are canonical Brownian motion processes, r denotes the risk free interest rate and d is the dividend yield. Their diffusion under the P-measure is given by

$$d\begin{bmatrix} s_t \\ Y_t \end{bmatrix} = \begin{bmatrix} a + bY_t \\ \kappa(\gamma - Y_t) \end{bmatrix} dt + \begin{bmatrix} \sqrt{(1-\rho^2)Y_t} & \rho\sqrt{Y_t} \\ 0 & \sigma\sqrt{Y_t} \end{bmatrix} d\begin{bmatrix} W_1^P(t) \\ W_2^P(t) \end{bmatrix},$$

In this model, $\Lambda = \left(\lambda_1\sqrt{(1-\rho^2)Y_t}, \lambda_2\sqrt{Y_t}\right)'$ is the market price of risk, $a = r - d$, $b = \lambda_1(1-\rho^2) - \frac{1}{2} + \lambda_2\rho$, $\kappa = \kappa' - \lambda_2\sigma$, and $\gamma = (\frac{\kappa+\lambda_2\sigma}{\kappa})\gamma'$.

The BS implied volatility proxy permits only the identification of the parameters $[\kappa, \gamma, \sigma, \rho, \lambda_1]$. Thus, the λ_2 parameter is arbitrarily fixed at 0, see, Aït-Sahalia and Kimmel (2007, p.426). The dividend yield is also fixed at 0, as Google is a non-dividend paying stock. The estimation results together with comparisons to those of Aït-Sahalia and Kimmel (2007) are reported in Table 2. It should be emphasized that Aït-Sahalia and Kimmel (2007) estimates were calculated using data on a different asset and over a different period.

Table 2. Parameter estimates of Heston's model.

	Our estimates	Aït-Sahalia and Kimmel's (2007) estimates
$\hat{\kappa}$	25.99	5.07
$\hat{\gamma}$	0.09	0.0457
$\hat{\sigma}$	1.14	0.48
$\hat{\rho}$	-0.45	-0.767
$\hat{\lambda}_1$	3.87	3.9

The estimated correlation coefficient, $\hat{\rho}$, between the stock price innovations and stochastic volatility is -0.45, which is smaller than the value of -0.767 reported in Aït-Sahalia and Kimmel (2007), for their analysis of the S&P data. The estimated long-run volatility, $\hat{\gamma}$, is 0.09, which is slightly larger than our restricted estimate $\hat{\sigma}^2_{\varepsilon,A} = 0.0716$. The estimated mean reverting speed, $\hat{\kappa}$, is approximately 26, which is larger than the value of 5.07 documented in Aït-Sahalia and Kimmel (2007). They also found that κ is typically overestimated in short samples (see p. 434 of their paper). In the context of the Heston model, this has also been documented by others, see for instance, by Pastorello, Renault and Touzi (2000). The standard deviation of the variance process is 1.14, compared with the value of 0.48 reported in Aït-Sahalia and Kimmel (2007). The estimated risk premium, $\hat{\lambda}_1$, is 3.87, which is practically identical to the value of 3.9 reported in Aït-Sahalia and Kimmel (2007).

For $\theta = [\rho, \sigma, \kappa, \lambda_1, \gamma]'$, the log-likelihood gradient value at the optimum is $[6.25846 * 10^{-10}, 1.6563 * 10^{-9}, -1.14798 * 10^{-10}, 1.49657 * 10^{-10}]'$, with the norm $2.26305 * 10^{-10}$ satisfying the commonly used tolerance of 10^{-6}.

2.2 Pricing

For the out-of-sample empirical experiment we took a one month period from December 2, 2013 to December 31, 2013, total of 21 trading days. We filtered the data by the following criteria:

1. only the most liquid options with volume above 20 transactions are included.

2. Quotes less than 0.5$ were eliminated.

3. Arbitrage violations were also eliminated. On a given trading day, as the closing stock prices may differ from the actual prices at the time when the last option transaction has occurred, we must eliminate the general arbitrage violations.

4. We excluded the short term options with less than one day to expiration.

With the above criteria we have a daily sample of 2287 options for 21 days, with approximately 109 option prices available on each day. We divided our sample into 9 moneyness-maturity categories, the number of contracts for each category is summarized in Table 3. For a similar division of the data, see Bakshi, Cao, and Chen. (1997, p. 2029).

Table 3. Number of options in each category.

	$m < 0.97$	$0.97 < m < 1.03$	$m > 1.03$
$n < 60$	407	1157	236
$60 < n < 120$	128	181	40
$n > 120$	69	44	25

To compute option prices from the Heston model we need to input the spot volatility values which are not observable from the market data. As in the previous section and as discussed in Aït-Sahalia and Kimmel (2007) we used the squared Black-Scholes implied volatility calculated from a short-maturity at the money option as a proxy for the unobservable volatility process.

The *ith* option price at time t is calculated from eq'n (29) of Aït-Sahalia and Kimmel (2007) (note that a π^{-1}-term seems to be missing from their formula), and is given by:

$$C_i(S_t, Y_t, K_i, \Delta, r, d; \rho, \sigma, \kappa', \gamma', \lambda_2) = S_t \cdot \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{\exp[w_0 + w_1 \cdot m_t + w_2 \cdot Y_t]}{\alpha(\alpha+1) - u^2 + (2\alpha+1)iu}\right] du$$

where α is an arbitrary scaling parameter, set to unity,
$m_t = \ln(\frac{S_t}{K_i})$,
$w_0 = \Delta((r-d)(\alpha+1) - r + (r-d)iu) + \frac{\kappa'\gamma'}{\sigma^2}(\Delta\gamma_1 - 2\ln(\gamma_2))$,
$w_1 = iu + \alpha$, $w_2 = (-u^2 + (2\alpha+1)iu + \alpha(\alpha+1))(1 - \frac{1}{\gamma_2})\frac{1}{\gamma_1}$,
$\gamma_0 = \sqrt{c_0 + c_1 u + c_2 u^2}$,
$\gamma_1 = \kappa - (iu + \alpha + 1)\rho\sigma + \gamma_0$, $\gamma_2 = 1 + (\frac{\gamma_1}{\gamma_0})(\frac{\exp(\Delta\gamma_0) - 1}{2})$,
$c_0 = (\kappa')^2 - \sigma(\alpha+1)(2\kappa'\rho - \sigma) - \sigma^2(\alpha+1)^2(1-\rho^2)$,
$c_1 = -i\sigma(2\sigma(\alpha+1)(1-\rho^2) + 2\kappa'\rho - \sigma)$, $c_2 = \sigma^2(1-\rho^2)$.

The results are summarized in Table 4 below. Overall, the mean and variance adjusted STUR based model is best in 5 out of the 9 categories (marked green in the PDF), SV is best in 3 out of the 9 categories (marked yellow in the PDF) and BS is best in the remaining category (marked blue in the PDF). All models systematically overprice options in all categories, with the greatest bias for the out of-the-money short maturity options. Apparently, the mean-variance adjusted STUR model outperforms the STUR model in all categories. In the short maturity categories ($n < 60$) the SV model outperforms the BS model at all moneyness categories. It also outperforms the STUR and mean-variance adjusted STUR models in the out-of-money and at-the-money categories, but is underperformed relative to the mean and variance adjusted STUR model in the in-the-money category. The SV model suffers the greatest bias in long maturity options ($n > 120$). The mean-variance adjusted STUR model performs particularly well for at the money ($0.97 < m < 1.03$) and in the money ($m > 1.03$) options, it has a smallest average percentage bias for in-the-money options in all maturity categories.

In addition, it outperforms the rest of the models for at the money options in the middle- and long maturity categories, giving in total the best results in 5 out of 9 categories. The BS model has the best results only for long maturity out-of-the-money options.

Table 4. Average Percentage Pricing Errors

		$m < 0.97$	$0.97 < m < 1.03$	$m > 1.03$
$n < 60$	SV	**−118.3**%	**−56.5**%	−5.8%
	BS	−216.7%	−94.1%	−5.9%
	$\hat{C}$	−449.7%	−225.6%	−8.1%
	$\hat{C}^*$	−285.7%	−86.9%	**−2.9**%
$60 < n < 120$	SV	**−49.4**%	−23.1%	−10.9%
	BS	−52%	−17.8%	−7%
	$\hat{C}$	−100%	−21.1%	−7.1%
	$\hat{C}^*$	−77.2%	**−15.6**%	**−4.6**%
$n > 120$	SV	−74.5%	−20.8%	−9.5%
	BS	**−47.1**%	−11.5%	−3.8%
	$\hat{C}$	−64.1%	−12.2%	−3.9%
	$\hat{C}^*$	−54.3%	**−9.7**%	**−2.6**%

Note: $\hat{C}$ and $\hat{C}^*$ denote STUR based pricing and mean and variance STUR based pricing. The average percentage pricing error is defined as $\frac{1}{n_j}\sum_{i\in j}\left(\frac{C_i-\hat{C}_i}{C_i}\right)$, where n_j denotes the total number of options in category j and C_i and $\hat{C}_i$ represent the market price and the estimated model price, respectively.

Additional Reference

Pastorello, S., E. Renault and N. Touzi (2000). Statistical inference for random-variance option pricing. *Journal of Business & Economic Statistics* 18, 358–367.

3 Proofs

Proof of Lemma 1. For $t = \lfloor ns \rfloor$ for any $s > 0$, we have

$$n^{-1/2} \sum_{j=1}^{t} \eta_j = B(t/n) + o_p(1),$$

so that

$$\begin{aligned} n^{-1/2} Y_t^* &= n^{-1/2} \sum_{s=1}^{t-1} e^{\frac{a'}{\sqrt{n}} \sum_{j=s+1}^{t} u_j} \varepsilon_s + O_p\left(n^{-1/2}\right) \\ &= n^{-1/2} e^{\frac{a'}{\sqrt{n}} \sum_{j=1}^{t} u_j} \sum_{s=1}^{t-1} e^{-\frac{a'}{\sqrt{n}} \sum_{j=1}^{s} u_j} \varepsilon_s + O_p\left(n^{-1/2}\right) \\ &= n^{-1/2} e^{\{a' B_u(t/n) + o_p(1)\}} \sum_{s=1}^{t-1} e^{\left\{-\frac{a'}{\sqrt{n}} \sum_{j=0}^{s-1} u_j - \frac{a'}{\sqrt{n}} u_s\right\}} \varepsilon_s + O_p\left(n^{-1/2}\right) \end{aligned}$$

$$\begin{aligned} &= n^{-1/2} e^{a' B_u(t/n)} \\ &\times \sum_{s=1}^{t-1} e^{-\{a' B_u((s-1)/n) + o_p(1)\}} \left(1 - \frac{a' u_s}{\sqrt{n}} + O_p\left(n^{-1}\right)\right) \varepsilon_s + o_p(1) \\ &= e^{a' B_u(t/n)} \sum_{s=1}^{t-1} e^{-a' B_u((s-1)/n)} \left(\frac{\sum_{j=1}^{s} \varepsilon_j}{\sqrt{n}} - \frac{\sum_{j=1}^{s-1} \varepsilon_j}{\sqrt{n}}\right) \\ &- e^{a' B_u(t/n)} \sum_{s=1}^{t-1} e^{-a' B_u((s-1)/n)} \left(\frac{a' u_s \varepsilon_s}{n}\right) + o_p(1). \end{aligned} \tag{1}$$

Setting $t = \lfloor nr \rfloor$ and noting that $\mathbb{E}\left(e^{-a' B_u(p)}\right)^2 < \infty$, the first term on the right hand side (rhs) of (1) has limit

$$e^{a' B_u\left(\frac{t}{n}\right)} \sum_{s=1}^{t-1} e^{-a' B_u\left(\frac{s-1}{n}\right)} dB_\varepsilon\left(\frac{s}{n}\right) \to_p e^{a' B_u(r)} \int_0^r e^{-a' B_u(p)} dB_\varepsilon(p) \equiv G_a^*(r), \tag{2}$$

and the second term is

$$
\begin{aligned}
&-e^{a'B_u(t/n)}\sum_{s=1}^{t-1}e^{-a'B_u\left(\frac{s-1}{n}\right)}\left(\frac{a'u_s\varepsilon_s}{n}\right)=-a'e^{a'B_u(t/n)}\sum_{s=1}^{t-1}e^{-a'B_u\left(\frac{s-1}{n}\right)}\\
&\times\left(\frac{u_s\varepsilon_s-\Sigma_{u\varepsilon}}{n}+\frac{\Sigma_{u\varepsilon}}{n}\right)\\
&=-a'\Sigma_{u\varepsilon}e^{a'B_u(t/n)}\frac{1}{n}\sum_{s=1}^{t-1}e^{-a'B_u((s-1)/n)}+O_p\left(n^{-1/2}\right)\\
&\underset{p}{\to}\ -a'\Sigma_{u\varepsilon}e^{a'B_u(r)}\int_0^r e^{-a'B_u(p)}dp.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
n^{-1/2}Y^*_{\lfloor nr\rfloor}\to_p G^*_a(r)-a'\Sigma_{u\varepsilon}e^{a'B_u(r)}&\int_0^r e^{-a'B_u(p)}dp\\
&=e^{a'B_u(r)}\left(\int_0^r e^{-a'B_u(p)}dB_\varepsilon(p)-a'\Sigma_{u\varepsilon}\int_0^r e^{-a'B_u(p)}dp\right)\\
&=:G_a(r),
\end{aligned}
\tag{3}
$$

as required for the first result. Next,

$$
\begin{aligned}
\frac{1}{n}\left\{\sum_{s=1}^{t-1}\left(\prod_{j=s+1}^{t}\beta_j\right)+1\right\}\mu&=\frac{1}{n}\left(\sum_{s=1}^{t-1}e^{\frac{a'}{\sqrt{n}}\sum_{j=s+1}^{t}u_j}+1\right)\mu\\
&\underset{p}{\to}\ \mu e^{a'B_u(r)}\int_0^r e^{-a'B_u(p)}dp,
\end{aligned}
$$

giving the second result. ■

Proof of Theorem 2. For the case $\mu=0$, the objective function is

$$
Q_n(a)=\sum_{t=2}^{n}\left\{Y_t-\beta_t(a)Y_{t-1}\right\}^2. \tag{4}
$$

In this model, letting $t=\lfloor nr\rfloor$, $n^{-1/2}Y_t\Rightarrow G_a(r)$, which is given by (3).

Minimizing (4) with respect to a yields

$$\begin{aligned}
\dot{Q}_n(\hat{a}_n) &= -2\sum_{t=2}^{n}\{Y_t - \beta_t(\hat{a}_n)Y_{t-1}\}\dot{\beta}_t(\hat{a}_n)Y_{t-1} = 0 \\
&\Longrightarrow \sum_{t=2}^{n}\{Y_t - \beta_t(\hat{a}_n)Y_{t-1}\}u_t\beta_t(\hat{a}_n)Y_{t-1} = 0 \\
&\Longrightarrow \sum_{t=2}^{n}Y_t u_t\beta_t(\hat{a}_n)Y_{t-1} = \sum_{t=2}^{n}u_t\beta_t^2(\hat{a}_n)Y_{t-1}^2. \qquad (5)
\end{aligned}$$

The third line in (5) is equivalent to

$$\sum_{t=2}^{n}u_t\beta_t(\hat{a}_n)\{\beta_t(a)Y_{t-1}+\varepsilon_t\}Y_{t-1} = \sum_{t=2}^{n}u_t\beta_t^2(\hat{a}_n)Y_{t-1}^2$$

or

$$\sum_{t=2}^{n}u_t\left(\beta_t^2(\hat{a}_n) - \beta_t(\hat{a}_n+a)\right)Y_{t-1}^2 = \sum_{t=2}^{n}u_t\varepsilon_t\beta_t(\hat{a}_n)Y_{t-1}.$$

As $\beta_t^2(\hat{a}_n) = \beta_t(2\hat{a}_n)$, the last expression boils down to

$$\sum_{t=2}^{n}u_t(\beta_t(2\hat{a}_n) - \beta_t(\hat{a}_n+a))Y_{t-1}^2 = \sum_{t=2}^{n}u_t\varepsilon_t\beta_t(\hat{a}_n)Y_{t-1}.$$

Expanding the last expression, we obtain

$$\begin{aligned}
&\sum_{t=2}^{n}u_t\left(\frac{u_t'(\hat{a}_n-a)}{\sqrt{n}} + \frac{(2\hat{a}_n'u_t)^2 - \left((\hat{a}_n+a)'u_t\right)^2}{2n} + O_p\left(n^{-3/2}\right)\right)Y_{t-1}^2 \qquad (6) \\
&= \sum_{t=2}^{n}u_t\varepsilon_t\left(1 + \frac{u_t'\hat{a}_n}{\sqrt{n}} + O_p\left(n^{-1}\right)\right)Y_{t-1}.
\end{aligned}$$

The first term on the lhs of (6) is

$$
\begin{aligned}
& \frac{1}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \left(\hat{a}_n - a\right) Y_{t-1}^2 \\
= \; & n^{3/2} \left(\Sigma_u \left(\hat{a}_n - a\right) \int_0^1 G_a^2 \left(r\right) dr + o_p \left(\left(\hat{a}_n - a\right)\right) \right) + n \sum_{t=2}^{n} \left(\frac{u_t u_t' - \Sigma_u}{\sqrt{n}} \right) \left(\hat{a}_n - a\right) \left(\frac{Y_{t-1}}{\sqrt{n}} \right)^2 \\
= \; & n^{3/2} \left(\Sigma_u \left(\hat{a}_n - a\right) \int_0^1 G_a^2 \left(r\right) dr + o_p \left(\left(\hat{a}_n - a\right)\right) \right) + O_p \left(n \left(\hat{a}_n - a\right)\right).
\end{aligned}
$$

The second term on the lhs of (6) is

$$
\begin{aligned}
& \frac{1}{2n} \sum_{t=2}^{n} u_t \left(\left(2\hat{a}_n' u_t\right)^2 - \left(\left(\hat{a}_n + a\right)' u_t\right)^2 \right) Y_{t-1}^2 \\
= \; & \frac{1}{2n} \sum_{t=2}^{n} u_t \left\{ \left(\hat{a}_n - a\right)' u_t \left(3\hat{a}_n + a\right)' u_t \right\} Y_{t-1}^2 \\
= \; & O_p \left(n \left(\hat{a}_n - a\right)\right).
\end{aligned}
$$

The remainder on the lhs of (6) is $O_p\left(\sqrt{n}\right)$.

The first term on the rhs of (6) is

$$
\sum_{t=2}^{n} u_t \varepsilon_t Y_{t-1} = n^{3/2} \Sigma_{u\varepsilon} \left(\int_0^1 G_a \left(r\right) dr + o_p \left(1\right) \right) + n \left(\int_0^1 G_a \left(r\right) dB_{u\varepsilon} \left(r\right) + o_p \left(1\right) \right),
$$

whereas the second term is

$$
\sum_{t=2}^{n} u_t \varepsilon_t \frac{u_t' \hat{a}_n}{\sqrt{n}} Y_{t-1} = n \left(E \left(\varepsilon_t u_t u_t' \hat{a}_n\right) \int_0^1 G_a \left(r\right) dr + o_p \left(1\right) \right).
$$

The remainder on the rhs of (6) is $O_p(\sqrt{n})$. Therefore, equation (6) is

$$\begin{aligned}
& n^{3/2}\left(\Sigma_u(\hat{a}_n - a)\int_0^1 G_a^2(r)\,dr + o_p((\hat{a}_n - a))\right) + O_p(n(\hat{a}_n - a)) + O_p\left(\sqrt{n}\right) \\
= \; & n^{3/2}\Sigma_{u\varepsilon}\left(\int_0^1 G_a(r)\,dr + o_p(1)\right) + n\left(\int_0^1 G_a(r)\,dB_{u\varepsilon}(r) + o_p(1)\right) \\
& + n\left(E(\varepsilon_t u_t u_t' \hat{a}_n)\int_0^1 G_a(r)\,dr + o_p(1)\right) + O_p\left(\sqrt{n}\right).
\end{aligned}$$

Considering the leading terms in the $\Sigma_{u\varepsilon} \neq 0$ case, we seek a solution to the equation

$$n^{3/2}\Sigma_u(\hat{a}_n - a)\int_0^1 G_a^2(r)\,dr = n^{3/2}\Sigma_{u\varepsilon}\int_0^1 G_a(r)\,dr.$$

The limit is therefore seen to be

$$(\hat{a}_n - a) \Rightarrow \frac{\int_0^1 G_a(r)\,dr}{\int_0^1 G_a^2(r)\,dr}\Sigma_u^{-1}\Sigma_{u\varepsilon}, \text{ if } \Sigma_{u\varepsilon} \neq 0,$$

which reduces to

$$\hat{a}_n \Rightarrow \frac{\int_0^1 B_\varepsilon(r)\,dr}{\int_0^1 B_\varepsilon^2(r)\,dr}\Sigma_u^{-1}\Sigma_{u\varepsilon}, \text{ if } \Sigma_{u\varepsilon} \neq 0 \text{ and } a = 0.$$

Considering the leading terms in the $\Sigma_{u\varepsilon} = 0$ case, we require a solution to the equation

$$\begin{aligned}
& n^{3/2}\Sigma_u(\hat{a}_n - a)\int_0^1 G_a^2(r)\,dr \\
= \; & n\left(\int_0^1 G_a(r)\,dB_{u\varepsilon}(r) + E(\varepsilon_t u_t u_t' \hat{a}_n)\int_0^1 G_a(r)\,dr\right).
\end{aligned}$$

We obtain

$$\sqrt{n}\left(\hat{a}_n - a\right) \Rightarrow \frac{1}{\int_0^1 G_a^2(r)\,dr}\Sigma_u^{-1}\left\{\{E(\varepsilon_t u_t u_t')\}\, a\int_0^1 G_a(r)\,dr + \int_0^1 G_a(r)\,dB_{u\varepsilon}(r)\right\},$$
$$\text{if } \Sigma_{u\varepsilon} = 0,$$

which simplifies in the $\Sigma_{u\varepsilon} = 0$ and $a = 0$ case to

$$\sqrt{n}\hat{a}_n \Rightarrow \frac{1}{\int_0^1 B_\varepsilon^2(r)\,dr}\Sigma_u^{-1}\int_0^1 B_\varepsilon(r)\,dB_{u\varepsilon}(r)\,,\ \text{if } \Sigma_{u\varepsilon} = 0 \text{ and } a = 0.$$

■

Proof of Theorem 3. For part (1), we have

$$\begin{aligned}
\hat{\sigma}_{\varepsilon,n} &= \frac{1}{n}\sum_{t=2}^{n}\left(Y_t - e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1}\right)^2 \\
&= \frac{1}{n}\sum_{t=2}^{n}\left(Y_t - e^{a' u_t/\sqrt{n}}e^{(\hat{a}_n - a)' u_t/\sqrt{n}}Y_{t-1}\right)^2 \\
&= \frac{1}{n}\sum_{t=2}^{n}\left(Y_t - e^{a' u_t/\sqrt{n}}\left(1 + \frac{(\hat{a}_n - a)' u_t}{\sqrt{n}}\right.\right. \\
&\quad \left.\left. + \frac{\left((\hat{a}_n - a)' u_t\right)^2}{2n} + o_p\left(\frac{1}{n}\right)\right)Y_{t-1}\right)^2
\end{aligned}$$

$$= \frac{1}{n}\sum_{t=2}^{n}\left(\varepsilon_t - e^{a'u_t/\sqrt{n}}\left(\frac{(\hat{a}_n - a)'\,u_t}{\sqrt{n}}\right.\right.$$

$$\left.\left. + \frac{\left((\hat{a}_n - a)'\,u_t\right)^2}{2n} + o_p\left(\frac{1}{n}\right)\right) Y_{t-1}\right)^2$$

$$= \frac{1}{n}\sum_{t=2}^{n}\varepsilon_t^2$$

$$- \frac{2}{n}\sum_{t=2}^{n}\varepsilon_t e^{a'u_t/\sqrt{n}}\left(\frac{(\hat{a}_n - a)'\,u_t}{\sqrt{n}} + \frac{\left((\hat{a}_n - a)'\,u_t\right)^2}{2n} + o_p\left(\frac{1}{n}\right)\right) Y_{t-1}$$

$$+ \frac{1}{n}\sum_{t=2}^{n} e^{2a'u_t/\sqrt{n}}\left(\frac{(\hat{a}_n - a)'\,u_t}{\sqrt{n}} + \frac{\left((\hat{a}_n - a)'\,u_t\right)^2}{2n} + o_p\left(\frac{1}{n}\right)\right)^2 Y_{t-1}^2. \qquad (7)$$

The first term on the rhs of (7) converges in probability to σ_ε^2, whereas the second term becomes

$$- \frac{2}{n}\sum_{t=2}^{n}\varepsilon_t\left(1 + \frac{a'u_t}{\sqrt{n}} + o_p\left(\frac{1}{\sqrt{n}}\right)\right)\left(\frac{(\hat{a}_n - a)'\,u_t}{\sqrt{n}} + \frac{\left((\hat{a}_n - a)'\,u_t\right)^2}{2n}\right.$$

$$\left. + o_p\left(\frac{1}{n}\right)\right) Y_{t-1}.$$

The dominant terms in the last expression are equal to

$$
\begin{aligned}
&= -\frac{2}{n}\left(\sum_{t=2}^{n}\left(\varepsilon_t\frac{(\hat{a}_n-a)'\,u_t}{\sqrt{n}}Y_{t-1}+\varepsilon_t\frac{\left((\hat{a}_n-a)'\,u_t\right)^2}{2n}Y_{t-1}\right)\right.\\
&\left.+\sum_{t=2}^{n}\left(\varepsilon_t\frac{a'u_t\,(\hat{a}_n-a)'\,u_t}{n}Y_{t-1}+\varepsilon_t\frac{a'u_t\left((\hat{a}_n-a)'\,u_t\right)^2}{2n^{3/2}}\right)Y_{t-1}\right)\\
&= -\frac{2}{\sqrt{n}}\left\{(\hat{a}_n-a)'\int_0^1 G_a(r)\,dB_{u\varepsilon}(r)\right.\\
&+\sqrt{n}\,(\hat{a}_n-a)'\,\Sigma_{u\varepsilon}\int_0^1 G_a(r)\,dr\\
&+\frac{1}{2}(\hat{a}_n-a)'\,E\left(\varepsilon_t u_t u_t'\right)(\hat{a}_n-a)\int_0^1 G_a(r)\,dr\\
&\left.+a'E\left(\varepsilon_t u_t u_t'\right)(\hat{a}_n-a)\int_0^1 G_a(r)\,dr+o_p(1)\right\}\\
&= -2\,(\hat{a}_n-a)'\,\Sigma_{u\varepsilon}\int_0^1 G_a(r)\,dr+o_p(1)\,. \qquad (8)
\end{aligned}
$$

The leading term in the third term on the rhs of (7) is seen to be

$$
\begin{aligned}
\frac{1}{n^2}\sum_{t=2}^{n}\left((\hat{a}_n-a)'\,u_t\right)^2Y_{t-1}^2 &= (\hat{a}_n-a)'\left(\frac{1}{n}\sum_{t=2}^{n}u_tu_t'\left(\frac{Y_{t-1}}{\sqrt{n}}\right)^2\right)(\hat{a}_n-a)\\
&= (\hat{a}_n-a)'\,\Sigma_u\,(\hat{a}_n-a)\int_0^1 G_a^2(r)\,dr+o_p(1)\,. \qquad (9)
\end{aligned}
$$

Using Theorem 2(1), the sum of (8) and (9) becomes

$$\begin{aligned}
&-2\left(\frac{\int_0^1 G_a(r)\,dr}{\int_0^1 G_a^2(r)\,dr}\Sigma'_{u\varepsilon}\Sigma_u^{-1}\right)\Sigma_{u\varepsilon}\int_0^1 G_a(r)\,dr \\
&+\left(\frac{\int_0^1 G_a(r)\,dr}{\int_0^1 G_a^2(r)\,dr}\Sigma'_{u\varepsilon}\Sigma_u^{-1}\right)\Sigma_u\left(\Sigma_u^{-1}\Sigma_{u\varepsilon}\frac{\int_0^1 G_a(r)\,dr}{\int_0^1 G_a^2(r)\,dr}\right)\int_0^1 G_a^2(r)\,dr + o_p(1) \\
&\Rightarrow -\frac{\left(\int_0^1 G_a(r)\,dr\right)^2}{\int_0^1 G_a^2(r)\,dr}\Sigma'_{u\varepsilon}\Sigma_u^{-1}\Sigma_{u\varepsilon}.
\end{aligned}$$

It follows that

$$\hat{\sigma}^2_{\varepsilon,n}-\sigma^2_{\varepsilon}\Rightarrow -\frac{\left(\int_0^1 G_a(r)\,dr\right)^2}{\int_0^1 G_a^2(r)\,dr}\Sigma'_{u\varepsilon}\Sigma_u^{-1}\Sigma_{u\varepsilon}.$$

For part (2), we have

$$\begin{aligned}
vech\left(\hat{\Sigma}_{u,n}\right) &= \frac{1}{n}\sum_{t=1}^{n} vech\left(u_t u_t'\right) = \frac{1}{n}\sum_{t=1}^{n} vech\left(u_t u_t' - \Sigma_u + \Sigma_u\right) \\
&= vech\left(\Sigma_u\right) + \frac{1}{n}\sum_{t=1}^{n} vech\left(u_t u_t' - \Sigma_u\right),
\end{aligned}$$

so that

$$\sqrt{n}\left(vech\left(\hat{\Sigma}_{u,n}\right) - vech\left(\Sigma_u\right)\right) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n} vech\left(u_t u_t' - \Sigma_u\right) \Rightarrow \xi(1),$$

follows directly from the functional law for centred partial sums of $u_t u_t'$. Let $e_t = Y_t - e^{\hat{a}_n' u_t/\sqrt{n}} Y_{t-1}$ be the least squares residual from the regression. For part (3), we have

$$\begin{aligned}\hat{\Sigma}_{u\varepsilon,n} &= \frac{1}{n}\sum_{t=2}^{n} e_t u_t = \frac{1}{n}\sum_{t=2}^{n}\left(Y_t - e^{\hat{a}_n' u_t/\sqrt{n}} Y_{t-1}\right) u_t \\ &= \frac{1}{n}\sum_{t=2}^{n}\left(Y_t - e^{a' u_t/\sqrt{n}} e^{(\hat{a}_n - a)' u_t/\sqrt{n}} Y_{t-1}\right) u_t \\ &= \frac{1}{n}\sum_{t=2}^{n}\left(Y_t - e^{a' u_t/\sqrt{n}}\right. \\ &\quad \times \left.\left(1 + \frac{(\hat{a}_n - a)' u_t}{\sqrt{n}} + \frac{\left((\hat{a}_n - a)' u_t\right)^2}{2n} + o_p\left(\frac{1}{n}\right)\right) Y_{t-1}\right) u_t. \end{aligned} \tag{10}$$

The dominant terms in the last expression become

$$\begin{aligned} &\frac{1}{n}\sum_{t=2}^{n}\left(\varepsilon_t - \left(1 + \frac{a' u_t}{\sqrt{n}}\right)\right. \\ &\left.\left(\frac{(\hat{a}_n - a)' u_t}{\sqrt{n}} + \frac{\left((\hat{a}_n - a)' u_t\right)^2}{2n}\right) Y_{t-1}\right) u_t \\ &= \Sigma_{u\varepsilon} - \frac{1}{n}\sum_{t=2}^{n}\left(\frac{(\hat{a}_n - a)' u_t}{\sqrt{n}} Y_{t-1} u_t + \frac{\left((\hat{a}_n - a)' u_t\right)^2}{2n} Y_{t-1} u_t\right. \\ &\left. + \frac{a' u_t (\hat{a}_n - a)' u_t}{n} Y_{t-1} u_t + \frac{a' u_t \left((\hat{a}_n - a)' u_t\right)^2}{2n^{3/2}} Y_{t-1} u_t\right) + o_p(1). \end{aligned} \tag{11}$$

By $[\Sigma_u]_j$ we denote the jth row of Σ_u, which is also equal to the jth column of Σ_u. The first term in the brackets of (11) behaves as

$$
\begin{aligned}
-\sum_{j=1}^{K}\left(\hat{a}_n-a\right)_j \frac{1}{n}\sum_{t=2}^{n} u_{j,t}u_t\frac{Y_{t-1}}{\sqrt{n}} &= -\sum_{j=1}^{K}\left(\hat{a}_n-a\right)_j \left[\Sigma_u\right]_j \frac{1}{n}\sum_{t=2}^{n}\frac{Y_{t-1}}{\sqrt{n}}+o_p\left(1\right) \\
&= -\Sigma_u\left(\hat{a}_n-a\right)\int_0^1 G_a(r)dr+o_p\left(1\right) \\
&\Rightarrow -\left(\int_0^1 G_a(r)dr\right)\Sigma_u\Sigma_u^{-1}\Sigma_{u\varepsilon}\frac{\int_0^1 G_a\left(r\right)dr}{\int_0^1 G_a^2\left(r\right)dr} \\
&= -\frac{\left(\int_0^1 G_a\left(r\right)dr\right)^2}{\int_0^1 G_a^2\left(r\right)dr}\Sigma_{u\varepsilon}.
\end{aligned}
$$

All other terms in the brackets of (11) are negligible and therefore

$$
\hat{\Sigma}_{u\varepsilon,n}-\Sigma_{u\varepsilon}\Rightarrow -\frac{\left(\int_0^1 G_a\left(r\right)dr\right)^2}{\int_0^1 G_a^2\left(r\right)dr}\Sigma_{u\varepsilon}.
$$

■

Proof of Theorem 4. In the $\mu\neq 0$ case, the least squares estimate of μ equals

$$
\hat{\mu}_n=\bar{Y}_n-\frac{1}{n}\sum_{t=2}^{n}e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1}
$$

and the plugged-in score function is

$$
\begin{aligned}
\dot{Q}_n\left(\hat{a}_n;\hat{\mu}_n\right) = -\frac{2}{\sqrt{n}}\sum_{t=2}^{n}\Big\{\left(Y_t-\bar{Y}_n\right)-\Big(&e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1} \\
&-\frac{1}{n}\sum_{t=2}^{n}e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1}\Big)\Big\}u_t e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1} \\
&= 0.
\end{aligned}
$$

This implies that

$$\sum_{t=2}^{n}\left(Y_t-\bar{Y}_n\right)u_t e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1} \tag{12}$$
$$=\sum_{t=2}^{n}\left(e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1}-\frac{1}{n}\sum_{t=2}^{n}e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1}\right)u_t e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1}.$$

It follows from Lieberman and Phillips (2014) that in this case

$$\frac{Y_t}{n}\Rightarrow \mu_0 e^{a' B_u(r)}\int_0^r e^{-a' B_u(p)}dp \equiv H_a\left(r\right).$$

Now,

$$\sum_{t=2}^{n}Y_t u_t e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1}=\sum_{t=2}^{n}\left(\mu_0+e^{a' u_t/\sqrt{n}}Y_{t-1}+\varepsilon_t\right)u_t e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1}.$$

We have

$$\begin{aligned}\mu_0\sum_{t=2}^{n}u_t e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1} &\sim \mu_0\sum_{t=2}^{n}u_t\left(1+\frac{\hat{a}_n' u_t}{\sqrt{n}}\right)Y_{t-1}\\ &\sim \mu_0\left(n^{3/2}\int_0^1 H_a\left(r\right)dB_u\left(r\right)+\frac{1}{\sqrt{n}}\sum_{t=2}^{n}u_t u_t' \hat{a}_n Y_{t-1}\right)\\ &\sim \mu_0\left(n^{3/2}\int_0^1 H_a\left(r\right)dB_u\left(r\right)\right.\\ &\left.+\frac{1}{\sqrt{n}}\Sigma_u \hat{a}_n n^2\left(\frac{\sum_{t=2}^{n}Y_{t-1}}{n^2}\right)\right)\\ &\sim \mu_0 n^{3/2}\left(\int_0^1 H_a\left(r\right)dB_u\left(r\right)+\Sigma_u\hat{a}_n\int_0^1 H_a\left(r\right)dr\right).\end{aligned} \tag{13}$$

Also,

$$\sum_{t=2}^{n} u_t e^{(\hat{a}_n+a)' u_t/\sqrt{n}} Y_{t-1}^2 \sim \sum_{t=2}^{n} u_t Y_{t-1}^2 + \frac{1}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \left(\hat{a}_n + a\right) Y_{t-1}^2$$
$$\sim n^{5/2} \int_0^1 H_a^2(r)\, dB_u(r) + \frac{1}{\sqrt{n}} \Sigma_u \left(\hat{a}_n + a\right) n^3 \sum_{t=2}^{n} \frac{Y_{t-1}^2}{n^3}$$
$$\sim n^{5/2} \left(\int_0^1 H_a^2(r)\, dB_u(r) + \Sigma_u \left(\hat{a}_n + a\right) \int_0^1 H_a^2(r)\, dr \right) \tag{14}$$

and

$$\sum_{t=2}^{n} \varepsilon_t u_t e^{\hat{a}_n' u_t/\sqrt{n}} Y_{t-1} \sim \sum_{t=2}^{n} \varepsilon_t u_t Y_{t-1} + \frac{1}{\sqrt{n}} \sum_{t=2}^{n} \varepsilon_t u_t u_t' \hat{a}_n Y_{t-1}$$
$$\sim \Sigma_{u\varepsilon} n^2 \int_0^1 H_a(r)\, dr + \left\{ E\left(\varepsilon_t u_t u_t'\right) \right\} \hat{a}_n n^{3/2} \int_0^1 H_a(r)\, dr \tag{15}$$
$$+ n^{3/2} \int_0^1 H_a(r)\, dB_{u\varepsilon}(r).$$

It follows from (13)-(15) that the first component of the lhs of (12) behaves as

$$\mu_0 n^{3/2} \left(\int_0^1 H_a(r)\, dB_u(r) + \Sigma_u \hat{a}_n \int_0^1 H_a(r)\, dr \right)$$
$$+ n^{5/2} \left(\int_0^1 H_a^2(r)\, dB_u(r) + \Sigma_u \left(\hat{a}_n + a\right) \int_0^1 H_a^2(r)\, dr \right)$$
$$+ \Sigma_{u\varepsilon} n^2 \int_0^1 H_a(r)\, dr + \left\{ E\left(\varepsilon_t u_t u_t'\right) \right\} \hat{a}_n n^{3/2} \int_0^1 H_a(r)\, dr$$
$$+ n^{3/2} \int_0^1 H_a(r)\, dB_{u\varepsilon}(r)$$
$$\sim n^{5/2} \left(\int_0^1 H_a^2(r)\, dB_u(r) + \Sigma_u \left(\hat{a}_n + a\right) \int_0^1 H_a^2(r)\, dr \right) \tag{16}$$
$$+ \Sigma_{u\varepsilon} n^2 \int_0^1 H_a(r)\, dr.$$

The second component of the lhs of (12) is

$$\bar{Y}_n \sum_{t=2}^{n} u_t e^{\hat{a}_n' u_t/\sqrt{n}} Y_{t-1} \sim n \int_0^1 H_a(r)\, dr \left(\sum_{t=2}^{n} u_t Y_{t-1} + \frac{1}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1} \right)$$

$$\sim n \int_0^1 H_a(r)\, dr \left(n^{3/2} \int_0^1 H_a(r)\, dB_u(r) \right.$$

$$\left. + \Sigma_u \hat{a}_n n^{3/2} \int_0^1 H_a(r)\, dr \right)$$

$$= n^{5/2} \int_0^1 H_a(r)\, dr \left(\int_0^1 H_a(r)\, dB_u(r) \right. \tag{17}$$

$$\left. + \Sigma_u \hat{a}_n \int_0^1 H_a(r)\, dr \right). \tag{18}$$

Combining (16) with (17) the lhs of (12) behaves as

$$n^{5/2} \left(\int_0^1 H_a^2(r)\, dB_u(r) + \Sigma_u (\hat{a}_n + a) \int_0^1 H_a^2(r)\, dr \right.$$

$$\left. - \int_0^1 H_a(r)\, dr \left(\int_0^1 H_a(r)\, dB_u(r) + \Sigma_u \hat{a}_n \int_0^1 H_a(r)\, dr \right) \right) \tag{19}$$

$$+ \Sigma_{u\varepsilon} n^2 \int_0^1 H_a(r)\, dr.$$

The first term on the rhs of (12) is

$$\sum_{t=2}^{n} u_t e^{2\hat{a}_n' u_t/\sqrt{n}} Y_{t-1}^2 \sim \sum_{t=2}^{n} u_t Y_{t-1}^2 + \frac{2}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1}^2$$

$$\sim n^{5/2} \left(\int_0^1 H_a^2(r)\, dB_u(r) + 2\Sigma_u \hat{a}_n \int H_a^2(r)\, dr \right). \tag{20}$$

The second term on the rhs of (12) is

$$\left(\frac{1}{n} \sum_{t=2}^{n} e^{\hat{a}_n' u_t/\sqrt{n}} Y_{t-1} \right) \sum_{t=2}^{n} u_t e^{\hat{a}_n' u_t/\sqrt{n}} Y_{t-1}.$$

We have

$$\frac{1}{n}\sum_{t=2}^{n} e^{\hat{a}_n' u_t/\sqrt{n}} Y_{t-1} \sim \frac{1}{n}\left(\sum_{t=2}^{n} Y_{t-1} + \frac{1}{\sqrt{n}}\sum_{t=2}^{n} \hat{a}_n' u_t Y_{t-1}\right)$$
$$\sim \frac{1}{n}\left(n^2 \int_0^1 H_a(r)\, dr + n\hat{a}_n' \int_0^1 H_a(r)\, dB_u(r)\right)$$
$$= n\int_0^1 H_a(r)\, dr + \hat{a}_n' \int_0^1 H_a(r)\, dB_u(r)\,. \tag{21}$$

Using (13) and (21), the second term on the rhs of (12) is

$$\left(\frac{1}{n}\sum_{t=2}^{n} e^{\hat{a}_n' u_t/\sqrt{n}} Y_{t-1}\right)\sum_{t=2}^{n} u_t e^{\hat{a}_n' u_t/\sqrt{n}} Y_{t-1}$$

$$\sim \left(n\int_0^1 H_a(r)\, dr + \hat{a}_n' \int_0^1 H_a(r)\, dB_u(r)\right)$$
$$\times\, n^{3/2}\left(\int_0^1 H_a(r)\, dB_u(r) + \Sigma_u \hat{a}_n \int_0^1 H_a(r)\, dr\right). \tag{22}$$

Using (20) and (22), the rhs of (12) behaves as

$$n^{5/2}\left(\int_0^1 H_a^2(r)\, dB_u(r) + 2\Sigma_u \hat{a}_n \int H_a^2(r)\, dr\right.$$
$$\left. - \int_0^1 H_a(r)\, dr \left(\int_0^1 H_a(r)\, dB_u(r) + \Sigma_u \hat{a}_n \int_0^1 H_a(r)\, dr\right)\right). \tag{23}$$

Equating (19) to (23), we obtain

$$
\begin{aligned}
& n^{5/2}\left(\int_0^1 H_a^2(r)\,dB_u(r) + \Sigma_u(\hat{a}_n + a)\int_0^1 H_a^2(r)\,dr\right. \\
& \left. - \int_0^1 H_a(r)\,dr\left(\int_0^1 H_a(r)\,dB_u(r) + \Sigma_u\hat{a}_n\int_0^1 H_a(r)\,dr\right)\right) \\
& + \Sigma_{u\varepsilon}n^2\int_0^1 H_a(r)\,dr \\
& = n^{5/2}\left(\int_0^1 H_a^2(r)\,dB_u(r) + 2\Sigma_u\hat{a}_n\int H_a^2(r)\,dr\right. \\
& \left. - \int_0^1 H_a(r)\,dr\left(\int_0^1 H_a(r)\,dB_u(r) + \Sigma_u\hat{a}_n\int_0^1 H_a(r)\,dr\right)\right).
\end{aligned}
$$

This leads to the result

$$
\sqrt{n}\left(\hat{a}_n - a\right) \Rightarrow \frac{\int_0^1 H_a(r)\,dr}{\int_0^1 H_a^2(r)\,dr}\Sigma_u^{-1}\Sigma_{u\varepsilon}.
$$

■

Proof of Theorem 5. We derive an expansion to the score equation in which the leading term is $O_p\left(n^{5/2}\right)$ and the smallest order term is $O_p\left(n^{3/2}\right)$. From Theorem 4, for the $\mu \neq 0$ and $a = 0$ case, $\hat{a}_n = O\left(n^{-1/2}\right)$ if $\Sigma_{u\varepsilon} \neq 0$ and $\hat{a}_n = o\left(n^{-1/2}\right)$ if $\Sigma_{u\varepsilon} = 0$. Tracking the proof of Theorem 4 and leaving all terms in summation form temporarily, we observe the following. The dominant term in (13) is

$$
\mu_0\sum_{t=2}^{n} u_t Y_{t-1} = O_p\left(n^{3/2}\right). \tag{24}
$$

In the expansion leading to (14), a term involving the square of $\hat{a}_n$ is $O_p(n)$, which is negligible, so the relevant term in (14) is

$$
\sum_{t=2}^{n} u_t Y_{t-1}^2 + \frac{1}{\sqrt{n}}\sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1}^2 = O_p\left(n^{5/2}\right).
$$

For (15), the relevant term is

$$\sum_{t=2}^{n} \varepsilon_t u_t Y_{t-1} = O_p\left(n^2\right). \tag{25}$$

It follows from (24)-(25) that for a test of $H_0 : a = 0$, the first component of the lhs of (12) behaves as

$$\mu_0 \sum_{t=2}^{n} u_t Y_{t-1} + \sum_{t=2}^{n} u_t Y_{t-1}^2 + \frac{1}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1}^2 + \sum_{t=2}^{n} \varepsilon_t u_t Y_{t-1} + O_p(n). \tag{26}$$

In the expansion leading to (17) the term involving the square of $\hat{a}_n$ is $O_p(n)$. Therefore, to the required order (17) is

$$\bar{Y}_n \left(\sum_{t=2}^{n} u_t Y_{t-1} \frac{1}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1} \right) = O_p\left(n^{5/2}\right).$$

It follows that the lhs of (12) is given by

$$\begin{aligned} &\mu_0 \sum_{t=2}^{n} u_t Y_{t-1} + \sum_{t=2}^{n} u_t Y_{t-1}^2 + \frac{1}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1}^2 + \sum_{t=2}^{n} \varepsilon_t u_t Y_{t-1} \\ &- \bar{Y}_n \left(\sum_{t=2}^{n} u_t Y_{t-1} + \frac{1}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1} \right). \end{aligned} \tag{27}$$

In the calculation of first term on the rhs of (12) to the required order, (20) is

$$\sum_{t=2}^{n} u_t Y_{t-1}^2 + \frac{2}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1}^2 + O_p(n).$$

The relevant term in (21) is

$$\bar{Y}_n + \frac{1}{n^{3/2}} \sum_{t=2}^{n} \hat{a}_n' u_t Y_{t-1} = O_p(n).$$

Thus, relevant term in (22) which is the required second term on the rhs of

(12) is

$$\bar{Y}_n \left(\sum_{t=2}^{n} u_t Y_{t-1} + \frac{1}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1} \right) = O_p\left(n^{5/2}\right). \tag{28}$$

Thus, the rhs of (12) behaves as

$$\begin{aligned}
&\sum_{t=2}^{n} u_t Y_{t-1}^2 + \frac{2}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1}^2 \\
&- \bar{Y}_n \left(\sum_{t=2}^{n} u_t Y_{t-1} + \frac{1}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1} \right) + O_p(n).
\end{aligned} \tag{29}$$

Equating (27) to (29), we obtain

$$\begin{aligned}
&\mu_0 \sum_{t=2}^{n} u_t Y_{t-1} + \sum_{t=2}^{n} u_t Y_{t-1}^2 + \frac{1}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1}^2 + \sum_{t=2}^{n} \varepsilon_t u_t Y_{t-1} \\
&- \bar{Y}_n \left(\sum_{t=2}^{n} u_t Y_{t-1} + \frac{1}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1} \right) + O_p(n) \\
&= \sum_{t=2}^{n} u_t Y_{t-1}^2 + \frac{2}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1}^2 \\
&- \bar{Y}_n \left(\sum_{t=2}^{n} u_t Y_{t-1} + \frac{1}{\sqrt{n}} \sum_{t=2}^{n} u_t u_t' \hat{a}_n Y_{t-1} \right) + O_p(n).
\end{aligned}$$

This leads to the result

$$n \left(\frac{\sum_{t=2}^{n} Y_{t-1}^2 u_t u_t'}{n^3} \right) \hat{a}_n - \sqrt{n} \frac{\sum_{t=2}^{n} \varepsilon_t u_t Y_{t-1}}{n^2} = \mu_0 \frac{\sum_{t=2}^{n} u_t Y_{t-1}}{n^{3/2}} + O_p\left(\frac{1}{\sqrt{n}} \right). \tag{30}$$

The last result can be rewritten as

$$\begin{aligned}
n \left(\frac{\sum_{t=2}^{n} Y_{t-1}^2 u_t u_t'}{n^3} \right) \hat{a}_n - \sqrt{n} \frac{\Sigma_{u\varepsilon} \sum_{t=2}^{n} Y_{t-1}}{n^2} &= \mu_0 \frac{\sum_{t=2}^{n} (u_t Y_{t-1})}{n^{3/2}} + \frac{\sum_{t=2}^{n} (u_t \varepsilon_t - \Sigma_{u\varepsilon}) Y_{t-1}}{n^{3/2}} \qquad (31) \\
&+ O_p\left(\frac{1}{\sqrt{n}} \right).
\end{aligned}$$

The first two terms on the rhs of (31) are $O_p(1)$. Now, under $H_0 : a = 0$,

$$\left(\frac{\sum_{t=2}^{n} Y_{t-1}^2 u_t u_t'}{n^3}\right) = \frac{\Sigma_u \sum_{t=2}^{n} Y_{t-1}^2}{n^3} + \frac{\sum_{t=2}^{n} (u_t u_t' - \Sigma_u) Y_{t-1}^2}{n^3}$$
$$= \frac{\Sigma_u \sum_{t=2}^{n} \left(\mu_0 (t-1) + \sum_{j=1}^{t-1} \varepsilon_j\right)^2}{n^3} + \frac{\sum_{t=2}^{n} (u_t u_t' - \Sigma_u) Y_{t-1}^2}{n^3}. \tag{32}$$

We therefore have

$$n\frac{\Sigma_u \sum_{t=2}^{n} Y_{t-1}^2}{n^3}\hat{a}_n - \sqrt{n}\frac{\Sigma_{u\varepsilon} \sum_{t=2}^{n} Y_{t-1}}{n^2} = \mu_0 \frac{\sum_{t=2}^{n} (u_t Y_{t-1})}{n^{3/2}} + \frac{\sum_{t=2}^{n} (u_t \varepsilon_t - \Sigma_{u\varepsilon}) Y_{t-1}}{n^{3/2}} - n\frac{\sum_{t=2}^{n} (u_t u_t' - \Sigma_u) Y_{t-1}^2}{n^3}\hat{a}_n + O_p\left(\frac{1}{\sqrt{n}}\right)$$

or

$$n\frac{\Sigma_u \mu_0^2}{3}\hat{a}_n - \sqrt{n}\frac{\Sigma_{u\varepsilon} \sum_{t=2}^{n} Y_{t-1}}{n^2} = \mu_0 \frac{\sum_{t=2}^{n} (u_t Y_{t-1})}{n^{3/2}} + \frac{\sum_{t=2}^{n} (u_t \varepsilon_t - \Sigma_{u\varepsilon}) Y_{t-1}}{n^{3/2}} - n\frac{\sum_{t=2}^{n} Y_{t-1}^2 (u_t u_t' - \Sigma_u)}{n^3}\hat{a}_n - 2n\mu_0 \Sigma_u \hat{a}_n \frac{\sum_{t=2}^{n} \left((t-1) \sum_{j=1}^{t-1} \varepsilon_j\right)}{n^3} + O_p\left(\frac{1}{\sqrt{n}}\right). \tag{33}$$

We note that the term involving $\left(\sum_{j=1}^{t-1} \varepsilon_j\right)^2$ appearing in (32) is absorbed into the $O_p\left(\frac{1}{\sqrt{n}}\right)$ term of the rhs of the last equation. Moreover,

$$\begin{aligned}
&\sqrt{n}\frac{\Sigma_{u\varepsilon}\sum_{t=2}^{n}Y_{t-1}}{n^2} \\
&=\sqrt{n}\frac{\Sigma_{u\varepsilon}\sum_{t=2}^{n}\left(\mu_0\left(t-1\right)+\sum_{j=1}^{t-1}\varepsilon_j\right)}{n^2} \\
&=\sqrt{n}\left(\frac{\Sigma_{u\varepsilon}\mu_0}{2}+\frac{\Sigma_{u\varepsilon}\sum_{t=2}^{n}\sum_{j=1}^{t-1}\varepsilon_j}{n^2}\right).
\end{aligned}$$

Therefore, we can write (33) as

$$\begin{aligned}
n\frac{\Sigma_u\mu_0^2}{3}\hat{a}_n-\sqrt{n}\frac{\Sigma_{u\varepsilon}\mu_0}{2} &= \mu_0\frac{\sum_{t=2}^{n}\left(u_tY_{t-1}\right)}{n^{3/2}} \\
&+\frac{\sum_{t=2}^{n}\left(u_t\varepsilon_t-\Sigma_{u\varepsilon}\right)Y_{t-1}}{n^{3/2}} \\
&-n\frac{\sum_{t=2}^{n}Y_{t-1}^2\left(u_tu_t'-\Sigma_u\right)}{n^3}\hat{a}_n \\
&-2n\mu_0\Sigma_u\hat{a}_n\frac{\sum_{t=2}^{n}\left(\left(t-1\right)\sum_{j=1}^{t-1}\varepsilon_j\right)}{n^3} \\
&+\frac{\Sigma_{u\varepsilon}\sum_{t=2}^{n}\sum_{j=1}^{t-1}\varepsilon_j}{n^{3/2}}+O_p\left(\frac{1}{\sqrt{n}}\right). \qquad (34)
\end{aligned}$$

By Remark 8 of the paper, under $H_0 : a = 0$,

$$\sqrt{n}\hat{a}_n \to_p \frac{3}{2\mu_0}\Sigma_u^{-1}\Sigma_{u\varepsilon}.$$

So, we can rewrite (34) as

$$
\begin{aligned}
n\frac{\Sigma_u \mu_0^2}{3}\hat{a}_n - \sqrt{n}\frac{\Sigma_{u\varepsilon}\mu_0}{2} &= \mu_0 \frac{\sum_{t=2}^{n}\left(u_t Y_{t-1}\right)}{n^{3/2}} \\
&+ \frac{\sum_{t=2}^{n}\left(u_t\varepsilon_t - \Sigma_{u\varepsilon}\right)Y_{t-1}}{n^{3/2}} \\
&- \frac{3}{2\mu_0}\frac{\sum_{t=2}^{n} Y_{t-1}^2\left(u_t u_t' - \Sigma_u\right)}{n^{5/2}}\Sigma_u^{-1}\Sigma_{u\varepsilon} \\
&- 3\Sigma_{u\varepsilon}\frac{\sum_{t=2}^{n}\left((t-1)\sum_{j=1}^{t-1}\varepsilon_j\right)}{n^{5/2}} \\
&+ \frac{\Sigma_{u\varepsilon}\sum_{t=2}^{n}\sum_{j=1}^{t-1}\varepsilon_j}{n^{3/2}} + O_p\left(\frac{1}{\sqrt{n}}\right). \qquad (35)
\end{aligned}
$$

This leads to the result given in the theorem. ■

Proof of Corollary 6. Equation (30) is

$$
nb_0\hat{a}_n - \sqrt{n}\frac{\sum_{t=2}^{n} u_t\varepsilon_t Y_{t-1}}{n^2} = b_2 + O_p\left(\frac{1}{\sqrt{n}}\right),
$$

leading to

$$
nb_0\hat{a}_n - \sqrt{n}b_1 = b_2 + b_3 + O_p\left(\frac{1}{\sqrt{n}}\right), \qquad (36)
$$

where

$$
\begin{aligned}
b_0 &= \frac{\sum_{t=2}^{n} Y_{t-1}^2 u_t u_t'}{n^3} \\
b_1 &= \frac{\hat{\Sigma}_{u\varepsilon,n}^0 \sum_{t=2}^{n} Y_{t-1}}{n^2} \\
b_2 &= \mu_0\frac{\sum_{t=2}^{n} u_t Y_{t-1}}{n^{3/2}} \\
b_3 &= \frac{\sum_{t=2}^{n}\left(u_t\varepsilon_t - \hat{\Sigma}_{u\varepsilon,n}^0\right)Y_{t-1}}{n^{3/2}}. \qquad (37)
\end{aligned}
$$

Now,

$$
\begin{aligned}
b_0 &= \frac{\sum_{t=2}^{n} Y_{t-1}^2 u_t u_t'}{n^3} \\
&= \frac{\hat{\Sigma}_{u,n} \sum_{t=2}^{n} Y_{t-1}^2}{n^3} + \frac{\sum_{t=2}^{n} \left(u_t u_t' - \hat{\Sigma}_{u,n} \right) Y_{t-1}^2}{n^3}. \qquad (38)
\end{aligned}
$$

Note that

$$
\begin{aligned}
n \frac{\sum_{t=2}^{n} Y_{t-1}^2 \left(u_t u_t' - \hat{\Sigma}_{u,n} \right)}{n^3} \hat{a}_n &= \left(\sum_{t=2}^{n} \frac{Y_{t-1}^2}{n^2} \frac{(u_t u_t' - \Sigma_u) + \left(\Sigma_u - \hat{\Sigma}_{u,n} \right)}{n^{1/2}} \right) \left(\sqrt{n} \hat{a}_n \right) \\
&= \left(\sum_{t=2}^{n} \frac{Y_{t-1}^2}{n^2} \frac{(u_t u_t' - \Sigma_u)}{n^{1/2}} \right) \left(\sqrt{n} \hat{a}_n \right) \\
&\quad + \left(\sum_{t=2}^{n} \frac{Y_{t-1}^2}{n^3} \right) \left(\sqrt{n} \left(\Sigma_u - \hat{\Sigma}_{u,n} \right) \right) \left(\sqrt{n} \hat{a}_n \right). \qquad (39)
\end{aligned}
$$

Both terms on the rhs of (39) are $O_p(1)$. In view of (38) and (39), we can rewrite (36) as

$$
n \frac{\left(\sum_{t=2}^{n} Y_{t-1}^2 \right) \hat{\Sigma}_{u,n}}{n^3} \hat{a}_n - \sqrt{n} b_1 = b_2 + b_3 - n \frac{\sum_{t=2}^{n} Y_{t-1}^2 \left(u_t u_t' - \hat{\Sigma}_{u,n} \right)}{n^3} \hat{a}_n + O_p \left(\frac{1}{\sqrt{n}} \right). \qquad (40)
$$

Moreover

$$
\begin{aligned}
\frac{1}{n^3} \sum_{t=2}^{n} Y_{t-1}^2 &= \frac{1}{n^3} \sum_{t=2}^{n} \left(\mu_0 (t-1) + \sum_{j=1}^{t-1} \varepsilon_j \right)^2 \\
&= \frac{1}{n^3} \left[\mu_0^2 \sum_{t=2}^{n} (t-1)^2 + 2\mu_0 \sum_{t=2}^{n} (t-1) \sum_{j=1}^{t-1} \varepsilon_j \right] + O_p\left(n^{-1}\right) \\
&= \frac{\mu_0^2}{3} + \frac{2\mu_0 \sum_{t=2}^{n} (t-1) \sum_{j=1}^{t-1} \varepsilon_j}{n^3} + O_p\left(n^{-1}\right).
\end{aligned}
$$

Hence, we can write (40) as

$$n\left[\frac{\mu_0^2}{3}+\frac{2\mu_0\sum_{t=2}^{n}(t-1)\sum_{j=1}^{t-1}\varepsilon_j}{n^3}\right]\hat{\Sigma}_{u,n}\hat{a}_n-\sqrt{n}b_1$$
$$=b_2+b_3-n\frac{\sum_{t=2}^{n}Y_{t-1}^2\left(u_tu_t'-\hat{\Sigma}_{u,n}\right)}{n^3}\hat{a}_n+O_p\left(\frac{1}{\sqrt{n}}\right),$$

or

$$n\frac{\mu_0^2}{3}\hat{\Sigma}_{u,n}\hat{a}_n-\sqrt{n}b_1$$
$$=b_2+b_3-n\frac{\sum_{t=2}^{n}Y_{t-1}^2\left(u_tu_t'-\hat{\Sigma}_{u,n}\right)}{n^3}\hat{a}_n-n\frac{2\mu_0\sum_{t=2}^{n}(t-1)\sum_{j=1}^{t-1}\varepsilon_j}{n^3}\hat{\Sigma}_{u,n}\hat{a}_n$$
$$+O_p\left(\frac{1}{\sqrt{n}}\right). \tag{41}$$

Furthermore, under the null we have

$$\begin{aligned}\hat{\mu}_n^0&=\frac{1}{n}\sum_{t=2}^{n}(Y_t-Y_{t-1})\\&=\frac{1}{n}\sum_{t=2}^{n}(\mu_0+Y_{t-1}+\varepsilon_t-Y_{t-1})\\&=\mu_0+\bar{\varepsilon}_n.\end{aligned}$$

Hence, we can write (41) as

$$n\frac{\left(\hat{\mu}_n^0-\bar{\varepsilon}_n\right)^2}{3}\hat{\Sigma}_{u,n}\hat{a}_n-\sqrt{n}b_1=b_2+b_3-n\frac{\sum_{t=2}^{n}Y_{t-1}^2\left(u_tu_t'-\hat{\Sigma}_{u,n}\right)}{n^3}\hat{a}_n$$
$$-n\frac{2\mu_0\sum_{t=2}^{n}(t-1)\sum_{j=1}^{t-1}\varepsilon_j}{n^3}\hat{\Sigma}_{u,n}\hat{a}_n+O_p\left(\frac{1}{\sqrt{n}}\right)$$

or

$$\frac{n}{3}\left(\left(\hat{\mu}_n^0\right)^2 - 2\bar{\varepsilon}_n\hat{\mu}_n^0\right)\hat{\Sigma}_{u,n}\hat{a}_n + O_p\left(n^{-1/2}\right) - \sqrt{n}b_1$$
$$= b_2 + b_3 - n\frac{\sum_{t=2}^n Y_{t-1}^2\left(u_t u_t' - \hat{\Sigma}_{u,n}\right)}{n^3}\hat{a}_n$$
$$- n\frac{2\mu_0\sum_{t=2}^n (t-1)\sum_{j=1}^{t-1}\varepsilon_j}{n^3}\hat{\Sigma}_{u,n}\hat{a}_n + O_p\left(\frac{1}{\sqrt{n}}\right).$$

This leads to

$$\frac{n\left(\hat{\mu}_n^0\right)^2}{3}\hat{\Sigma}_{u,n}\hat{a}_n - \sqrt{n}b_1 = b_2 + b_3 - n\frac{\sum_{t=2}^n Y_{t-1}^2\left(u_t u_t' - \hat{\Sigma}_{u,n}\right)}{n^3}\hat{a}_n$$
$$- n\frac{2\mu_0\sum_{t=2}^n (t-1)\sum_{j=1}^{t-1}\varepsilon_j}{n^3}\hat{\Sigma}_{u,n}\hat{a}_n + \frac{2n\bar{\varepsilon}_n\hat{\mu}_n^0}{3}\hat{\Sigma}_{u,n}\hat{a}_n + O_p\left(\frac{1}{\sqrt{n}}\right). \quad (42)$$

Now,

$$b_1 = \frac{\left(\sum_{t=2}^n Y_{t-1}\right)\hat{\Sigma}_{u\varepsilon,n}^0}{n^2} = \frac{\left(\sum_{t=2}^n\left(\mu_0(t-1) + \sum_{j=1}^{t-1}\varepsilon_j\right)\right)\hat{\Sigma}_{u\varepsilon,n}^0}{n^2}$$
$$= \frac{\mu_0}{2}\hat{\Sigma}_{u\varepsilon,n}^0 + \frac{\sum_{t=2}^n\sum_{j=1}^{t-1}\varepsilon_j}{n^2}\hat{\Sigma}_{u\varepsilon,n}^0 + O_p\left(\frac{1}{n}\right).$$

So, eq'n (42) becomes

$$\frac{n\left(\hat{\mu}_n^0\right)^2}{3}\hat{\Sigma}_{u,n}\hat{a}_n - \sqrt{n}\left(\frac{\mu_0}{2}\hat{\Sigma}_{u\varepsilon,n}^0 + \frac{\sum_{t=2}^n\sum_{j=1}^{t-1}\varepsilon_j}{n^2}\hat{\Sigma}_{u\varepsilon,n}^0\right)$$
$$= b_2 + b_3 - n\frac{\sum_{t=2}^n Y_{t-1}^2\left(u_t u_t' - \hat{\Sigma}_{u,n}\right)}{n^3}\hat{a}_n - n\frac{2\mu_0\sum_{t=2}^n (t-1)\sum_{j=1}^{t-1}\varepsilon_j}{n^3}\hat{\Sigma}_{u,n}\hat{a}_n$$
$$+ \frac{2n\bar{\varepsilon}_n\hat{\mu}_n^0}{3}\hat{\Sigma}_{u,n}\hat{a}_n + O_p\left(\frac{1}{\sqrt{n}}\right),$$

or

$$\frac{n\left(\hat{\mu}_n^0\right)^2}{3}\hat{\Sigma}_{u,n}\hat{a}_n-\sqrt{n}\frac{\mu_0}{2}\hat{\Sigma}_{u\varepsilon,n}^0=b_2+b_3$$
$$-n\frac{\sum_{t=2}^n Y_{t-1}^2\left(u_tu_t'-\hat{\Sigma}_{u,n}\right)}{n^3}\hat{a}_n$$
$$-n\frac{2\mu_0\sum_{t=2}^n(t-1)\sum_{j=1}^{t-1}\varepsilon_j}{n^3}\hat{\Sigma}_{u,n}\hat{a}_n$$
$$+\frac{2n\bar{\varepsilon}_n\hat{\mu}_n^0}{3}\hat{\Sigma}_{u,n}\hat{a}_n$$
$$+\sqrt{n}\frac{\sum_{t=2}^n\sum_{j=1}^{t-1}\varepsilon_j}{n^2}\hat{\Sigma}_{u\varepsilon,n}^0+O_p\left(\frac{1}{\sqrt{n}}\right). \quad (43)$$

Using again $\mu_0=\hat{\mu}_n^0-\bar{\varepsilon}_n$, eq'n (43) becomes

$$\frac{n\left(\hat{\mu}_n^0\right)^2}{3}\hat{\Sigma}_{u,n}\hat{a}_n-\sqrt{n}\frac{\hat{\mu}_n^0}{2}\hat{\Sigma}_{u\varepsilon,n}^0=b_2+b_3-n\frac{\sum_{t=2}^n Y_{t-1}^2\left(u_tu_t'-\hat{\Sigma}_{u,n}\right)}{n^3}\hat{a}_n$$
$$-n\frac{2\mu_0\sum_{t=2}^n(t-1)\sum_{j=1}^{t-1}\varepsilon_j}{n^3}\hat{\Sigma}_{u,n}\hat{a}_n+\frac{2n\bar{\varepsilon}_n\hat{\mu}_n^0}{3}\hat{\Sigma}_{u,n}\hat{a}_n$$
$$+\sqrt{n}\frac{\sum_{t=2}^n\sum_{j=1}^{t-1}\varepsilon_j}{n^2}\hat{\Sigma}_{u\varepsilon,n}^0-\sqrt{n}\frac{\bar{\varepsilon}_n}{2}\hat{\Sigma}_{u\varepsilon,n}^0+O_p\left(\frac{1}{\sqrt{n}}\right). \quad (44)$$

Standard manipulations give

$$\sum_{t=2}^{n}(t-1)\sum_{j=1}^{t-1}\varepsilon_j=\frac{(n-1)\,n}{2}\sum_{t=1}^{n-1}\varepsilon_t-\sum_{t=1}^{n-1}\frac{(t-1)\,t}{2}\varepsilon_t,$$

and

$$\sum_{t=2}^{n}\sum_{j=1}^{t-1}\varepsilon_j=\sum_{t=1}^{n-1}(n-t)\,\varepsilon_t.$$

So, eq'n (44) becomes

$$\frac{n\left(\hat{\mu}_n^0\right)^2}{3}\hat{\Sigma}_{u,n}\hat{a}_n - \sqrt{n}\frac{\hat{\mu}_n^0}{2}\hat{\Sigma}_{u\varepsilon,n}^0 = b_2 + b_3 - n\frac{\sum_{t=2}^n Y_{t-1}^2\left(u_t u_t' - \hat{\Sigma}_{u,n}\right)}{n^3}\hat{a}_n$$
$$- n\frac{2\mu_0\left(\frac{(n-1)n}{2}\sum_{t=1}^{n-1}\varepsilon_t - \sum_{t=1}^{n-1}\frac{(t-1)t}{2}\varepsilon_t\right)}{n^3}\hat{\Sigma}_{u,n}\hat{a}_n + \frac{2n\bar{\varepsilon}_n\hat{\mu}_n^0}{3}\hat{\Sigma}_{u,n}\hat{a}_n$$
$$+ \sqrt{n}\frac{\sum_{t=1}^{n-1}(n-t)\varepsilon_t}{n^2}\hat{\Sigma}_{u\varepsilon,n}^0 - \sqrt{n}\frac{\bar{\varepsilon}_n}{2}\hat{\Sigma}_{u\varepsilon,n}^0 + O_p\left(\frac{1}{\sqrt{n}}\right).$$

Substituting (37), we get

$$\frac{n\hat{a}_n\hat{\sigma}_u^2\left(\hat{\mu}_n^0\right)^2}{3} - \sqrt{n}\frac{\hat{\sigma}_{u\varepsilon}\hat{\mu}_n^0}{2} = \mu_0\frac{\sum_{t=2}^n\left(u_t Y_{t-1}\right)}{n^{3/2}} + \frac{\sum_{t=2}^n\left(u_t\varepsilon_t - \hat{\Sigma}_{u\varepsilon,n}^0\right)Y_{t-1}}{n^{3/2}}$$
$$- n\frac{\sum_{t=2}^n Y_{t-1}^2\left(u_t u_t' - \hat{\Sigma}_{u,n}\right)}{n^3}\hat{a}_n - n\frac{2\mu_0\left(\frac{(n-1)n}{2}\sum_{t=1}^{n-1}\varepsilon_t - \sum_{t=1}^{n-1}\frac{(t-1)t}{2}\varepsilon_t\right)}{n^3}\hat{\Sigma}_{u,n}\hat{a}_n$$
$$+ \frac{2n\bar{\varepsilon}_n\hat{\mu}_n^0}{3}\hat{\Sigma}_{u,n}\hat{a}_n + \sqrt{n}\frac{\sum_{t=1}^{n-1}(n-t)\varepsilon_t}{n^2}\hat{\Sigma}_{u\varepsilon,n}^0 - \sqrt{n}\frac{\bar{\varepsilon}_n}{2}\hat{\Sigma}_{u\varepsilon,n}^0 + O_p\left(\frac{1}{\sqrt{n}}\right). \tag{45}$$

Finally, as $\mu_0 = \hat{\mu}_n^0 + O_p\left(n^{-1/2}\right)$, we can replace μ_0 by $\hat{\mu}_n^0$ in the first and fourth terms on the rhs of (45) without affecting the $O_p(1)$ order of magnitude of the rhs. The desired result is is given in the Corollary. ■

Proof of Theorem 7. With $\bar{Y}_n = n^{-1}\sum_{t=2}^n Y_t$, the least squares estimator of μ is given by

$$\hat{\mu}_n = \bar{Y}_n - \frac{1}{n}\sum_{t=2}^n e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1}$$
$$= \frac{Y_n - Y_1}{n} - \frac{1}{n}\sum_{t=2}^n\left(\frac{\hat{a}_n' u_t}{\sqrt{n}} + \frac{\left(\hat{a}_n' u_t\right)^2}{2n} + o_p\left(\frac{1}{n}\right)\right)Y_{t-1}.$$

Now,

$$\frac{Y_n}{n} \Rightarrow \mu e^{a' B_u(1)}\int_0^1 e^{-a' B_u(r)}dr = \mu H_a^*(1)$$

and $Y_1/n = O_p(n^{-1})$. Furthermore, using Theorem 4,

$$\frac{1}{n^{3/2}}\hat{a}_n' \sum_{t=2}^{n} u_t Y_{t-1} \Rightarrow \mu a' \int_0^1 H_a^*(r)\, dB_u(r)$$

and $\frac{1}{n^2}\hat{a}_n'\left(\sum_{t=2}^{n} u_t u_t' Y_{t-1}\right)\hat{a}_n \Rightarrow \mu a'\Sigma_u a \int_0^1 H_a^*(r)\, dr$. Hence,

$$\hat{\mu}_n \Rightarrow \mu\left(H_a^*(1) - a'\int_0^1 H_a^*(r)\, dB_u(r) - \frac{1}{2}a'\Sigma_u a\int_0^1 H_a^*(r)\, dr\right) = \mu B(a).$$

■

Proof of Theorem 8. By definition, $\mu_n^* = \frac{\hat{\mu}_n}{B(a)}$. For part (1),

$$\begin{aligned}
\left(\hat{\sigma}_{\varepsilon,n}^{\mu}\right)^2 &= \frac{1}{n}\sum_{t=2}^{n}\left(Y_t - \mu_n^* - e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1}\right)^2 = \frac{1}{n}\sum_{t=2}^{n}\left(Y_t - \mu - e^{\hat{a}_n' u_t/\sqrt{n}}Y_{t-1} + o_p(1)\right)^2 \\
&= \frac{1}{n}\sum_{t=2}^{n}\left(Y_t - \mu - e^{a'u_t/\sqrt{n}}e^{(\hat{a}_n - a)'u_t/\sqrt{n}}Y_{t-1} + o_p(1)\right)^2 = \frac{1}{n}\sum_{t=2}^{n}\left(\varepsilon_t + o_p(1)\right. \\
&\left. - e^{a'u_t/\sqrt{n}}\left(\frac{(\hat{a}_n - a)'u_t}{\sqrt{n}} + \frac{\left((\hat{a}_n - a)'u_t\right)^2}{2n} + o_p\left(\frac{1}{n}\right)\right)Y_{t-1}\right)^2. \qquad (46)
\end{aligned}$$

Now,

$$\begin{aligned}
&-\frac{2}{n}\sum_{t=2}^{n}\varepsilon_t e^{a'u_t/\sqrt{n}}\left(\frac{(\hat{a}_n - a)'u_t}{\sqrt{n}} + \frac{\left((\hat{a}_n - a)'u_t\right)^2}{2n} + o_p\left(\frac{1}{n}\right)\right)Y_{t-1} \\
&= -\frac{2}{n}\sum_{t=2}^{n}\varepsilon_t\left(1 + \frac{a'u_t}{\sqrt{n}} + o_p\left(\frac{1}{\sqrt{n}}\right)\right) \\
&\times\left(\frac{(\hat{a}_n - a)'u_t}{\sqrt{n}} + \frac{\left((\hat{a}_n - a)'u_t\right)^2}{2n} + o_p\left(\frac{1}{n}\right)\right)Y_{t-1}. \qquad (47)
\end{aligned}$$

By Theorem 4, $\hat{a}_n - a = O_p\left(n^{-1/2}\right)$. Therefore,

$$-\frac{2}{n}\sum_{t=2}^{n}\varepsilon_t\frac{(\hat{a}_n - a)' u_t}{\sqrt{n}}Y_{t-1} = -2(\hat{a}_n - a)'\int_0^1 H_a(r)\,dB_{u\varepsilon}(r) + o_p\left(\frac{1}{\sqrt{n}}\right)$$
$$= O_p\left(\frac{1}{\sqrt{n}}\right),$$

$$-\frac{2}{n}\sum_{t=2}^{n}\varepsilon_t\frac{\left((\hat{a}_n - a)' u_t\right)^2}{2n}Y_{t-1} = O_p\left(\frac{1}{n}\right),$$
$$-\frac{2}{n}\sum_{t=2}^{n}\varepsilon_t\frac{a'u_t}{\sqrt{n}}\frac{(\hat{a}_n - a)' u_t}{\sqrt{n}}Y_{t-1} = O_p\left(\frac{1}{\sqrt{n}}\right),$$
$$-\frac{2}{n}\sum_{t=2}^{n}\varepsilon_t\frac{a'u_t}{\sqrt{n}}\frac{\left((\hat{a}_n - a)' u_t\right)^2}{2n}Y_{t-1} = O_p\left(\frac{1}{n^{3/2}}\right).$$

It follows that (47) converges in probability to zero. The last term in (46) is

$$\frac{1}{n}\sum_{t=2}^{n}e^{2a'u_t/\sqrt{n}}\left(\frac{(\hat{a}_n - a)' u_t}{\sqrt{n}} + \frac{\left((\hat{a}_n - a)' u_t\right)^2}{2n} + o_p\left(\frac{1}{n}\right)\right)^2 Y_{t-1}^2. \tag{48}$$

The leading term in the last expression is

$$\frac{1}{n}\sum_{t=2}^{n}\left(\frac{(\hat{a}_n - a)' u_t}{\sqrt{n}}\right)^2 Y_{t-1}^2 = (\hat{a}_n - a)' E(u_t u_t')\left(\sum_{t=2}^{n}\left(\frac{Y_{t-1}}{n}\right)^2\right)(\hat{a}_n - a)$$
$$\Rightarrow \left(\frac{\int_0^1 H_a(r)\,dr}{\int_0^1 H_a^2(r)\,dr}\Sigma_{u\varepsilon}'\Sigma_u^{-1}\right)\Sigma_u\left(\int_0^1 H_a^2(r)\,dr\right)$$
$$\times\left(\frac{\int_0^1 H_a(r)\,dr}{\int_0^1 H_a^2(r)\,dr}\Sigma_u^{-1}\Sigma_{u\varepsilon}\right)$$
$$= \frac{\left(\int_0^1 H_a(r)\,dr\right)^2}{\int_0^1 H_a^2(r)\,dr}\Sigma_{u\varepsilon}'\Sigma_u^{-1}\Sigma_{u\varepsilon}.$$

All other terms in (48) converge in probability to zero. Hence,

$$\left(\hat{\sigma}^{\mu}_{\varepsilon,n}\right)^2 - \sigma^2_{\varepsilon} \Rightarrow \frac{\left(\int_0^1 H_a(r)\,dr\right)^2}{\int_0^1 H_a^2(r)\,dr}\Sigma'_{u\varepsilon}\Sigma_u^{-1}\Sigma_{u\varepsilon},$$

as required.

For part (2),

$$\begin{aligned}
\Sigma^{\mu}_{u\varepsilon,n} &= \frac{1}{n}\sum_{t=2}^{n} u_t\left(Y_t - \mu_n^* - e^{\hat{a}'_n u_t/\sqrt{n}}Y_{t-1}\right) \\
&= \frac{1}{n}\sum_{t=2}^{n} u_t\left(Y_t - \mu + o_p(1) - e^{a'u_t/\sqrt{n}}e^{(\hat{a}_n - a)'u_t/\sqrt{n}}Y_{t-1}\right) \\
&= \frac{1}{n}\sum_{t=2}^{n} u_t\left(\varepsilon_t + o_p(1) - \left(1 + \frac{a'u_t}{\sqrt{n}} + o_p\left(\frac{1}{\sqrt{n}}\right)\right)\right. \\
&\quad \times \left.\left(\frac{(\hat{a}_n - a)'u_t}{\sqrt{n}} + \frac{\left((\hat{a}_n - a)'u_t\right)^2}{2n} + o_p\left(\frac{1}{n}\right)\right)Y_{t-1}\right). \qquad (49)
\end{aligned}$$

Now, $n^{-1}\sum_{t=1}^{n} u_t\varepsilon_t \rightarrow_p \Sigma_{u\varepsilon}$ and

$$\begin{aligned}
\frac{1}{n}\sum_{t=2}^{n} u_t\frac{(\hat{a}_n - a)'u_t}{\sqrt{n}}Y_{t-1} &= \frac{1}{n}\sum_{t=2}^{n} u_t u'_t\frac{Y_{t-1}}{n}\left(\sqrt{n}(\hat{a}_n - a)\right) \\
&\Rightarrow \left(\Sigma_u\int_0^1 H_a(r)\,dr\right)\left(\frac{\int_0^1 H_a(r)\,dr}{\int_0^1 H_a^2(r)\,dr}\Sigma_u^{-1}\Sigma_{u\varepsilon}\right) \\
&= \frac{\left(\int_0^1 H_a(r)\,dr\right)^2}{\int_0^1 H_a^2(r)\,dr}\Sigma_{u\varepsilon}.
\end{aligned}$$

All other terms in (49) converge in probability to zero. Hence,

$$\Sigma^{\mu}_{u\varepsilon,n} - \Sigma_{u\varepsilon} \Rightarrow -\frac{\left(\int_0^1 H_a(r)\,dr\right)^2}{\int_0^1 H_a^2(r)\,dr}\Sigma_{u\varepsilon}$$

and the proof of the theorem is completed. ■

Proof of Equations (26) - (27). We can write eq'n (14) of the paper as

$$
\begin{aligned}
Y_n(r) &= \mu_T e^{a'B_u(r)} \int_0^r e^{-a'B_u(p)} dp + \sqrt{n} G_a(r) \\
&= \mu_T e^{a'B_u(r)} \int_0^r e^{-a'B_u(p)} dp + \sqrt{n} e^{a'B_u(r)} \left(\left[\Sigma^{1/2}\right]_2 \int_0^r e^{-a'B_u(p)} dB^*(p) \right. \\
&\left. -\frac{1}{n} a' (n\Sigma_{u\varepsilon}) \int_0^r e^{-a'B_u(p)} dp \right) = \mu_T e^{a'B_u(r)} \int_0^r e^{-a'B_u(p)} dp \\
&+ e^{a'B_u(r)} \left(\left[\Sigma_T^{1/2}\right]_2 \int_0^r e^{-a'B_u(p)} dB^*(p) - \frac{1}{\sqrt{n}} a' \Sigma_{u\varepsilon,T} \int_0^r e^{-a'B_u(p)} dp \right).
\end{aligned} \tag{50}
$$

Further, $a'B_u(r) = a'\Sigma_u^{1/2} B_u^*(r) = (T/n)^{1/2} a' \Sigma_{u,A}^{1/2} B_u^*(r)$. Hence, (50) becomes

$$
\begin{aligned}
Y_n(r) &= \mu_T e^{a_n' \Sigma_{u,A}^{1/2} B_u^*(r)} \int_0^r e^{-a_n' \Sigma_{u,A}^{1/2} B_u^*(p)} dp \\
&+ e^{a_n' \Sigma_{u,A}^{1/2} B_u^*(r)} \left(\left[\Sigma_T^{1/2}\right]_2 \int_0^r e^{-a_n' \Sigma_{u,A}^{1/2} B_u^*(p)} dB^*(p) \right. \\
&\left. -\frac{1}{\sqrt{n}} a' \Sigma_{u\varepsilon,T} \int_0^r e^{-a_n' \Sigma_{u,A}^{1/2} B_u^*(r)} dp \right),
\end{aligned} \tag{51}
$$

giving the required result. ■

Proof of the Stochastic Differential Equation for the Process $G(r)$ given in Equation (30). By stochastic differentiation of $\log(S(r))$ and using $[S]_r$ to represent the quadratic variation of $S(r)$ we have

$$
\begin{aligned}
dG(r) &= \frac{dS(r)}{S(r)} - \frac{1}{2S^2(r)} d[S]_r \\
&= \frac{1}{S(r)} \left\{ T\mu_A + \sqrt{T} \left(\frac{a_n' \Sigma_{u,A} a_n}{2} G_{a_n,A}(r) - a_n' \Sigma_{u\varepsilon,A} \right) \right\} S(r)\, dr \\
&+ \frac{1}{S(r)} S(r) \sqrt{T} b_A(r)' \left[\Sigma_A^{1/2}\right] dB^*(r) \\
&- \frac{1}{2S^2(r)} \left\{ T S^2(r)\, b_A(r)' \left[\Sigma_A^{1/2}\right] \left[\Sigma_A^{1/2}\right]' b_A(r) \right\} dr
\end{aligned}
$$

$$
= \left\{ T\mu_A + \sqrt{T}\frac{a_n'\Sigma_{u,A}a_n}{2}G_{a_n,A}(r) - \sqrt{T}a_n'\Sigma_{u\varepsilon,A} \right.
$$
$$
\left. -\frac{T}{2}b_A(r)'\Sigma_A b_A(r)\,dr \right\} + \sqrt{T}b_A(r)'\Sigma_A^{1/2}dB^*(r).
$$

Now,

$$
b_A(r)'\Sigma_A b_A(r) = G^2_{a_n,A}(r)\,a_n'\Sigma_{u,A}a_n + 2G_{a_n,A}(r)\,a_n'\Sigma_{u\varepsilon,A} + \sigma^2_{\varepsilon,A} \tag{52}
$$

and thus, because

$$
\frac{a'}{\sqrt{n}}\Sigma_{u\varepsilon,T} = \frac{a'}{\sqrt{n}}T\Sigma_{u\varepsilon,A} = \sqrt{T}a_n'\Sigma_{u\varepsilon,A}
$$

we get

$$
\begin{aligned}
dG(r) &= \left(T\mu_A + \sqrt{T}\left(\frac{a_n'\Sigma_{u,A}a_n}{2}G_{a_n,A}(r) - a_n'\Sigma_{u\varepsilon,A}\right)\right. \\
&\left. -\frac{T}{2}\left(G^2_{a_n,A}(r)\,a_n'\Sigma_{u,A}a_n + 2G_{a_n,A}(r)\,a_n'\Sigma_{u\varepsilon,A} + \sigma^2_{\varepsilon,A}\right)\right) dr \\
&+ \sqrt{T}b_A(r)'\Sigma_A^{1/2}dB^*(r) \\
&= \left(T\left(\mu_A - \frac{\sigma^2_{\varepsilon,A}}{2}\right) - \sqrt{T}a_n'\Sigma_{u\varepsilon,A}\right) dr \\
&+ \left(\sqrt{T}\left(\frac{a_n'\Sigma_{u,A}a_n}{2} - \sqrt{T}a_n'\Sigma_{u\varepsilon,A}\right) G_{a_n,A}(r) \right. \\
&\left. -T\frac{a_n'\Sigma_{u,A}a_n}{2}G^2_{a_n,A}(r)\,dr\right) + \sqrt{T}b_A(r)'\Sigma_A^{1/2}dB^*(r).
\end{aligned}
$$

■

Proof of Equation (33) for $S(r)$. From the above derivation and equation (30) of the paper, we have

$$
\begin{aligned}
&G(r) - G(0) = \log(S(r)) - \log(S(0)) \\
&= \int_0^r \left(T\left(\mu_A - \frac{\sigma^2_{\varepsilon,A}}{2}\right) - \sqrt{T} a_n' \Sigma_{u\varepsilon,A} \right) dr \\
&+ \int_0^r \left(\sqrt{T}\left(\frac{a_n' \Sigma_{u,A} a_n}{2} - \sqrt{T} a_n' \Sigma_{u\varepsilon,A}\right) G_{a_n,A}(s) \right. \\
&\left. - T\frac{a_n' \Sigma_{u,A} a_n}{2} G^2_{a_n,A}(s) \right) ds + \sqrt{T} \int_0^r b_A(s)' \Sigma_A^{1/2} dB(s) \\
&= \left(T\left(\mu_A - \frac{\sigma^2_{\varepsilon,A}}{2}\right) - \sqrt{T} a_n' \Sigma_{u\varepsilon,A} \right) r \\
&+ \int_0^r \left\{ \sqrt{T}\left(\frac{a_n' \Sigma_{u,A} a_n}{2} - \sqrt{T} a_n' \Sigma_{u\varepsilon,A}\right) G_{a_n,A}(s) \right. \\
&\left. - T\frac{a_n' \Sigma_{u,A} a_n}{2} G^2_{a_n,A}(s) \right\} ds \\
&+ \sqrt{T} a_n' \int_0^r G_{a_n,A}(s) \Sigma_{u,A}^{1/2}(s) dB_u^*(s) + \sqrt{T} \Sigma_{2,A}^{1/2} B^*(r),
\end{aligned}
$$

and result given in (30) immediately follows. ■

Proof of the Details in Section 4.2. We must have $df(r,x) = dV(r)$ and by direct calculation

$$
\begin{aligned}
dV(r) &= \alpha_S(r) \left\{ \left(T\mu_A + \sqrt{T}\left(\frac{a_n' \Sigma_{u,A} a_n}{2} G_{a_n,A}(r) - a_n' \Sigma_{u\varepsilon,A} \right) \right) S(r)\, dr \right. \\
&\left. + \sqrt{T} S(r) b_A(r)' \Sigma_A^{1/2} dB^*(r) \right\} + T r_{f,A} \alpha_Z(r) \gamma(r)\, dr \\
&= \left\{ \alpha_S(r) \left(T\mu_A + \sqrt{T}\left(\frac{a_n' \Sigma_{u,A} a_n}{2} G_{a_n,A}(r) - a_n' \Sigma_{u\varepsilon,A} \right) \right) S(r) \right. \\
&\left. + T r_{f,A} \alpha_Z(r) \gamma(r) \right\} dr + \sqrt{T} \alpha_S(r) S(r) b_A(r)' \Sigma_A^{1/2} dB^*(r). \qquad (53)
\end{aligned}
$$

Now, in view of equation (29) of the paper, we have

$$
(d(S(r)))^2 = T S^2(r) b_A(r)' \Sigma_A b_A(r)\, dr, \qquad (54)
$$

and since $df(r,x) = f_r dr + f_x dS(r) + \frac{1}{2} f_{xx} (d(S(r)))^2$, we deduce that

$$\begin{aligned}
df(r, S(r)) &= f_r dr + f_x \{T\mu_A \\
&\quad + \sqrt{T}\left(\frac{a_n' \Sigma_{u,A} a_n}{2} G_{a_n,A}(r) - a_n' \Sigma_{u\varepsilon,A}\right)\Big\} S(r)\, dr \\
&\quad + f_x \sqrt{T} S(r)\, b_A(r)' \Sigma_A^{1/2} dB^*(r) + \frac{T}{2} f_{xx} S^2(r)\, b_A(r)' \Sigma_A b_A(r)\, dr \\
&= \Bigg\{ f_r + f_x \left\{ T\mu_A + \sqrt{T}\left(\frac{a_n' \Sigma_{u,A} a_n}{2} G_{a_n,A}(r) - a_n' \Sigma_{u\varepsilon,A}\right)\right\} S(r) \\
&\quad + \frac{T}{2} f_{xx} S^2(r)\, b_A(r)' \Sigma_A b_A(r) \Bigg\} dr + \sqrt{T} f_x S(r)\, b_A(r)' \Sigma_A^{1/2} dB^*(r).
\end{aligned} \tag{55}$$

Equating the coefficients of dr and of the stochastic component in (53) and (55) gives

$$\begin{aligned}
&\alpha_S(r) \left\{ T\mu_A + \sqrt{T}\left(\frac{a_n' \Sigma_{u,A} a_n}{2} G_{a_n,A}(r) - a_n' \Sigma_{u\varepsilon,A}\right)\right\} S(r) \\
&+ T r_{f,A} \alpha_Z(r) \gamma(r) \\
&= f_r + f_x \left\{ T\mu_A + \sqrt{T}\left(\frac{a_n' \Sigma_{u,A} a_n}{2} G_{a_n,A}(r) - a_n' \Sigma_{u\varepsilon,A}\right)\right\} S(r) \\
&+ \frac{T}{2} f_{xx} S^2(r)\, b_A(r)' \Sigma_A b_A(r)
\end{aligned} \tag{56}$$

and

$$\sqrt{T}\alpha_S(r) S(r)\, b_A(r)' \Sigma_A^{1/2} dB^*(r) = \sqrt{T} f_x S(r)\, b_A(r)' \Sigma_A^{1/2} dB^*(r). \tag{57}$$

The latter yields

$$\alpha_S(r) = f_x \tag{58}$$

which is the condition in the classic case. Using this condition in (56) we have

$$
\begin{aligned}
&\alpha_S(r)\left\{T\mu_A+\sqrt{T}\left(\frac{a_n'\Sigma_{u,A}a_n}{2}G_{a_n,A}(r)-a_n'\Sigma_{u\varepsilon,A}\right)\right\}S(r) \\
&+Tr_{f,A}\alpha_Z(r)\gamma(r) \\
&=f_r+\alpha_S(r)\left\{T\mu_A+\sqrt{T}\left(\frac{a_n'\Sigma_{u,A}a_n}{2}G_{a_n,A}(r)-a_n'\Sigma_{u\varepsilon,A}\right)\right\}S(r) \\
&+\frac{T}{2}f_{xx}S^2(r)\,b_A(r)'\,\Sigma_A b_A(r).
\end{aligned}
$$

which implies

$$
Tr_{f,A}\alpha_Z(r)\gamma(r)=f_r+\frac{T}{2}f_{xx}S^2(r)\,b_A(r)'\,\Sigma_A b_A(r)
$$

and equation (36) of the paper follows.■

4 Figures 1-8

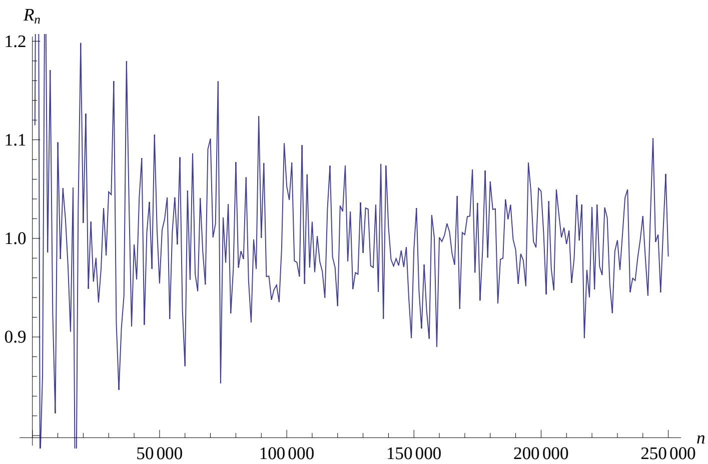

Figure 1: Plot of R_n, given in (57), against n: the $\mu \neq 0$ and $a = 0$ case.

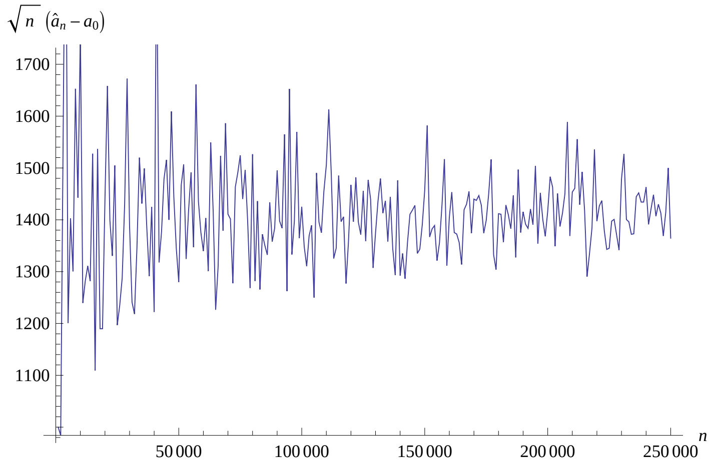

Figure 2: Plot of $\sqrt{n}\,(\hat{a}_n - a_0)$ against n when $a_0 = 0.15$: the $\mu \neq 0$ case.

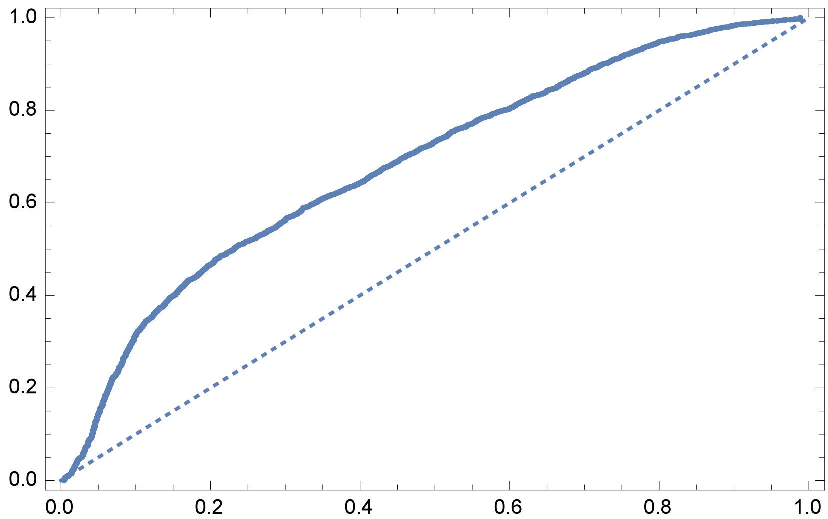

Figure 3: PP-Plot of T_n against the asymptotic distribution, $n = 1000$, $rep = 5000$.

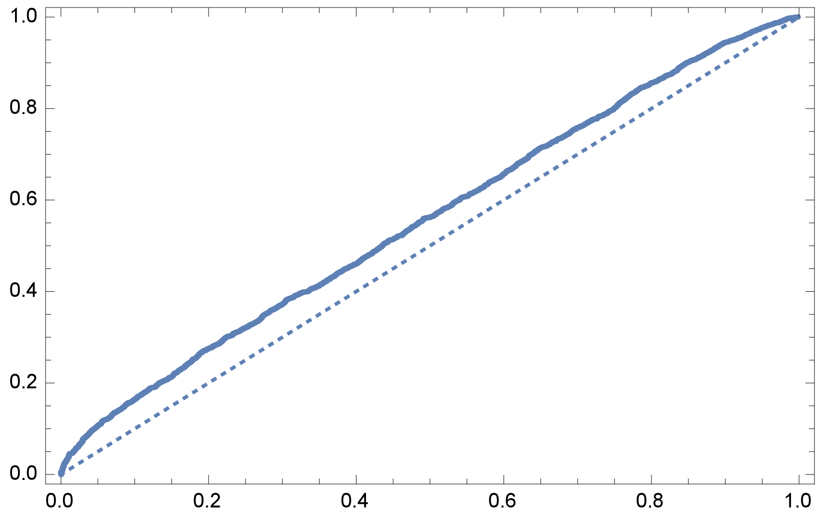

Figure 4: PP-Plot of T_n against the asymptotic distribution, $n = 10000$, $rep = 5000$.

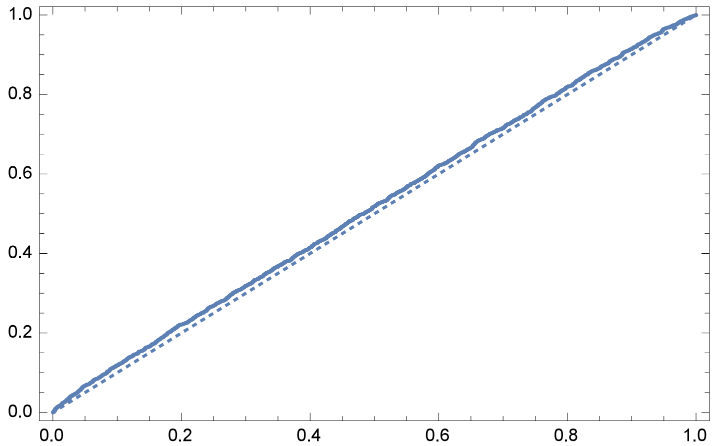

Figure 5: PP-Plot of T_n against the asymptotic distribution, $n = 100000$, $rep = 5000$.

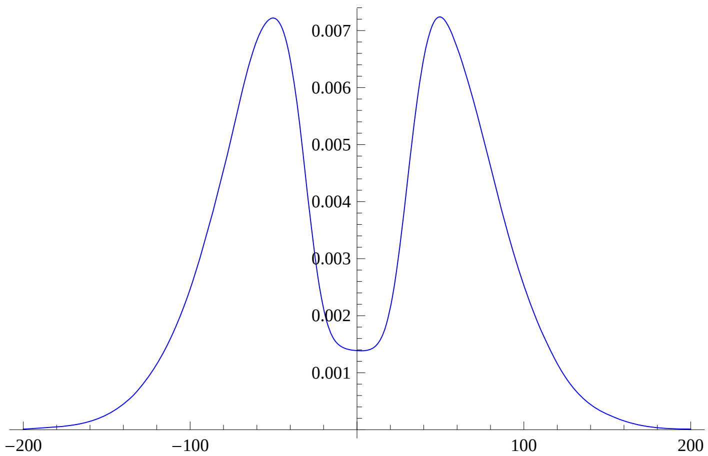

Figure 6: Kernel density estimate of the asymptotic distribution of $\hat{a}_n$ in the $\mu = 0$, $\sigma_{u\varepsilon} \neq 0$ and $a = 0$ case, with $n = 1231$, 100000 replications and historical estimates of the Google-Nasdaq data.

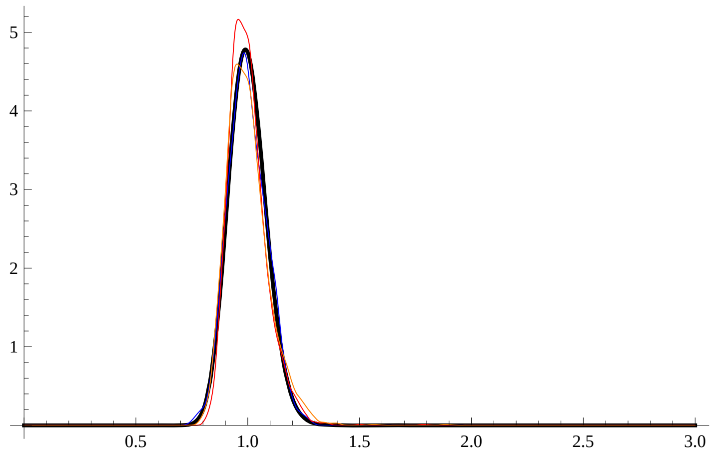

Figure 7: Kernel density estimates of the λ's: Theoretical $\lambda_{BS}^{Q}(1)$ (Black), Simulated $\lambda_{BS}^{Q}(1)$ (BLUE), $\lambda^{*Q}(1)$ (Red), $\bar{\lambda}^{Q}(1)$ (orange). Based on the multivariate model (59) with $n = 36$.

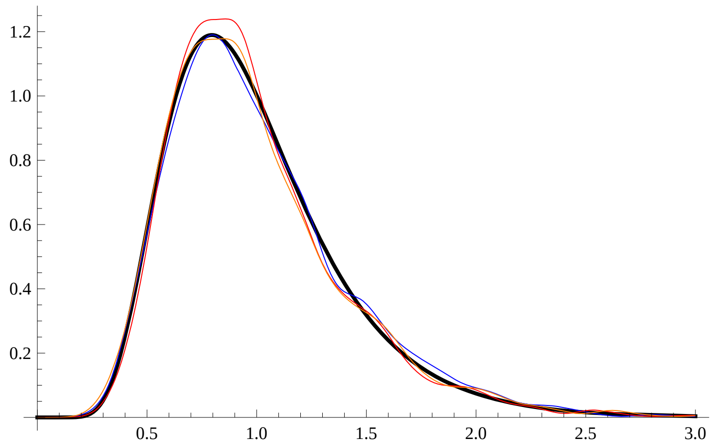

Figure 8: Kernel density estimates of the λ's: Theoretical $\lambda_{BS}^{Q}(1)$ (Black), Simulated $\lambda_{BS}^{Q}(1)$ (BLUE), $\lambda^{*Q}(1)$ (Red), $\bar{\lambda}^{Q}(1)$ (orange). Based on the model (58) with $n = 778$ and $\rho = 0.95$.