Manipulation-resistant false-name-proof facility location mechanisms for complex graphs

Ilan Nehama · Taiki Todo · Makoto Yokoo

Abstract In many real-life scenarios, a group of agents needs to agree on a common action, e.g., on a location for a public facility, while there is some consistency between their preferences, e.g., all preferences are derived from a common metric space. The *facility location* problem models such scenarios and it is a well-studied problem in social choice. We study mechanisms for facility location on unweighted undirected graphs that are resistant to manipulations (*strategy-proof*, *abstention-proof*, and *false-name-proof*) by both individuals and coalitions on one hand and anonymous and efficient (*Pareto-optimal*) on the other.

We define a new family of graphs, *ZV-line graphs*, and show a general facility location mechanism for these graphs that satisfies all these desired properties. This mechanism can also be computed in polynomial time and it can equivalently be defined as the first Pareto-optimal location according to some predefined order. Our main result, the ZV-line graphs family and the mechanism we present for it, unifies all works in the literature of false-name-proof facility location on discrete graphs including the preliminary (unpublished) works we are aware of. In particular, we show mechanisms for *all graphs of at most five vertices*, *discrete trees*, *bicliques,* and *clique tree graphs.*

Finally, we discuss some generalizations and limitations of our result for facility location problems on other structures: Weighted graphs, large discrete cycles, infinite graphs; and for facility location problems concerning infinite societies.

Keywords Facility location · Strategy-proofness · False-name-proofness · ZV-line graphs

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This work initialized while the first author was a member of the Multi-agent laboratory managed by the third author at Kyushu University, Japan.

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1 Introduction

Reaching an agreement could be hard. The seminal works of Gibbard [11] and Satterthwaite [25] show that one cannot devise a general procedure for aggregating the preferences of strategic agents to a single outcome, besides trivial procedures that a-priori ignore all agents except one (that is, the outcome is based on the preference of a predefined agent) or a-priori rule out all outcomes except two (that is, regardless of the preferences of the agents, the outcome is one of two predefined outcomes). The problem is that agents might act strategically aiming to get an outcome that they prefer, so there might be scenarios in which for any profile of actions (a possible agreement) at least one of the agents prefers changing her action. Note that while we refer to a *procedure* and later to a *mechanism*, this impossibility is not technical but conceptual. We identify a procedure with the conceptual mapping induced by the procedure from the opinions of the agents to an agreement, while the procedure itself could be complex and abstract, e.g., to have several rounds or include a deliberation process between the agents (cheap-talk). For simplicity of terms, we refer to the *direct mechanism* that implements this mapping. That is, we think of an exogenous entity, the *designer*, who receives as input the opinions of the agents and returns as output the aggregated decision. This assumption does not hurt the generality, as according to the revelation principle [20] any general procedure is equivalent (w.r.t. the properties we study) to such a direct mechanism.

But in many natural scenarios, it is exogenously given that the preferences satisfy some additional rationality property, i.e., the mechanism need not be defined for any profile of preferences, giving rise to mechanisms that are not prone to the above drawbacks. Two prominent examples are *VCG mechanisms* and *generalized-median mechanisms*. VCG mechanisms [4, 24, 12, 29] are the mechanisms that are resistant to manipulations like the ones described above for scenarios in which the preferences of agents are quasi-linear with respect to money [16, Def. 3.b.7], and monetary transfers are allowed (that is, the outcome space is closed under monetary exchanges between the agents or between the agents and the designer). The second example, *Generalized-median mechanisms*, does not include monetary transfers and has more of an ordinal flavor. Generalized-median mechanisms [17] are the mechanisms that are resistant to manipulations like above when it is known that the preferences are *single-peaked* w.r.t. the real line [3]. That is, the outcomes are *locations* on the real line, each agent has a unique optimal location, $\ell^\star$, and her preference over the locations to the right of $\ell^\star$ is derived by the proximity to $\ell^\star$, and similarly for the locations to the left of $\ell^\star$. For example, in the Euclidean single-peaked case, the preferences for all agents are minimizing the distance to their respective optimal locations.

The facility location problem

A natural generalization of the second scenario is the *facility location* problem. In this problem, we are given a metric space over the outcomes (that is, a distance function between outcomes) and it is assumed that the preference of each of the agents is defined by the distance to her optimal outcome: An agent with an optimal outcome $\ell^\star$ prefers an outcome a over an outcome b if and only if a is closer to

$\ell^\star$ than b. For ease of presentation, throughout this paper, we assume that there are finitely many agents and finitely many locations, and in Section 6.1 discuss the extension to the infinite case. In the finite case, a natural way to represent the common metric space is using a weighted undirected graph. That is, having a vertex (location) for each outcome and weighted edges between vertices s.t. the distance between any two locations is equal to the distance between the two respective vertices (or generally to the length of the shortest path between them). Roughly speaking, given such a graph one seeks to find a mechanism that on one hand does not a-priori ignore some of the agents or rule out some of the locations, and on the other hand, is resistant to manipulations of the agents. Facility location problems and, moreover, facility location problems for complex combinatorial structures model many real-life scenarios of group decision making in which it is natural to assume some homogeneity between the preferences of different agents (e.g., an additional rationality assumption). These examples include not only locating a common facility, like a school, a bus-stop, or a library, but also more general agreement scenarios with a common metric, e.g., a partition of a common budget to several tasks, committee selection, or group decision making with multi-dimensional criteria. Following the common facility problem, we sometimes refer to the outcome of the mechanism as *the facility.* In this work, we look for mechanisms that satisfy the following desired properties:

Anonymity: The mechanism should not a-priori ignore agents and, moreover, we desire it to treat the agents equally in the following strong sense. The mechanism should be a function of the agents' votes (which we also refer to as *ballots*) but not their identities. Formally, the outcome of the mechanism should be invariant to any permutation of the ballots. In practice, most voting systems satisfy this property by first accumulating the different (physical) ballots, thus losing the agents' identities, and next applying the mechanism on the identity-less ballots.

We would also like the mechanism to treat the locations in an a-priori fair manner. Note that it is unreasonable to require that all locations are treated equally (i.e., neutrality of the mechanism) due to the inherent asymmetry induced by the graph. Instead, we require the following much weaker property of non-imposition.

Citizen Sovereignty/Non-imposition *[1, 19]:* The mechanism should not a-priori rule out a location, and each location should be the outcome of some profile. Formally, the mapping to a facility location should be an onto function.

Furthermore, the mechanism should respect the preferences of the agents and aim to optimize the aggregated welfare of the agents.

Pareto-optimality: The mechanism should not return a location ℓ if there exists another location ℓ' s.t. switching from ℓ to ℓ' benefits one of the agents (move the facility closer to her) while not hurting any of the other agents. In particular, if there exists a unique location that is unanimously most-preferred by all agents, then it must be the outcome. Note that any reasonable notion of aggregated welfare optimization entails Pareto-optimality.

Strategy-proofness: An agent should not be able to change the outcome to a location she strictly prefers by reporting a location different than her true location.

Abstention-proofness:[1] An agent should not be able to change the outcome to a location she strictly prefers by not casting a ballot.

False-name-proofness: An agent should not be able to change the outcome to a location she strictly prefers by casting more than one ballot.

False-name-manipulations received less attention in the classic social choice literature since in most voting scenarios there exists a central authority that can enforce a 'one person, one vote' principle (but cannot enforce participation or truthful voting). In contrast, many of the voting and aggregation scenarios nowadays are run in a distributed manner on some network and include virtual identities or avatars, which can be easily generated, so a manipulation of an agent pretending to represent many voters is eminent.

Resistance to group manipulations: We also consider a generalization of the above three properties dealing with manipulations of a coalition of agents. We define the *preference of a coalition* as the unanimous preference of its members. That is, a coalition C weakly prefers an outcome a over an outcome b if all the agents in C weakly prefer a over b. Equivalently, C strictly prefers a over b if ***(i)*** all the agents in C weakly prefer a over b (C weakly prefers a over b), and ***(ii)*** at least one agent in C strictly prefers a over b (C does not weakly prefer b over a). We require that a coalition should not be able to change the outcome to a location it strictly prefers by its members casting untruthful ballots, abstaining, or casting more than one ballot. We note that for onto mechanisms this property entails Pareto-optimality. Nevertheless, we prefer to think of Pareto-optimality as an efficiency requirement and not as a manipulation-resistance requirement.

Our contribution

Besides the work of Todo et al. [28], who characterized the false-name-proof mechanisms for facility location on the continuous line and on continuous trees, we are not aware of other works characterizing false-name-proof mechanisms for facility location on a graph. Moreover, as far as we know, a false-name-proof mechanism is known to the community only for very few simple graphs, and the current knowledge is still highly preliminary.[2]

In this paper, we define a family of unweighted undirected graphs, which we name ZV-line graphs, and show a general mechanism for facility location over these graphs that satisfies the desired properties. To the best of our knowledge, this is the first work to show a general false-name-proof mechanism for a general family of graphs. Our mechanism for the ZV-line graphs family unifies the few mechanisms that are known and it induces mechanisms for many other graphs. The mechanism

[1] In the voting literature (e.g., [5, 18, 10]) this property is also referred to as **voluntary participation** and the **no-show paradox**. This property is also equivalent to **individual-rationality** which takes the different point of view of mechanism design.

[2] When starting to work on this problem, we initially devised mechanisms for few of the examples we describe below - cycles, cliques, and the $2 \times n$ grid. We are not aware of any other previously-known positive results besides these graphs or small perturbations of them.

is Pareto-optimal and in particular, it satisfies citizen sovereignty and does not a-priori rule out any location; It is anonymous, so in particular, no agent is ignored; But on the other hand, it is resistant to all the above manipulations.

Roughly speaking, in a ZV-line graph there are two types of locations, Z-locations and V-locations (and we also refer to them as Z-vertices and V-vertices), and the facility is commonly (except if all agents unanimously agree differently) located on a Z-location. For instance, the Z-locations could represent commercial locations for locating a public mall, or the set of status-quo outcomes.

Below, we give a series of common elementary graphs that are ZV-line graphs to demonstrate the richness and naturality of this family. The full formal definition of ZV-line graphs is given in Section 3, and we describe more examples for ZV-line graphs in Section 5. Consider the following family of graphs (which is a sub-family of the ZV-line graphs family and captures the gist of our mechanism). Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a bipartite unweighted undirected graph with a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$. That is, there exists a partition of the vertices $\mathcal{V} = V \mathbin{\dot\cup} Z$ s.t. there are no edges between V-vertices and no edges between Z-vertices. We require that ***(a)*** there exists a predefined order over the Z-vertices, which we refer to as a left-to-right order, and that ***(b)*** any of the V-vertices is connected to a contiguous sequence of Z-vertices. Similarly to the single-peaked consistency case [3], one can think of this constraint as a homogeneity constraint over the preferences of agents, i.e., as representing a restriction over the possible preference profiles, focusing on scenarios where voters' preferences are derived from some common structure.

Our mechanism for these graphs:

- ▶ The mechanism returns the leftmost Pareto-optimal Z-location if one exists.[3]
- ▶ If no location in Z is Pareto-optimal, then necessarily all agents voted for the same location, and the mechanism returns this location.

For example, bicliques (full bipartite graphs) can be represented as a ZV-line graph in which each V-vertex is connected to all the Z-vertices as follows (and we use below ◯ for Z-vertices and ⧫ for V-vertices):

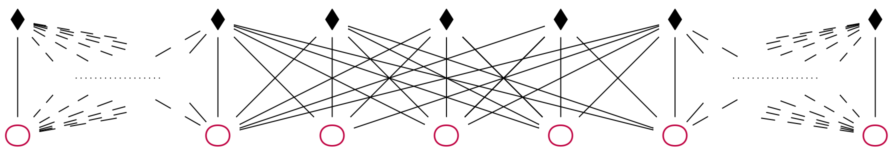


Fig. 1: Bicliques

Our mechanism for biclique graphs:

- ▶ If all agents voted unanimously for the same location, the mechanism returns this location.
- ▶ If all agents voted for V-locations, the mechanism returns the leftmost Z-location.
- ▶ Otherwise, the mechanism returns the leftmost Z-location that was voted for.

Notice that in this case, the order over the Z-locations is arbitrary (as well as the choice of one of the sides to be the Z-locations) in the sense that it is not derived from the graph but a parameter of the mechanism. For instance, the order might represent the social norm of the society.

[3] An outcome $o \in \mathcal{V}$ is Pareto-optimal if there exists no location $o' \in \mathcal{V}$ s.t. switching the outcome to be ℓ' benefits one of the agents while not hurting any of the other agents.

A second example is the discrete line graph, which can be represented as a ZV-line graph in which every two consecutive Z-vertices are connected by a unique V-vertex,

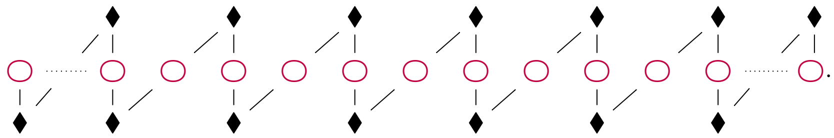


Fig. 2: The line graph

In particular, we show mechanisms for facility location on the discrete line that are strategy-proof, false-name-proof, anonymous, Pareto-optimal, and are far from *generalized-median mechanisms* (for instance, in the common case the output of the mechanism belongs to a subset consisting of only half of the locations). This, in contrast to the characterization of these mechanisms for the continuous line of Todo et al. [28, Thm. 2], who showed that generalized median mechanisms are the strategy-proof, false-name-proof, anonymous, Pareto-optimal for the continuous line, and in contrast to the characterization of strategy-proof mechanisms for the discrete line of Dokow et al. [7, Thm. 3.4]. In this work, Dokow et al., characterized the strategy-proof mechanisms for the discrete line as a superset of generalized-median mechanisms. Hence, we get a strict subset of their characterization (and, moreover, a small fraction of their characterization) due to adding the requirement of false-name-proofness.

Two elementary graphs that are generalizations of (the ZV-line graph representation of) the discrete line graph are

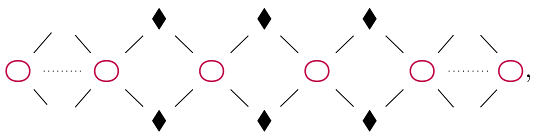


Fig. 3

in which every two consecutive Z-vertices are connected by two V-vertices, and the $2 \times n$ grid

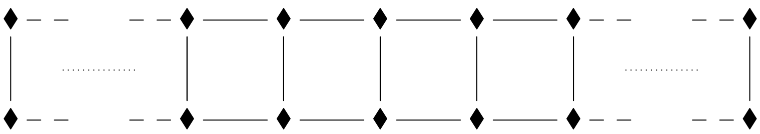


Fig. 4: The $2 \times n$ grid

which can be represented as a ZV-line graph in which every three consecutive Z-vertices are connected by a unique V-vertex, i.e.,

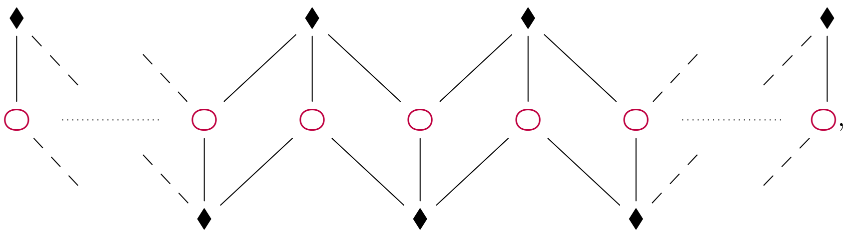
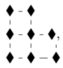


Fig. 5: The $2 \times n$ grid

A common property to all the above examples is their regularity. All the V-vertices have the same degree and similarly all the Z-vertices have the same degree. An example we encountered of a non-regular graph for which a mechanism exists is  which can be represented as a non-regular bipartite ZV-line graph in the following way:

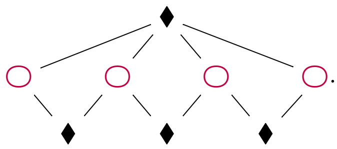


Fig. 6

In the definition of the ZV-line graphs family (Def. 4) we extend the above family (and extend the mechanism accordingly) in two different ways: allowing edges between the Z-vertices (under a similar connectivity constraint to the constraint we had on the neighborhood of the V-vertices), and replacing vertices by a tree, a clique, or any other ZV-line graph. For example, the following ZV-line graph is the outcome of taking a graph of the type of Fig. 3 and ***(a)*** adding an edge between two consecutive Z-vertices and ***(b)*** replacing some of the vertices by cliques or trees that are ZV-line graphs (See also Fig. 8 and the description next to it).

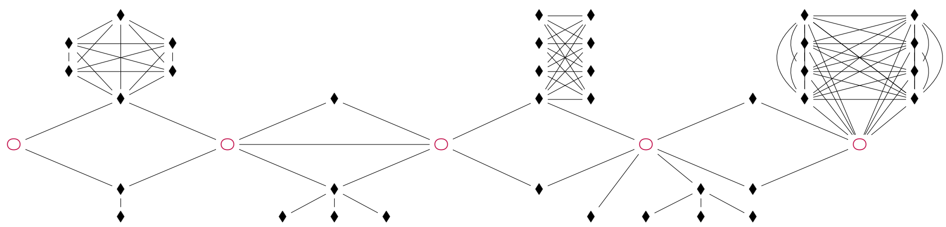


Fig. 7

In particular, as we discuss in Section 5, the ZV-line graphs family includes trees, cliques, block graphs [13], all connected graphs of at most five vertices (except the cycle of five vertices), and all graphs for which (as far as we found) a false-name-proof mechanism was known to the community. In Section 5, we also show that for recursive sub-families like trees, cliques, or block graphs, a recursive simple mechanism can be easily derived; that there are no group-manipulation-resistant, Pareto-optimal, anonymous mechanisms for cycles of size larger than 5; and the family of group-manipulation-resistant, Pareto-optimal, anonymous mechanisms for cycles of size five which is similar to the mechanism we present for ZV-line graphs, although this graph is not a ZV-line graph.

Related work

Problems of facility location on discrete graphs were also studied by Dokow et al. [7], who characterized the strategy-proof mechanisms for the discrete line and discrete cycle. Other variants of the facility location problem were also considered in the literature. For instance, Schummer and Vohra [26] considered the case of continuous graphs, Lu et al. [14] and Lu et al. [15] studied variants in which several facilities need to be located and scenarios in which an agent is located on several locations, and Feldman et al. [9] studied the impact of constraining the input language of the agents.

False-name-proofness was first introduced by Yokoo et al. [30, (based on a series of previous conference papers)] in the framework of combinatorial auctions. In this work, the authors showed that the VCG mechanism does not satisfy false-name-proofness in the general case, and they proposed a property of the preferences under which this mechanism becomes false-name-proof. A similar concept was also studied in the framework of peer-to-peer systems by Douceur [8] under the name *Sybil attacks*. Later, Conitzer [5] analyzed false-name-proof mechanisms in voting scenarios, Todo et al. [27] characterized other false-name-proof mechanisms for combinatorial auctions, and Todo et al. [28] characterized the false-name-proof mechanisms for facility location on the continuous line and facility location on continuous trees. The proof techniques we use here are different than the techniques used by Todo et al. [28] for continuous lines and trees. In their work, they essentially note that any manipulation-resistant mechanism is in particular strategy-proof, and using this insight they reduce the characterization problem to previous characterizations of strategy-proof mechanisms. In our work, we do not use previous proofness characterizations but prove the properties directly. In a recent work, Ono et al. [22] showed, in the framework of facility location on the discrete line, a relation between false-name-proofness and the property of *population monotonicity*.

Approximate mechanism design

The characterization of manipulation-resistant mechanisms for facility location is highly related to problems in *Approximate mechanism design without money* [23]. In these problems, agents are characterized using cardinal utilities and the designer seeks to find an outcome maximizing a desired target function (e.g., the sum of utilities, the product of utilities, or the minimal utility). These works bound the trade-of between the target function and manipulation-resistance. They bound the loss to the target function due to manipulation-resistance constraints. Similar bounds were derived for false-name-proof facility location mechanisms on the continuous line and continuous trees by Todo et al. [28], strategy-proof facility location on the continuous cycle by Alon et al. [2], and for strategy-proof facility location on the discrete cycle by Dokow et al. [7].

In this work, we do not analyze the approximation implications of the characterization and in particular, we do not assume a specific cardinal representation of the preference of agents. Yet, we claim that for most natural representations and target functions the approximation ratio is expected to be bad. For example, recall the mechanism for biclique graphs (Fig. 1). In this mechanism, the facility might be located on an 'extremely' left Z-location. Moreover, the facility might be very far from the vast majority of the agents, resulting in a very bad approximation ratio for most reasonable target functions. This phenomenon is not specific for biclique graphs. For most ZV-line graphs, due to the false-name-proof requirement, the mechanism might be located on a location extremely far from almost all agents, resulting in a very bad approximation ratio (roughly, the number of agents times the diameter of the graph) for most reasonable target functions.

2 Model

Consider a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ with a set of vertices $\mathcal{V}$ and a set of, neither weighted nor directed, edges $\mathcal{E} \subseteq \binom{\mathcal{V}}{2}$, and we refer to the vertices $v \in \mathcal{V}$ also as *locations* and use the two terms interchangeably. The distance between two vertices $v, u \in \mathcal{V}$, notated $d(v,u)$, is the length of the shortest path connecting v and u, and the distance between a vertex $v \in \mathcal{V}$ and a set of vertices $S \subseteq \mathcal{V}$, $d(v,S)$, is defined as the minimal distance between v and a vertex in S. For simplicity, we assume the graph is connected, so the distances are finite, and in Section 6.1 discuss the extension to unconnected graphs. We define $\mathcal{B}(v,d)$, the *ball* of radius $d \geqslant 0$ around a vertex $v \in \mathcal{V}$, to be the set of vertices of distance at most d from v,

$$\mathcal{B}(v,d) = \{u \in \mathcal{V} \mid d(v,u) \leqslant d\} .$$

We say that two vertices are *neighbors* if there is an edge connecting them and notate by $N(v)$ the set of neighbors of a vertex v.

An instance of the *facility location problem over* $\mathcal{G}$ is comprised of n agents who are located on vertices of $\mathcal{V}$. Formally, we represent it by a *location profile* $\boldsymbol{x} \in \mathcal{V}^n$ where x_i is the location of Agent i. We also use the notations x_C for the location profile of the agents in a given coalition $C \subseteq \mathcal{N}$ and x_C for the location profile of the agents outside of the coalition C. Given an instance $\boldsymbol{x}$, we would like to locate a facility on a vertex of the graph while taking into account the preferences of the agents over the locations. In this work, we assume the preference of an agent is defined by her distance to the facility: An agent located on $x \in \mathcal{V}$ strictly prefers the facility being located on $v \in \mathcal{V}$ over it being located on $u \in \mathcal{V}$ iff $d(x,v) < d(x,u)$.

A *general facility location mechanism* (or shortly a *mechanism*) defines for any location profile of any number of agents a location for the facility. We introduce the notation $\mathcal{V}^\star = \bigcup_{t \geqslant 0} \mathcal{V}^t$ for the set of all profiles of a finite number of agents. Hence, we represent the mechanism by a function $F \colon \mathcal{V}^\star \to \mathcal{V}$. We also think on F as a voting procedure: Each agent votes for a location (and we also refer to his vote as a *ballot*), and based on the ballots, F returns a location for the facility. We say that a mechanism is **anonymous** if the outcome $F(\boldsymbol{x})$ does not depend on the identities of the agents, i.e., it can be defined as a function of the ballot tally, the number of votes for each of the locations.

Manipulation-resistance

A strategic agent might act untruthfully if she thinks it might cause the mechanism to return a location she prefers (that is, a location closer to her). In this work, we consider the following manipulation types: • **Misreport**: An agent might report to the mechanism a location different from her true location; • **False-name-vote**: An agent might pretend to be several agents and submit several (not necessarily identical) ballots;[4] • **Abstention**: An agent might choose not to participate in the mechanism at all. A mechanism in which no agent benefits from these manipulations, regardless to the ballots of the other agents, is said to be **strategy-proof**, **false-name-proof**, and **abstention-proof**, respectively. We also consider a generalization of these manipulations to manipulations of a coalition, and say a mechanism is **group-manipulation-resistant** if no coalition can change the outcome, by misreporting, false-name-voting, or abstaining, to a different location which they unanimously agree is no worse than the original outcome (that is, when they vote truthfully) and at least one agent in the coalition strictly prefers the new outcome over the original outcome. Note that this is a rather strong manipulation-resistance requirement. A coalition cannot find a deviation that is beneficial for one of its members without hurting one of its other members, not even one in which different agents use different types of individual deviations.

Definition 1 (Group-manipulation-resistance[5]) An anonymous mechanism F is group-manipulation-resistant if there exists no coalition of agents $C \subseteq \{1, \ldots, n\}$, a profile of locations $\boldsymbol{x} \in \mathcal{V}^n$, and a set of ballots[6] $\boldsymbol{A} \in \mathcal{V}^\star$ s.t. ***(i)*** all the agents in C weakly prefer $F(\boldsymbol{A}, \boldsymbol{x}_{-C})$, that is, the outcome when the agents outside of C do not change their vote and the agents of C replace their ballots by $\boldsymbol{A}$, over $F(\boldsymbol{x})$ and ***(ii)*** at least one agent in C strictly prefers $F(\boldsymbol{A}, \boldsymbol{x}_{-C})$ over $F(\boldsymbol{x})$.

We note that for C being a singleton, this definition coincides with resistance to misreporting for $|\boldsymbol{A}| = 1$, with resistance to false-name-voting for $|\boldsymbol{A}| > 1$, and with resistance to abstention for $\boldsymbol{A} = \varnothing$.

The revelation principle

One could consider more general mechanisms in which the agents vote using more abstract ballots, and define similar manipulation-resistance notions for the general framework. Applying a simple direct revelation principle [20] shows that any such general manipulation-resistant mechanism is equivalent to a manipulation-resistant mechanism in our framework: The two mechanisms implement the same mapping of the private preferences of the agents to a location for the facility, and since the above properties are defined for the mapping they are invariant to this transformation. That is, given some general mechanism M that maps abstract actions to a location for the facility and a behavior protocol D that maps types of the agents (i.e., locations) to actions of M, if D satisfies the generalized desiderata, then the direct mechanism $M \circ D$ satisfies our desiderata.

[4] A special case of false-name-voting which is considered in the literature is **double-voting**: Casting the same (truthful) ballot several times to increase its impact.

[5] For simplicity of notations, we give the formal definition for anonymous mechanisms.

[6] Since F is an anonymous mechanisms, we define the deviation $\boldsymbol{A}$ as a set of ballots ignoring identities.

Efficiency

So far, we defined the desired manipulation-resistance properties of a mechanism. On the other hand, we would also like the mechanism to respect the preferences of the agents. We would like to avoid a scenario in which, after the mechanism has been used, the agents can agree that a different location is preferable. Given a location profile $\boldsymbol{x} \in \mathcal{V}^\star$, the set of *Pareto-optimal* locations, $PO(\boldsymbol{x})$, is the set of all locations which the agents cannot agree to rule out. Formally, given a location profile $\boldsymbol{x} \in \mathcal{V}^\star$ and two locations $v, u \in \mathcal{V}$, we say that u *Pareto-dominates* v *w.r.t.* $\boldsymbol{x}$ if ***(i)*** all agents weakly prefer u over v and ***(ii)*** at least one agent strictly prefers u over v. We say that v is *Pareto-optimal w.r.t.* $\boldsymbol{x}$ ($v \in PO(\boldsymbol{x})$) if it is not Pareto-dominated by any other location in $\mathcal{V}$. We say a mechanism is **Pareto-optimal** if for any ballot profile (a location profile) $\boldsymbol{x}$ $F(\boldsymbol{x}) \in PO(\boldsymbol{x})$. In particular, Pareto-optimality entails **unanimity**, whenever all the agents unanimously vote for the same location, the mechanism outputs this location; and **citizen sovereignty**, the mechanism is onto and does not a-priori rule out any location.

Relaxing false-name-proofness

The false-name-proofness property might seem to be a too strong desired property since we do not bound the number of false-name-ballots a manipulator might cast. Addressing this concern, we show that assuming either abstention-proofness or strategy-proofness, this property is equivalent to resistance to only one additional false-name-ballot of the manipulator.

Let $\boldsymbol{\mathcal{F}}^\star : \mathcal{V}^\star \to \mathcal{V}$ be a mechanism and $\boldsymbol{x} \in \mathcal{V}^\star$ a profile s.t. Agent i has a false-name-manipulation in $\boldsymbol{x}$. That is, there exists a multi-set $A = \{a_1, \ldots, a_k\} \subseteq \mathcal{V}$ s.t. Agent i strictly prefers the outcome $F(A, \boldsymbol{x}_{-i})$ over the outcome $F(x_i, \boldsymbol{x}_{-i})$. We define the following sequence of $(k+2)$ profiles:
- $\boldsymbol{x}^{(0)} = \langle x_i, \boldsymbol{x}_{-i}\rangle$
- For $t = 1, \ldots, k$: $\boldsymbol{x}^{(t)} = \langle x_i, a_1, \ldots, a_t, \boldsymbol{x}_{-i}\rangle$
- $\boldsymbol{x}^{(k+1)} = \langle a_1, \ldots, a_t, \boldsymbol{x}_{-i}\rangle$

Since Agent i strictly prefers the outcome $F(A, \boldsymbol{x}_{-i})$ over the outcome $F(x_i, \boldsymbol{x}_{-i})$, then necessarily
- Either there exists $t \in \{1, \ldots, k\}$ s.t. Agent i strictly prefers the outcome $F\left(\boldsymbol{x}^{(t)}\right)$ over the outcome $F\left(\boldsymbol{x}^{(t-1)}\right)$, i.e., she can manipulate using one additional false-name-ballot,
- or the following two statements hold
  - Agent i strictly prefers the outcome $F\left(\boldsymbol{x}^{(k+1)}\right)$ over the outcome $F\left(\boldsymbol{x}^{(k)}\right)$, i.e., she can manipulate by abstaining; and
  - Agent i strictly prefers the outcome $F\left(\boldsymbol{x}^{(k+1)}\right)$ over the outcome $F\left(\boldsymbol{x}^{(k-1)}\right)$, i.e., she can manipulate by misreporting her location.

3 Main Result

In this work, we define a family of graphs, *ZV-line graphs*, and present a simple and general mechanism for this family. This family is defined by introducing a simple combinatorial structure: A partition of the locations into two types and a connectivity constraint. One could think of the partition as representing a social agreement or a social norm according to which the mechanism is defined, e.g., a subset of status-quo locations or an a-priori priority hierarchy over the locations. The connectivity constraint (as the graph in general) represents homogeneity over the preferences of different agents, that is, a restriction over the possible preference profiles, focusing on scenarios where the preferences share some common structure. This allows us to construct a group-manipulation-resistant mechanism.

Definition 2 (ZV-ordered partitions)
Given an unweighted undirected connected graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$, we say that a sequence of non-empty sets of vertices $Z, V_1, \ldots, V_k \subseteq \mathcal{V}$ ($k \geqslant 0$) is a ZV-ordered partition of $\mathcal{G}$ if the following hold.

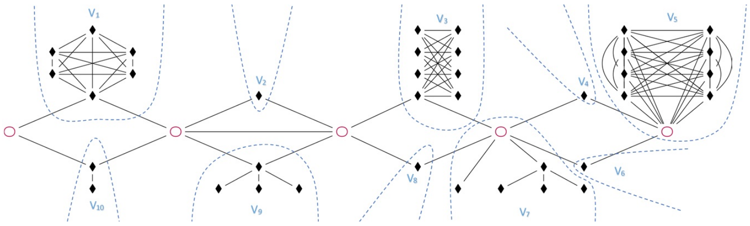
1. $(V_i \backslash Z) \cap (V_j \backslash Z) = \varnothing \quad$ for $i \neq j$.
2. The sequence is a cover of $\mathcal{V}$,
$$Z \cup V_1 \cup \cdots \cup V_k = \mathcal{V},$$
and $V_i \nsubseteq Z$ for $i = 1, \ldots, k$.
3. For $i = 1, \ldots, k \quad V_i$ is a connected component of $\mathcal{G}$.
4. For $i = 1, \ldots, k$, there is a unique vertex in V_i which is closest (in V_i) to Z. We refer to it as *the root of* V_i and denote it by $\mathcal{R}(V_i)$,
$$\mathcal{R}(V_i) = \operatorname*{argmin}_{v \in V_i} d(v, Z).$$
5. All paths between vertices of V_i and vertices outside of V_i include the root $\mathcal{R}(V_i)$ and intersect Z.
6. Z is equipped with a full linear order. For simplicity of description, we refer to this order as an order from left to right.

We use the notions V_i-subgraphs, V-vertices, and Z-vertices (or Z-locations & Z-locations)for the respective sets of vertices. Note that we do not require the sets of Z-vertices and V-vertices to be disjoint, but from Condition 4 we see that for all i if $V_i \cap Z \neq \varnothing$, it includes only one vertex, $\mathcal{R}(V_i)$. For example, in Figures 1,2,3,5,&6 in the introduction: The V_i-subgraphs consist of single vertices (the ⧫ vertices), which are also the roots of the respective subgraphs; The Z-vertices are not connected to each other and are ordered on a horizontal line. In the last example in the introduction, Fig. 7, there are 10 disjoint V_i-subgraphs and for two of them the root is also a Z-vertex (See Figure 8a). One could also define a different ZV-ordered partition of this graph in which instead of the V_i-subgraph V_7 in Figure 8a, there are two disjoint V_i-subgraphs (See Figure 8b).

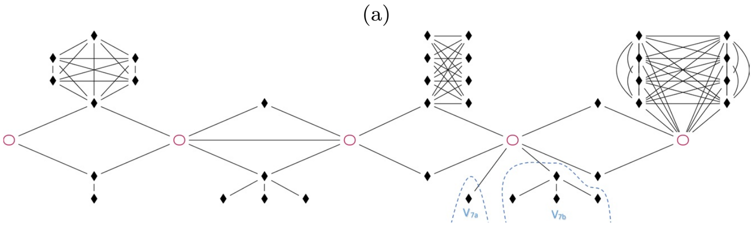
Given a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ with a ZV-ordered partition $\mathcal{V} = Z \cup (V_1 \dot\cup \cdots \dot\cup V_k)$ and mechanisms $F_i \colon \boldsymbol{x} \in V_i^\star \to V_i$ for $i = 1, \ldots, k$, we define the following mechanism $\boldsymbol{\mathcal{F}}^\star \colon \mathcal{V}^\star \to \mathcal{V}$.

Definition 3 ($\boldsymbol{\mathcal{F}}^\star$) Given a ballot profile (location reports of the agents) $\boldsymbol{x} \in \mathcal{V}^\star$,
- ▶ If all ballots belong to the same V_i-subgraph, return $F_i(\boldsymbol{x})$.
- ▶ Otherwise, return the leftmost Pareto-optimal location in Z.

Note that $\boldsymbol{\mathcal{F}}^\star$ is defined w.r.t. a given ZV-ordered partition, so when there are several different ZV-ordered partitions for $\mathcal{G}$, e.g., when $\mathcal{G}$ is a biclique, different



(a)



(b)

Fig. 8

mechanisms could arise. It is also important to note that we do not assume that the agents know the ZV-ordered partition of the graph, but they know the mechanism $\boldsymbol{\mathcal{F}}^{\star}$. In other words, we see this structure as a combinatorial property of a graph which derives the preferences of agents which could represent some homogeneity of the preferences or a social norm of giving priority to the Z-locations.

Note that $\boldsymbol{\mathcal{F}}^{\star}$ satisfies the following desired properties:

$\boldsymbol{\mathcal{F}}^{\star}$ is well-defined: Unless all ballots belong to the same V_i-subgraph, there exist two locations that belong to two different V_i-subgraphs and a shortest path between them s.t. all its vertices are in $PO(\boldsymbol{x})$, so $PO(\boldsymbol{x}) \cap Z \neq \emptyset$.

$\boldsymbol{\mathcal{F}}^{\star}$ runs in polynomial time: Finding $PO(\boldsymbol{x})$ can be done in time $|\mathcal{V}|^2 \cdot |\mathcal{N}|$ by iterating over all location pairs to find the Pareto-dominated locations.

Order representation of $\boldsymbol{\mathcal{F}}^{\star}$: If $F_1, \ldots, F_k$ can be defined as the first Pareto-optimal location (in V_i) according to some order, then an equivalent way to define $\boldsymbol{\mathcal{F}}^{\star}$ is as the first Pareto-optimal location in the following order: First, go over the Z-locations from left to right; Then, iterate over the V_i-subgraphs, and for each V_i-subgraph go over its locations according to the order of F_i.

We define ZV-line graphs by introducing an additional connectivity constraint.

Definition 4 (ZV-line graphs)

An unweighted undirected connected graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is a ZV-line graph w.r.t. $\mathcal{V} = Z \cup (V_1 \,\dot\cup \cdots \dot\cup\, V_k)$, if $\langle Z, V_1, \ldots, V_k \rangle$ is a ZV-ordered partition of $\mathcal{G}$ and

– $k > 0$ and
  - (a) For any vertex $z \in Z$, $\mathcal{B}(z, 1) \cap Z$ is a contiguous sequence of Z-vertices, and for $i = 1, \ldots, k$
  - (b) $\mathcal{B}(\mathcal{R}(V_i), 1) \cap Z$ is a contiguous sequence of Z-vertices.
  - (c) The induced V_i-*subgraph* $\mathcal{G}_i = \left\langle V_i, \mathcal{E} \cap \binom{V_i}{2} \right\rangle$ is a ZV-line graph.

(*d*) $\mathcal{R}(V_i)$ is a Z-vertex of $\mathcal{G}_i$ (that is, a Z-vertex in the ZV-ordered partition of $\mathcal{G}_i$) and it is the leftmost Z-vertex of $\mathcal{G}_i$.

– Or $k = 0$ (i.e., $\mathcal{V} = Z$) and for any vertex $z \in \mathcal{V}$, $\mathcal{B}(z, 1)$ is a contiguous sequence of Z-vertices (according to the order over Z).

We say that a graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is a ZV-line graph, if it is a ZV-line graph w.r.t. some ZV-ordered partition of $\mathcal{G}$.

For example, in Figures 1,2,3,5,&6 in the introduction:
– The Z-vertices are not connected to each other, so $\mathcal{B}(z, 1) \cap Z = \{z\}$ for all $z \in Z$;
– For any V_i-subgraph, V_i is a singleton, so $\left\langle V_i, \mathcal{E} \cap \binom{V_i}{2} \right\rangle$ is the trivial ZV-line graph which consists of a single vertex; and
– Each V-vertex is connected to a contiguous sequence of Z-vertices.

Note that bicliques (Fig. 1) are ZV-line graphs w.r.t. any order over the Z-vertices.

We note that the second case in Definition 4 ($k = 0$) can be replaced by $\mathcal{G}$ being the singleton graph.

Proposition 1 *Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a connected graph over $\mathcal{V} = \{1, \ldots, m\}$ for $m > 1$ s.t. $\mathcal{G}$ is a ZV-line graph w.r.t. $Z = \mathcal{V}$ and $k = 0$. Then $\mathcal{G}$ is also a ZV-line graph w.r.t. $Z = \{1, \ldots, m-1\}$ and $V_1 = \{m\}$.*

Proof $\mathcal{G}$ is a ZV-line graph w.r.t. $Z = \mathcal{V} = \{1, \ldots, m\}$ and $k = 0$.

$$\iff \text{For } v = 1, \ldots, m\text{: } \mathcal{B}(v, 1) = \{v\} \cup N(v) \text{ is a contiguous sequence of vertices.}$$

$$\iff \begin{cases} \text{For } v = 1, \ldots, m-1 : \mathcal{B}(v, 1) \text{ is a contiguous sequence of vertices.} \\ \mathcal{B}(m, 1) = \{m\} \cup N(m) \text{ is a contiguous sequence of vertices.} \end{cases}$$

$$\implies \begin{cases} \text{For } v = 1, \ldots, m-1 : \mathcal{B}(v, 1) \text{ is a contiguous sequence of vertices.} \\ N(m) \text{ is a contiguous sequence of vertices.} \end{cases}$$

$$\iff \mathcal{G} \text{ is a } ZV\text{-line graph w.r.t. } Z = \{1, \ldots, m-1\} \text{ and } V_1 = \{m\}.$$ □

The case $k = 0$ serves as a base step of the inductive definition of ZV-line graphs. For this reason, we prefer to enrich it over defining the base step as only the singleton graph. Two immediate examples of ZV-line graphs in the base case are the line graph with the natural order (note this is a different representation as a ZV-line graph than the one in Figure 2) and the clique graph with any order over the vertices.[7] It turns out that an equivalent definition of this case is a definition as a composition of a line and cliques.

Proposition 2 *Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a connected graph over $\mathcal{V} = \{1, \ldots, m\}$. Then the following four statements are equivalent*

1. *$\mathcal{G}$ is a ZV-line graph w.r.t. $Z = \mathcal{V} = \{1, \ldots, m\}$ and $k = 0$.*
2. *For any vertex $v \in \mathcal{V}$ and $h = \max \mathcal{B}(v, 1)$, the projection of $\mathcal{G}$ on $\{v, \ldots, h\}$ is a clique, i.e., for any $v \leqslant a < b \leqslant h \quad (a, b) \in \mathcal{E}$.*
3. *For any vertex $v \in \mathcal{V}$ and $\ell = \min \mathcal{B}(v, 1)$, the projection of $\mathcal{G}$ on $\{\ell, \ldots, v\}$ is a clique, i.e., for any $\ell \leqslant a < b \leqslant v \quad (a, b) \in \mathcal{E}$.*
4. *$\mathcal{G}$ can be represented as a '**line of cliques**.' That is, there exists a sequence of integer ranges $\{(a_t, b_t)\}_{t=1}^{r}$* s.t.

$$a_t < a_{t+1} \leqslant b_t < b_{t+1} \qquad t = 1, \ldots, r-1 \qquad \text{and}$$

[7] So the mechanism $\mathcal{F}^\star$ for the line graph returns the leftmost location that was voted for and $\mathcal{F}^\star$ for the clique graph returns the first location that was voted for.

$$\text{for any } 1 \leqslant u < v \leqslant m \qquad (u,v) \in \mathcal{E} \iff \exists t \in \{1, \ldots r\} \text{ s.t. } a_t \leqslant u < v \leqslant b_t.$$

Proof We show that (1) ⇒ (2) ⇒ (4) ⇒ (1). Showing that (1) ⇒ (3) ⇒ (4) is symmetric.

1 ⇒ 2: Let $v \in \mathcal{V}$ and $v \leqslant a < b \leqslant h$. v is connected to h, and since $\mathcal{B}(v,1)$ is a contiguous sequence of vertices, v is also connected to b. Similarly, b is connected to v, and since $\mathcal{B}(b,1)$ is a contiguous sequence of vertices, it is also connected to a.

2 ⇒ 4: We define two sequences $\{a_t\}_{t=1}^{|\mathcal{V}|}$ and $\{b_t\}_{t=1}^{|\mathcal{V}|}$ by $a_t = t$ and $b_t = \max N(t)$ (Since $\mathcal{G}$ is connected, $N(t) \neq \varnothing$) and get that for any $1 \leqslant u < v \leqslant m$:
- If $(u,v) \in \mathcal{E}$, then $a_u = u < v \leqslant \max N(u) = b_u$.
- If $\exists t \in \{1, \ldots m\}$ s.t. $a_t \leqslant u < v \leqslant b_t$, then the projection of $\mathcal{G}$ on $\{a_t, \ldots, b_t\}$ is a clique and in particular $(u,v) \in \mathcal{E}$.

That is,
$$(u,v) \in \mathcal{E} \iff \exists t \in \{1, \ldots m\} \text{ s.t. } a_t \leqslant u < v \leqslant b_t.$$

We note that we can omit without hurting this property from the sequences pairs for which $a_t > b_t$ and pairs that are included in other pairs ($a_{t'} \leqslant a_t < b_t \leqslant b_{t'}$). Hence, we get that

$$\forall t \quad \begin{cases} a_t < a_{t+1} \\ b_t < b_{t+1} \end{cases}$$

and since $\mathcal{G}$ is a connected graph also

$$\forall t \quad a_{t+1} \leqslant b_t.$$

4 ⇒ 1: Let $v \in \mathcal{V}$ and $u \in \mathcal{B}(v,1)$. If $u < v$, then there exists an index $t \in \{1, \ldots r\}$ s.t. $a_t \leqslant u < v < b_t$. Furthermore, v is connected to any vertex $w \in \mathcal{V}$ s.t. $a_t \leqslant u \leqslant w < v$. Similarly, if $u > v$, then v is connected to any vertex $w \in \mathcal{V}$ s.t. $v < w \leqslant u$. Hence, $\mathcal{B}(v,1) = \{v\} \cup N(v)$ is a contiguous sequence of vertices. □

Given a ZV-line graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$, applying Def. 3 recursively on $\mathcal{G}$ and its V_i-subgraphs (with the case $\mathcal{V} = Z$ as the base of the recursion) defines a mechanism $\boldsymbol{\mathcal{F}}^\star : \mathcal{V}^\star \to \mathcal{V}$. Our main result shows that this resulted mechanism satisfies the desired properties.

Theorem 1 (Main result)

Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a ZV-line graph (w.r.t. some ZV-ordered partition) and let $\boldsymbol{\mathcal{F}}^\star : \mathcal{V}^\star \to \mathcal{V}$ be the result of applying Definition. 3 recursively on $\mathcal{G}$. Then $\boldsymbol{\mathcal{F}}^\star$ is an anonymous, Pareto-optimal, group-manipulation-resistant mechanism. That is, $\boldsymbol{\mathcal{F}}^\star$ satisfies:

For any location profile $\boldsymbol{x} \in \mathcal{V}^n$, a coalition of agents C, and a set of ballots $\boldsymbol{A} \in \mathcal{V}^\star$, $\boldsymbol{A}$ is not a beneficial deviation for C.[8]

A special case of ZV-line graphs which we found interesting on their own merits are the *Bipartite ZV-line graphs.*

Definition 5 (Bipartite ZV-line graphs)

A bipartite graph $\mathcal{G} = \langle \mathcal{V} = L \dot\cup R, \mathcal{E} \rangle$ (that is, $\mathcal{E} \subseteq L \times R$) is a ZV-line graph (w.r.t. $Z = L$ and $V_1 = R$), if • L is equipped with an order and • For any vertex $v \in R \quad N(v)$ is a contiguous sequence of vertices of L.

[8] Since $\boldsymbol{\mathcal{F}}^\star$ is onto, this property entails Pareto-optimality. Yet, we prefer to state explicitly Pareto-optimality as a desired efficiency property.

For bipartite ZV-line graphs, an immediate corollary of the main theorem:

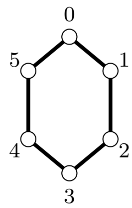
Corollary 1

Let $\mathcal{G} = \langle \mathcal{V} = L \,\dot\cup\, R, \mathcal{E}\rangle$ be a bipartite ZV-line graph and let $\boldsymbol{\mathcal{F}}^\star$ be the following mechanism:
- ▶ *If all ballots are identical, return this location as the outcome.*
- ▶ *If one of the ballots is a location in L, return the leftmost location in L that was voted for.*
- ▶ *Otherwise, return the leftmost location in L.*

Then $\boldsymbol{\mathcal{F}}^\star$ is an anonymous, Pareto-optimal, group-manipulation-resistant mechanism.

Note that the bipartiteness of the graph (and the existence of a ZV-ordered partition in general) is not a sufficient condition for group-manipulation-resistance of $\boldsymbol{\mathcal{F}}^\star$. For example, C_6, the cycle of size 6, is a bipartite graph but there exists no mechanism for C_6 which satisfies the properties of the theorem (In Section 5.1, we generalize this and show that for any $n > 5$ there exists no anonymous Pareto-optimal mechanism for C_n which is resistant even to manipulations a single agent).

Proof

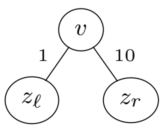


Assume towards a contradiction that F is a Pareto-optimal, anonymous, group-manipulation-resistant mechanism for C_6. We notate the vertices of C_6 by $\{0, 1, 2, 3, 4, 5\}$, and w.l.o.g. assume that for the profile of six agents who vote for all six locations the outcome is 0.

For the profile $\langle \mathbf{2, 4, 5} \rangle$: From resistance to false-name manipulations of the first and last agents, the outcome must be either 0 or 4 (Since any of them can change the result to be 0 by adding false-ballots). From the Pareto-optimality of F, the outcome cannot be 0 which is Pareto-dominated by 4. Hence, the outcome for the profile $\langle \mathbf{2, 4, 5} \rangle$ is 4.

Similarly, for the symmetric profile $\langle \mathbf{1, 2, 4} \rangle$ the outcome must be 2. From false-name-resistance the outcome for the profile $\langle \mathbf{2, 4} \rangle$ must also be 2 (Otherwise, the first agent will cast an additional false-ballot 1 to get the outcome to be 2).

But, the second agent in the profile $\langle \mathbf{2, 4} \rangle$ (who is located on 4) can change the outcome to be 4 that she strictly prefers by casting one additional false-ballot 5. So we get a contradiction. □



$V = \{v\}$
$Z = \{z_\ell, z_r\}$

We also note that the theorem does not hold for weighted graphs. Consider the weighted graph on the right and a profile in which Alice is located on z_r and Bob is located on v. Then the outcome of $\boldsymbol{\mathcal{F}}^\star$ is z_r, but Bob can move the facility to a preferred location z_ℓ both ***(i)*** by misreporting z_ℓ; hence, $\boldsymbol{\mathcal{F}}^\star$ is not strategy-proof; and ***(ii)*** by false-name-voting z_ℓ besides his truthful ballot, hence, $\boldsymbol{\mathcal{F}}^\star$ is not false-name-proof.[9]

Last, we note that there exist simple mechanisms that satisfy subsets of the properties of Thm. 1:
- The *fixed mechanism*, which always locates the facility on a pre-defined location ignoring the votes of the agents, is trivially group-manipulation-resistant and anonymous for any graph, but it is not onto and hence not Pareto-optimal.

[9] The mechanism that returns the leftmost ballot according to the order $z_\ell - v - z_r$ satisfies the desiderata. Notice that this mechanism can be defined as $\boldsymbol{\mathcal{F}}^\star$ w.r.t. a ZV-ordered partition with $Z = \{z_\ell, v, z_r\}$ and $k = 0$.

- A *dictatorship* of the first agent, i.e., a mechanism that always locates the facility on the location reported by the first agent, is not anonymous, but it is group-manipulation-resistant for any graph.[10]
- The *median mechanism*, which minimizes the sum of distances between the facility and the ballots, is an anonymous Pareto-optimal mechanism for any graph. For some graphs, e.g., the discrete line, it also satisfies strategy-proofness and abstention-proofness both against one manipulator and against a coalition, but an agent can benefit by casting multiple identical ballots.
- Also the *mean mechanism*, which minimizes the sum of squares of the distances between the facility and the ballots, is an anonymous Pareto-optimal mechanism for any graph, but it might not be strategy-proof or false-name-proof even against one agent, e.g., for the discrete line graph (it is abstention-proof, though).

4 Proof of Main Result (Thm. 1)

We prove a stronger result (Thm. 2) showing a general method to construct from anonymous, Pareto-optimal, manipulation-resistant mechanisms for the V_i-subgraphs, an anonymous, Pareto-optimal, manipulation-resistant mechanism $\boldsymbol{\mathcal{F}}^\star$ for $\mathcal{G}$. Theorem 1 is the outcome of applying Thm. 2 on the recursive steps of the definition of $\boldsymbol{\mathcal{F}}^\star$ (Def. 3).

Theorem 2
Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a graph with a ZV-ordered partition $\mathcal{V} = Z \cup (V_1 \dot\cup \cdots \dot\cup V_k)$ and let $F_i \colon V_i^\star \to V_i$ be a sequence of mechanisms s.t. for $i = 1, \ldots, k$

- *F_i is an anonymous Pareto-optimal mechanism;*
- *For an infinite number of $\tau \in \mathbb{N}$, there exists a profile $\boldsymbol{x} \in V_i^\star$ in which there are at least τ ballots for any location in V_i s.t. $F_i(\boldsymbol{x}) = \mathcal{R}(V_i)$; and*
- *F_i is a group-manipulation-resistant mechanism. I.e., for any profile of locations $\boldsymbol{x} \in V_i^\star$, a coalition of agents C, and a set of ballots $\boldsymbol{A} \in V_i^\star$, $\boldsymbol{A}$ is not a beneficial deviation for C in $\boldsymbol{x}$.*

Then for $\boldsymbol{\mathcal{F}}^\star \colon \mathcal{V}^\star \to \mathcal{V}$ as defined in Definition 3, $\boldsymbol{\mathcal{F}}^\star$ is an anonymous Pareto-optimal mechanism and

(***I***) *If $\mathcal{G}$ is a ZV-line graph w.r.t. $\mathcal{V} = Z \cup (V_1 \dot\cup \cdots \dot\cup V_k)$, then also $\boldsymbol{\mathcal{F}}^\star$ is a group-manipulation-resistant mechanism.*

(***II***) *If for $i = 1, \ldots, k$ $\mathcal{R}(V_i) \in Z$ and the mechanism $F_Z \colon Z^\star \to Z$ that returns the leftmost Pareto-optimal location (that is, the restriction of $\boldsymbol{\mathcal{F}}^\star$ to $Z^\star$) is a group-manipulation-resistant mechanism, then also $\boldsymbol{\mathcal{F}}^\star$ is a group-manipulation-resistant mechanism.*

Proof of Theorem 2
Since the mechanisms F_i and F_Z are anonymous, $\boldsymbol{\mathcal{F}}^\star$ is also an anonymous mechanism.

Unless all ballots belong to the same V_i-subgraph, from the definition of $\boldsymbol{\mathcal{F}}^\star$, the outcome lies in Z and it is Pareto-optimal. If all agents belong to the same

[10] While we did not formally define false-name-proofness for non-anonymous mechanisms, assuming a false-name-ballot cannot be counted as the vote of the first agent, no agent can benefit from casting additional ballots.

V_i-subgraph, then all of them strictly prefer $\mathcal{R}(V_i)$ over any location outside of V_i, so $PO(\boldsymbol{x}) \subseteq V_i$. Furthermore, any location $v \in V_i \backslash PO(\boldsymbol{x})$ is Pareto-dominated by a location $y \in PO(\boldsymbol{x}) \subseteq V_i$. Hence, the Pareto-optimal set when considering only the locations in V_i equals to the Pareto-optimal set when considering all locations. Since the mechanisms F_i are Pareto-optimal mechanisms, we get that also $\boldsymbol{\mathcal{F}}^\star$ is a Pareto-optimal mechanism.

To prove the main part of the theorem, we assume towards a contradiction that there exists a profile of locations (ballots) $\boldsymbol{x} \in \mathcal{V}^n$, a coalition of agents C, and a set of ballots $\boldsymbol{A} \in \mathcal{V}^\star$, s.t. the coalition C can cast $\boldsymbol{A}$ and change the outcome to be $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ that it strictly prefers. That is, all the agents in C weakly prefer $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ over $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}) = \boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}_C, \boldsymbol{x}_{-C})$ and at least one agent in C, Agent i for $i \in C$, strictly prefers $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ over $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$. $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}) \in PO(\boldsymbol{x})$ and in particular the coalition of all agents does not strictly prefer $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ over $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$. Hence, there exists an Agent j, for $j \notin C$, who strictly prefers $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$ over $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$.

If $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$ is not in Z: Then necessarily, all the locations in $\boldsymbol{x}$ and $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$ belong to the same V_i-subgraph, w.l.o.g. V_1, so $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}) = F_1(\boldsymbol{x})$. Since F_1 is resistant to false-name manipulations of Agent i and since Agent i can achieve $\mathcal{R}(V_1)$ by casting enough false ballots, we get that Agent i weakly prefers $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$ over $\mathcal{R}(V_1)$ and hence Agent i strictly prefers $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ over $\mathcal{R}(V_1)$. Since for any location u outside of V_1 it holds that $d(x_i, \mathcal{R}(V_1)) < d(x_i, u)$, we get that $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C}) \in V_1 \backslash \{\mathcal{R}(V_1)\} \subseteq V_1 \backslash Z$. Hence, $\boldsymbol{A} \subseteq V_1$ and $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C}) = F_1(\boldsymbol{A}, \boldsymbol{x}_{-C})$, in contradiction to the group-manipulation-resistance of F_1.

Similarly, if $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ is not in Z: Then necessarily, $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ and all the locations in $\boldsymbol{A}$ and $\boldsymbol{x}_{-C}$ belong to the same V_i-subgraph, w.l.o.g. V_1, so $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C}) = F_1(\boldsymbol{A}, \boldsymbol{x}_{-C})$. Since F_1 is resistant to false-name manipulations of Agent j and since Agent j can achieve $\mathcal{R}(V_1)$ by casting enough false ballots, we get that Agent j weakly prefers $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ over $\mathcal{R}(V_1)$ and strictly prefers $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$ over $\mathcal{R}(V_1)$. Since for any location u outside of V_1 it holds that $d(x_j, \mathcal{R}(V_1)) < d(x_j, u)$, we get that $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}) \in V_1 \backslash \{\mathcal{R}(V_1)\} \subseteq V_1 \backslash Z$. Hence, $\boldsymbol{x} \subseteq V_1$ and $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}) = F_1(\boldsymbol{x})$, in contradiction to the group-manipulation-resistance of F_1.

If both $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$ and $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ are in Z: We deal with this case using two different argumentations for the two scenarios of the theorem.

(I) $\mathcal{G}$ is a ZV-line graph w.r.t. $\mathcal{V} = Z \cup (V_1 \dot\cup \cdots \dot\cup V_k)$:

First, we prove the following two auxiliary lemmas.

Lemma i *For any $v \in \mathcal{V}$ and $d \geqslant 0$, $\mathcal{B}(v, d) \cap Z$ is a contiguous sequence of Z-vertices.*

Proof If $d < d(v, Z)$, $\mathcal{B}(v, d) \cap Z = \varnothing$. Henceforth, we assume that $d \geqslant d(v, Z)$.

First, we prove the lemma for vertices $v \in \mathcal{V}$ s.t. $d(v, Z) \leqslant 1$ (i.e., $v \in Z$ or v is the root of some V_i-subgraph) by induction over d.

For $d = 0$: $\mathcal{B}(v, 0) \cap Z$ is either the empty set (if $v \notin Z$) or $\{v\}$ (if $v \in Z$).

For $d = 1$: $\mathcal{B}(v, 1) \cap Z$ is a contiguous sequence of Z-vertices by the definition of ZV-line graphs.

For $d \geqslant 2$: We note that if $u \in \mathcal{B}(v, d) \cap Z$, then there exists a path from v to u of length at most d s.t. all the vertices of the path are either Z-vertices or roots of V_i-subgraphs. Also, there are no edges between two vertices of distance 1 from Z. Hence,

- If $v \in Z$: $\mathcal{B}(v,d) \cap Z = \left(\{v\} \cup \left(\bigcup_{\substack{u\in N(v) \text{ s.t.} \\ d(u,Z)\leqslant 1}} \mathcal{B}(u,d-1)\right)\right) \cap Z$
$$= \{v\} \cup \left(\bigcup_{\substack{u\in N(v) \text{ s.t.} \\ d(u,Z)\leqslant 1}} \mathcal{B}(u,d-1) \cap Z\right),$$
and since for any location $u \in N(v)$ $\quad v \in \mathcal{B}(u,d-1) \cap Z$ and $\mathcal{B}(u,d-1) \cap Z$ is a contiguous sequence of Z-vertices, also $\mathcal{B}(v,d) \cap Z$ is a contiguous sequence of Z-vertices as the union of intersecting contiguous sequences of Z-vertices.
- If $v \notin Z$: $\mathcal{B}(v,d) \cap Z = \left(\{v\} \cup \left(\bigcup_{u\in N(v)\cap Z} \mathcal{B}(u,d-1)\right)\right) \cap Z$
$$= \bigcup_{u\in N(v)\cap Z} \mathcal{B}(u,d-1) \cap Z,$$
and since $N(v) \cap Z$ is a contiguous sequence of Z-vertices and for any $u \in N(v) \cap Z$ $\quad \mathcal{B}(u,d-1) \cap Z$ is a contiguous sequence of Z-vertices, also $\mathcal{B}(v,d) \cap Z$ is a contiguous sequence of Z-vertices.

Last, if $v \notin Z$ and it is not a root of a V_i-subgraph, then $d \geqslant d(v,Z) > 1$. We take u to be the root of v's V_i-subgraph and since all paths from v to locations in Z pass through u, we get that

$$\mathcal{B}(v,d) \cap Z = \mathcal{B}(u, d - d(v,u)) \cap Z$$

which is a contiguous sequence of Z-vertices. □

Lemma ii *Let $\boldsymbol{x}$ be a location profile s.t. $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}) \in Z$ and let $v \in Z$ be a location s.t. Agent i strictly prefers v over $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$. Then $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$ is to the left of v.*

Proof If $x_i \in Z$, then $x_i \in PO(\boldsymbol{x}) \cap Z$ and by the definition of $\boldsymbol{\mathcal{F}}^\star$, $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$ is to the left of x_i. Since $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}) \notin \mathcal{B}(x_i, d(x_i,v)) \cap Z$ and since $\mathcal{B}(x_i, d(x_i,v)) \cap Z$ is a contiguous sequence of Z-vertices that includes x_i, we get that $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$ is to the left of $\mathcal{B}(x_i, d(x_i,v)) \cap Z$ and in particular to the left of v.

If all the ballots in $\boldsymbol{x}$ belong to the same V_i-subgraph, w.l.o.g. V_1, then $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}) \in V_1 \cap Z$, and hence $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}) = \mathcal{R}(V_1)$, and $v \in Z \backslash V_1$. This in contradiction to the fact that Agent i is located in V_1 and she strictly prefers v over $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$.

Last, if both former cases do not hold, i.e., $x_i \notin Z$ and there exists an Agent k for which x_k is not in the same V_i-subgraph as x_i, then there exists a location $u \in Z$ s.t. u is on a shortest path from x_i to x_k, $u \in Z$, $u \in PO(\boldsymbol{x})$, and $d(x_i,u) {=} d(x_i,Z) {\leqslant} d(x_i,v)$ and so

$$u \in \mathcal{B}(x_i, d(x_i,u)) \cap Z \subseteq \mathcal{B}(x_i, d(x_i,v)) \cap Z.$$

Both sets are contiguous sequences of Z-vertices, $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$ is to the left of u (or equal to it), and $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}) \notin \mathcal{B}(x_i, d(x_i,v)) \cap Z$. Hence, $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$ is to the left of v. □

By applying Lemma (ii) for the profile $\boldsymbol{x}$ and Agent i, we get that $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$ is to the left of $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$; and by applying Lemma (ii) for the profile $(\boldsymbol{A}, \boldsymbol{x}_{-C})$ and Agent j, we get that $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ is to the left of $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$. Hence, we get a contradiction.

(II) $\mathcal{R}(V_i) \in Z$ for $i = 1, \ldots, k$ and
F_Z is a group-manipulation-resistant mechanism:

Given a location profile $\boldsymbol{y} \in \mathcal{V}^\star$, we use the notation $\widehat{\boldsymbol{y}} \in Z^\star$ for the location profile generated from $\boldsymbol{y}$ by replacing each ballot outside of Z with the root of its V_i-subgraph. The preference of an agent whose location is in a V_i-subgraph over the locations of Z is identical to the preference of an agent whose location is $\mathcal{R}(V_i)$. Hence, for any profile $\boldsymbol{y}$ s.t. $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{y}) \in Z$, $\boldsymbol{\mathcal{F}}^\star(\boldsymbol{y}) = F_Z(\widehat{\boldsymbol{y}})$. Therefore, for the profile $\widehat{\boldsymbol{x}} \in Z^n$ the coalition C can, by casting $\widehat{\boldsymbol{A}}$, get an outcome $F_Z\left(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{x}}_{-C}\right) = \boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-C})$ that it strictly prefers over $F_Z(\widehat{\boldsymbol{x}}) = \boldsymbol{\mathcal{F}}^\star(\boldsymbol{x})$, in contradiction to the group-manipulation-resistance of F_Z. □

The same proof shows that also weaker notions of manipulation-resistance can be lifted from the mechanisms for the V_i-subgraphs to the mechanism $\boldsymbol{\mathcal{F}}^\star$, e.g., resistance against manipulations of only some coalitions and resistance against only some manipulation types.

Theorem 3

Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a graph with a ZV-ordered partition $\mathcal{V} = Z \cup (V_1 \dot\cup \cdots \dot\cup V_k)$ and let $F_i \colon V_i^\star \to V_i$ be a sequence of mechanisms s.t. for $i = 1, \ldots, k$

- *F_i is an anonymous Pareto-optimal mechanism;*
- *For an infinite number of $\tau \in \mathbb{N}$, there exists a profile $\boldsymbol{x} \in V_i^\star$ in which there are at least τ ballots for any location in V_i s.t. $F_i(\boldsymbol{x}) = \mathcal{R}(V_i)$; and*
- *F_i is resistant against false-name-voting of any single agent.*

Then for $\boldsymbol{\mathcal{F}}^\star \colon \mathcal{V}^\star \to \mathcal{V}$ as defined in Definition 3, $\boldsymbol{\mathcal{F}}^\star$ is an anonymous Pareto-optimal mechanism and for any profile of locations $\boldsymbol{x} \in \mathcal{V}^\star$, a coalition of agents C, and a set of ballots $\boldsymbol{A} \in \mathcal{V}^\star$

- *If there exists a V_i-subgraph s.t. $\begin{cases} \boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}) \in V_i \backslash Z \\ \boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-c}) \notin V_i \backslash Z \end{cases}$ or $\begin{cases} \boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-c}) \in V_i \backslash Z \\ \boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}) \notin V_i \backslash Z \end{cases}$, then $\boldsymbol{A}$ is not a beneficial deviation for C in $\boldsymbol{x}$.*
- *If there exists a V_i-subgraph s.t. $\begin{cases} \boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-c}) \in V_i \backslash Z \\ \boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}) \in V_i \backslash Z \end{cases}$, then necessarily both $\boldsymbol{x}$ and $\boldsymbol{A}$ are included in V_i and $\boldsymbol{A}$ is a beneficial deviation for C in $\boldsymbol{x}$ (under $\boldsymbol{\mathcal{F}}^\star$) iff $\boldsymbol{A}$ is a beneficial deviation for C in $\boldsymbol{x}$ under F_i.*
- *If $\begin{cases} \boldsymbol{\mathcal{F}}^\star(\boldsymbol{A}, \boldsymbol{x}_{-c}) \in Z \\ \boldsymbol{\mathcal{F}}^\star(\boldsymbol{x}) \in Z \end{cases}$,*
  - ***(I)*** *If $\mathcal{G}$ is a ZV-line graph w.r.t. $\mathcal{V} = Z \cup (V_1 \dot\cup \cdots \dot\cup V_k)$, then $\boldsymbol{A}$ is not a beneficial deviation for C in $\boldsymbol{x}$.*
  - ***(II)*** *If for $i = 1, \ldots, k$ $\mathcal{R}(V_i) \in Z$, then for the mechanism $F_Z \colon Z^\star \to Z$ that returns the leftmost Pareto-optimal location (that is, the restriction of $\boldsymbol{\mathcal{F}}^\star$ to $Z^\star$): If $\boldsymbol{A}$ is a beneficial deviation for C in $\boldsymbol{x}$, then there exists a beneficial deviation for C under F_Z (in some profile $\boldsymbol{x}' \in Z^\star$).*

5 Examples of ZV-line graphs

In this section, we study several graph families which we show are ZV-line graphs, by that also illustrating the richness of the family of ZV-line graphs. We also present the mechanisms for these graphs that are derived from our main result. As we saw previously, cliques, lines, and bicliques are ZV-line graphs.

Claim For any $n \geqslant 1$, the clique over n vertices,

$$K_n = \langle \mathcal{V} = \{1, \ldots n\}, \mathcal{E} = \{(i, j) \mid i \neq j\} \rangle$$

(that is, the graph over n vertices with an edge between any two vertices), is a ZV-line graph w.r.t. the Z-vertices being all n vertices with any order over them.

Claim For any $n \geqslant 1$, the discrete line of n vertices,

$$P_n = \langle \mathcal{V} = \{1, \dots n\}, \mathcal{E} = \{(i, i+1) \mid i = 1, \dots, n-1\} \rangle,$$

is a ZV-line graph w.r.t. the following two ZV-ordered partitions (and others)

- w.r.t. the Z-vertices being all n vertices with the natural order over them; and
- w.r.t. the Z-vertices being the $\lceil n/2 \rceil$ odd-indexed vertices with the natural order over them and each of the even-indexed vertices being a singleton V_i-subgraph.

(Note that these two ZV-ordered partitions result in two different mechanisms).

Claim For any $n, m \geqslant 1$, the biclique over n and m vertices,

$$K_{m,n} = \langle \mathcal{V} = \{1, \dots n\} \dot\cup \{n+1, \dots n+m\}, \mathcal{E} = \{(i, j) \mid i \leqslant n < j\} \rangle$$

(that is, the graph with n vertices on one side, m vertices on the other side and an edge between any two vertices of opposite sides), is a ZV-line graph w.r.t. Z-vertices being the first n vertices with any order over them and each of the other m vertices being a singleton V_i-subgraph.

Furthermore, the following three propositions show that also small perturbations of cliques, the outcome of adding vertices or removing edges, are ZV-line graphs.

Proposition 3

Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a graph with a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$, and let $\mathcal{V} = V^\star \dot\cup V^{(1)} \dot\cup V^{(2)} \dot\cup \cdots \dot\cup V^{(k)}$ (for $k \geqslant 1$ and $V^\star \neq \emptyset$) be a partition of the vertex set $\mathcal{V}$ s.t.

- *The restriction of $\mathcal{G}$ to $\mathcal{V} \backslash V^\star$ is a clique. That is, for any two vertices $v, u \notin V^\star$ $(v, u) \in \mathcal{E}$; and*
- *For any vertex $v \in V^\star$, $N(v) = V^{(i)}$ for some partition element $V^{(i)}$ $(i \in \{1, \dots, k\})$.*

Then $\mathcal{G}$ is a ZV-line graph.

Proof

We name the vertices of $V^\star$

$$V^\star = \left\{v_1, v_2, \dots, v_{|V^\star|}\right\},$$

and claim that $\mathcal{G}$ is a ZV-line graph w.r.t. the singleton subgraphs $V_i = \{v_i\}$ for $i = 1, \dots, |V^\star|$, $Z = \mathcal{V} \backslash V^\star = \bigcup_{i=1}^{k} V^{(i)}$, and an order over Z s.t. for $i = 1, \dots, k$ $V^{(i)}$ is a contiguous sequence of vertices. Note that such an order exists since the sets $V^{(i)}$ are disjoint.

* The sets V_i are disjoint, $\mathcal{V} = Z \cup V_1 \cup \cdots \cup V_{|V^\star|}$, and for $i = 1, \dots, |V^\star|$ $V_i \nsubseteq Z$.
* There are no edges between different V_i-subgraphs.
* Since the V_i-subgraphs are singletons, $\mathcal{R}(V_i) = v_i$ and the induced V_i-subgraphs are ZV-line graphs.
* For any vertex $z \in Z$, $\mathcal{B}(z, 1) \cap Z = \mathcal{V} \backslash V^\star = Z$.
* For $i = 1, \dots, |V^\star|$ $\mathcal{B}(\mathcal{R}(V_i), 1) \cap Z = N(v_i)$ which is a contiguous sequence of Z-vertices by our choice of the order.

Hence, $\mathcal{G}$ is a ZV-line graph. □

Proposition 4 *Let $K_n \backslash e$ be the outcome of removing one edge from a clique of size n. I.e., $K_n \backslash e := \langle \mathcal{V}, \mathcal{E} \rangle$ s.t. $|\mathcal{V}| = n$ and there exist two vertices $u, v \in \mathcal{V}$ s.t. $(u, v) \notin \mathcal{E}$ and $\mathcal{E} \,\dot{\cup}\, \{e\} = \binom{\mathcal{V}}{2}$. Then $\mathcal{G}$ is a ZV-line graph.*

Proof
We name the vertices $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ s.t. $(v_1, v_2) \notin \mathcal{E}$. Then by choosing $V^{\star} = \{v_1\}$, $V^{(1)} = \{v_2, \ldots, v_n\}$ and applying Prop. 3 we get that $K_n \backslash e$ is a ZV-line graph (w.r.t. $Z = \{v_2, \ldots, v_n\}$ with any order over Z and $V_1 = \{v_1\}$). □

Generalizing Proposition 4, we get the following proposition.

Proposition 5 *For $1 < m < n$, let $K_n \backslash K_m$ be the outcome of removing a clique of size m from the clique of size n. I.e.,*

$$K_n \backslash K_m := \langle \mathcal{V} = \{1, \ldots n\}, \mathcal{E} = \{(i, j) \mid i \neq j\} \setminus \{(i, j) \mid i, j \leqslant m \text{ and } i \neq j\} \rangle .$$

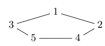
Then $\mathcal{G}$ is a ZV-line graph.

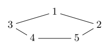
Proof
We define $V^{\star} = \{v_1, \ldots, v_m\}$, $V^{(1)} = \{v_{m+1}, \ldots, v_n\}$. We note that for any vertex $v \in V^{\star}$ $\;N(v) = V^{(1)}$ and that the graph restricted to $V^{(1)}$ is a clique of size $(n - m)$. Hence, by applying Prop. 3 we get that $K_n \backslash K_m$ is a ZV-line graph (w.r.t. $Z = \{m + 1, \ldots, n\}$ with any order over Z and each of the other m vertices being a singleton V_i-subgraph). □

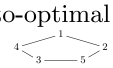
5.1 The discrete cycle over n vertices (C_n)

In this section, we characterize the discrete cycle graphs over n vertices, C_n, for which a group-manipulation-resistant, anonymous, Pareto-optimal mechanism exists. In particular, we show that for a large enough cycle ($n \geqslant 6$) there is no anonymous Pareto-optimal mechanism that is resistant even to manipulations of a single agent. On the other hand, we show that for smaller cycles one can construct group-manipulation-resistant, anonymous, Pareto-optimal mechanisms.

***C_2*, *C_3*, *C_4*:** These three graphs are ZV-line graphs: C_2 and C_3 are cliques and hence can be defined as ZV-line graphs with only Z-locations (and any order over them), C_4 is a $(2, 2)$-biclique and hence can be defined as a ZV-line graph w.r.t. taking two non-adjacent locations to be the Z-vertices (and any order over them) and two singleton V_i-subgraphs consisting of the other two locations. Moreover, these are the only group-manipulation-resistant, anonymous, Pareto-optimal mechanisms for these graphs.

***C_5*:** It is not hard to verify that a mechanism which returns the first Pareto-optimal location according to one of the following orders - 

, 

, 

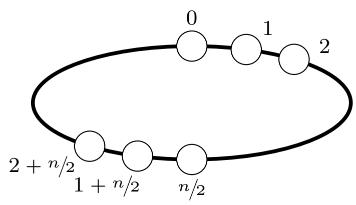
- is a group-manipulation-resistant mechanism. These three mechanisms are of the template of Def. 3 with all locations being Z-locations, but these ZV-ordered partitions do not satisfy the connectivity constraints of Def. 4 and the cycle of size 5 is not a ZV-line graph. It is a bit exhaustive but not hard to verify that these mechanisms (and their rotations and reflections) are the only group-manipulation-resistant, anonymous, Pareto-optimal mechanisms for C_5.

***C_n* for *$n \geqslant 6$*:**

Proposition 6 *For $n \geqslant 6$ there is no anonymous Pareto-optimal mechanism for C_n which is resistant even to manipulations of a single agent.*

Proof
The proof generalizes the proof for C_6 which we showed on page 16. For simplicity of notations we divide the proof into three cases:
- Cycles of even size $n \geqslant 6$,
- Large cycles (C_n for $n \geqslant 6$ except $n = 7$, $n = 11$, and $n \in \{6, 8, 10, 12, 14, 16, 20\}$),
- and last C_7 and C_{11}.

C_n for even $n \geqslant 6$:



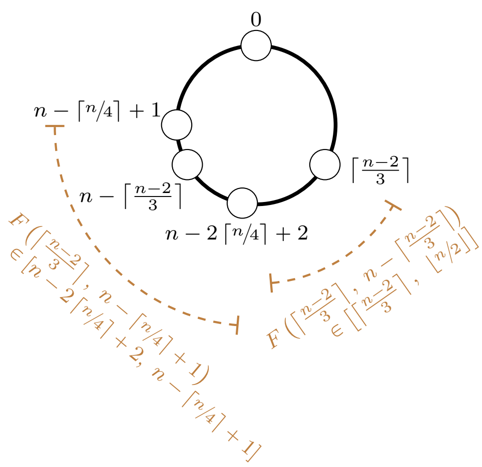
Assume towards a contradiction that F is a Pareto-optimal, anonymous, manipulation-resistant mechanism for C_n. We notate the vertices of C_n by $\{0, 1, 2, \ldots, n-1\}$, and w.l.o.g. assume that for the profile of n agents who vote for all n locations the outcome is 0.

For the profile $\langle \mathbf{2},\ \mathbf{1} + {}^{\boldsymbol{n}}\!/\mathbf{2},\ \mathbf{2} + {}^{\boldsymbol{n}}\!/\mathbf{2} \rangle$: From resistance to false-name manipulations of the first and last agents, the outcome must be either 0 or 4 (Since any of them can change the result to be 0 by adding false-ballots). From the Pareto-optimality of F, the outcome cannot be 0 which is Pareto-dominated by 4. Hence, the outcome for the profile $\langle \mathbf{2},\ \mathbf{1} + \lfloor {}^{\boldsymbol{n}}\!/\mathbf{2} \rfloor,\ \mathbf{2} + \lfloor {}^{\boldsymbol{n}}\!/\mathbf{2} \rfloor \rangle$ is 4.

Similarly, for the profile $\langle \mathbf{1},\ \mathbf{2},\ \mathbf{1} + {}^{\boldsymbol{n}}\!/\mathbf{2} \rangle$ the outcome must be 2. From false-name-resistance of $\boldsymbol{\mathcal{F}}^{\star}$, the outcome for the profile $\langle \mathbf{2},\ \mathbf{1} + {}^{\boldsymbol{n}}\!/\mathbf{2} \rangle$ must also be 2 (Otherwise, the first agent can cast an additional false-ballot 1 to get the outcome to be 2).

But, the second agent in the profile $\langle \mathbf{2},\ \mathbf{1} + {}^{\boldsymbol{n}}\!/\mathbf{2} \rangle$ (who is located on $1 + {}^{n}\!/2$) can change the outcome to be 4 which is closer to her by casting one additional false-ballot $2 + {}^{n}\!/2$. So we get a contradiction. □

C_n for $n \geqslant 6$ except $n = 7$, $n = 11$, and $n \in \{6, 8, 10, 12, 14, 16, 20\}$:



Assume towards a contradiction that F is a Pareto-optimal, anonymous, manipulation-resistant mechanism for C_n. We notate the vertices of C_n by $\{0, 1, 2, 3, \ldots, n-1\}$, and w.l.o.g. assume that for the profile of n agents who vote for all n locations the outcome is 0.

For the profile $\left\langle \left\lceil \frac{\boldsymbol{n}-\mathbf{2}}{\mathbf{3}} \right\rceil,\ \boldsymbol{n} - \left\lceil \frac{\boldsymbol{n}-\mathbf{2}}{\mathbf{3}} \right\rceil \right\rangle$: From the Pareto-optimality of F, since

$$\left(n - \left\lceil \frac{n-2}{3} \right\rceil\right) - \left\lceil \frac{n-2}{3} \right\rceil < \frac{n}{2},$$

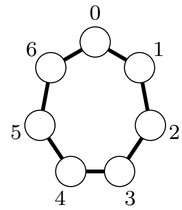
the outcome must be in $\left[\left\lceil \frac{n-2}{3} \right\rceil, n - \left\lceil \frac{n-2}{3} \right\rceil\right]$. W.l.o.g. assume it is in $\left[\left\lceil \frac{n-2}{3} \right\rceil, \lfloor {}^{n}\!/2 \rfloor\right]$.

Next, we consider the profile $\left\langle \left\lceil \frac{\boldsymbol{n-2}}{\boldsymbol{3}} \right\rceil \boldsymbol{,\ n - \lceil n/4 \rceil + 1} \right\rangle$: From the Pareto-optimality of F, since

$$(n - \lceil n/4 \rceil + 1) - \left\lceil \frac{n-2}{3} \right\rceil < \frac{n}{2},$$

the outcome must be in $\left[\left\lceil \frac{n-2}{3} \right\rceil, n - \lceil n/4 \rceil + 1\right]$. Since the second agent in this profile can change the result to be 0 by adding false-ballots, by false-name-proofness the outcome must be in $[n - 2\lceil n/4 \rceil + 2, n - \lceil n/4 \rceil + 1]$. Hence, the second agent in the profile $\left\langle \left\lceil \frac{\boldsymbol{n-2}}{\boldsymbol{3}} \right\rceil \boldsymbol{,\ n -} \left\lceil \frac{\boldsymbol{n-2}}{\boldsymbol{3}} \right\rceil \right\rangle$ (who is located on $n - \left\lceil \frac{n-2}{3} \right\rceil$) can change the outcome to be $F\left(\left\lceil \frac{n-2}{3} \right\rceil, n - \lceil n/4 \rceil + 1\right)$ which is closer to her by changing her vote to $n - \lceil n/4 \rceil + 1$. So we get a contradiction. □

<u>C_7:</u>

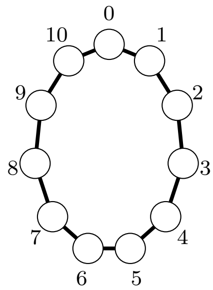


Assume towards a contradiction that F is a Pareto-optimal, anonymous, manipulation-resistant mechanism for C_7. We notate the vertices of C_7 by $\{0, 1, 2, 3, 4, 5, 6\}$, and w.l.o.g. assume that for the profile of seven agents who vote for all seven locations the outcome is 0.

For the profile $\langle \mathbf{2,\ 5} \rangle$: From resistance to false-name manipulations of the first and last agents, the outcome must be 0, 3, or 4 (Since any of them can change the result to be 0 by adding false-ballots). From the Pareto-optimality of F, the outcome cannot be 0 which is Pareto-dominated by 4. Hence, the outcome for the profile $\langle \mathbf{2,\ 5} \rangle$ is either 3 or 4. W.l.o.g. assume it is 3. By strategy-proofness, also the outcome for the profile $\langle \mathbf{3,\ 5} \rangle$ must be 3 (Otherwise, the first agent in this profile has a beneficial manipulation of voting 2).

Now, let us consider the profile $\langle \mathbf{3,\ 6} \rangle$: From the Pareto-optimality of F, the outcome cannot be 0 which is Pareto-dominated by 5. Since the second agent in this profile can change the result to be 0 by adding false-ballots, by false-name-proofness the outcome must be either 5 or 6. Hence, the second agent in the profile $\langle \mathbf{3,\ 5} \rangle$ (who is located on 5) can change the outcome to be $F(3, 6)$ which is closer to her by changing her vote to 6. So we get a contradiction. □

<u>C_{11}:</u>



Assume towards a contradiction that F is a Pareto-optimal, anonymous, manipulation-resistant mechanism for C_{11}. We notate the vertices of C_{11} by $\{0, 1, 2, 3, \ldots, 10\}$, and w.l.o.g. assume that for the profile of eleven agents who vote for all eleven locations the outcome is 0.

For the profile $\langle \mathbf{4,\ 7} \rangle$: From the Pareto-optimality of F, the outcome must lie in $\{4, 5, 6, 7\}$ (The locations 0, 1, 2, and 3 are Pareto-dominated by 5. the locations 8, 9, and 10 are Pareto-dominated by 6). W.l.o.g. we assume it is either 4 or 5.

For the profile $\langle \mathbf{4,\ 9} \rangle$: From resistance to false-name manipulations of the first and last agents, the outcome must be 0, 7, or 8 (Since any of them can change the result to be 0 by adding false-ballots). From the Pareto-optimality of F, the outcome cannot be 0 which is Pareto-dominated by 7. Hence, the outcome for the profile $\langle \mathbf{4,\ 9} \rangle$ is either 7 or 8.

Hence, the second agent in the profile $\langle \mathbf{4}, \mathbf{7} \rangle$ (who is located on 7) can change the outcome to be $F(4,9)$ which is closer to her by changing her vote to 9. So we get a contradiction. □

5.2 Recursive families of ZV-line graphs

Many graph families are defined using a recursive definition: That is, stating a base case consisting of a small initial family of simple graphs and an inductive step defining graphs in the family as a simple amalgamation of other (more basic) graphs in the family (for example, condition (c) in the definition of ZV-line graphs, Def. 4). Given a recursive family of ZV-line graphs, the mechanism $\boldsymbol{\mathcal{F}}^{\star}$ of Thm. 1 is a recursive (and hence commonly simple) mechanism that satisfies our desiderata.

Example (i)*: Rooted trees*

A simple example of a recursive family of ZV-line graphs are rooted trees which can be defined recursively as follows:

Base: A tree of height 0 is a single vertex (and it is also the root of the tree).
Step: A tree of height $h > 0$ is comprised of a vertex (the root) which is connected to the roots of a non-empty set of trees of maximal height $(h - 1)$.

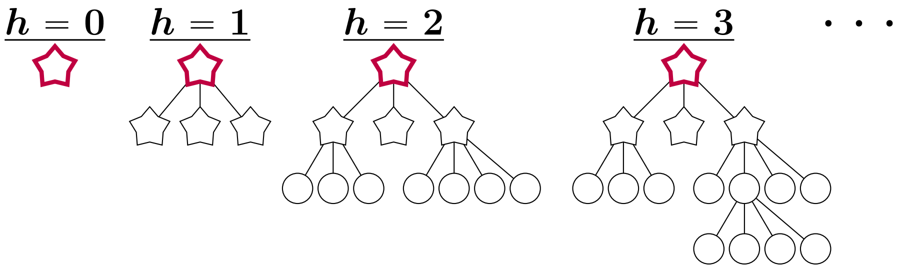


Fig. 9: Trees of height 0, 1, 2, and 3.
The new vertex (the root) added in each step is notated by .
The roots of the sub-graphs of each step are notated by .

We can see that indeed these graphs are ZV-line graphs w.r.t. the only Z-vertex being the root, by noticing that the recursive step of the definition satisfies the recursive connectivity constraint of Definition 4. Hence, we get, as a corollary of Thm. 1, that the mechanism that returns the lowest common ancestor of the ballots is an anonymous, Pareto-optimal, group-manipulation-resistant mechanism. Noting that given a tree graph, it is a rooted graph w.r.t. the root being any of the vertices, we get that any mechanism F that returns the lowest common ancestor of the ballots w.r.t. some arbitrary root r, or equivalently

$$F(\boldsymbol{x}) = \operatorname*{argmin}_{v \in PO(\boldsymbol{x})} d(v, r),$$

is an anonymous, Pareto-optimal, group-manipulation-resistant mechanism. These are also the mechanisms that Todo et al. [28] characterized as the false-name-proof, anonymous, Pareto-optimal mechanisms for the continuous tree.

Example (ii): A generalization of rooted trees

We show an anonymous, Pareto-optimal, group-manipulation-resistant mechanism for the following family of rooted graphs (that is, $\langle \mathcal{V}, \mathcal{E}, r \rangle$ s.t. $\mathcal{E} \subseteq \binom{\mathcal{V}}{2}$ and $r \in \mathcal{V}$).

Definition 6 ($\mathcal{F}$)

Base: $\langle \{v\}, \varnothing, v \rangle \in \mathcal{F}$.

Step: For any $k, \ell \geqslant 1$: If $\{\langle \mathcal{V}_i, \mathcal{E}_i, r_i \rangle\}_{i=1}^{k}$ are graphs in $\mathcal{F}$ (and the $\mathcal{V}_i$ are pairwise disjoint), then also the graph

$$\left\langle \{\widehat{r}_j\}_{j=1}^{\ell} \,\dot{\cup} \left(\dot{\bigcup_{i=1}^{k}} \mathcal{V}_i \right), \left(\dot{\bigcup_{i=1}^{k}} \mathcal{E}_i \right) \dot{\cup} \{(\widehat{r}_j, r_i)\}_{\substack{i=1\ldots k \\ j=1\ldots\ell}}, \widehat{r}_1 \right\rangle$$

is in $\mathcal{F}$. That is, adding a new layer of ℓ pre-roots, a biclique between the pre-roots and the roots of the sub-graphs, and defining the new root to be one of the pre-roots.

We'll use the notation $h(\mathcal{G})$ for the minimal number of steps needed to generate $\mathcal{G}$ and call it the complexity of $\mathcal{G}$.

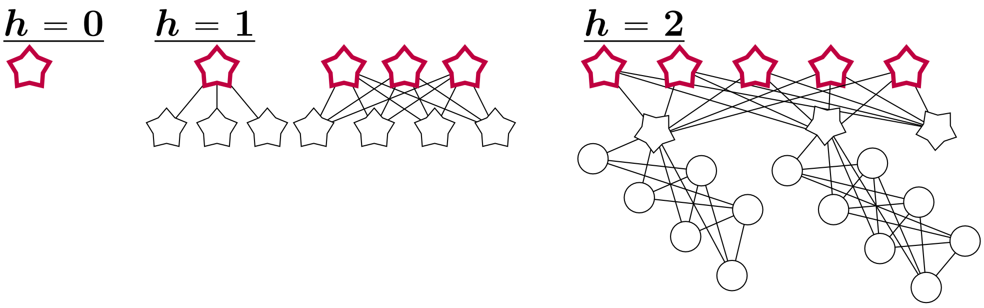


Fig. 10: Graphs of complexity 0, 1, and 2.
The new pre-roots added in each step are notated by .
The roots of the sub-graphs of each step are notated by .

We note that the graphs of complexity $h(\mathcal{G}) = 1$ are the bicliques and that by taking $\ell = 1$ in the recursive step of the definition we get the family of rooted trees. Hence, both are sub-families of this family of graphs. We can see that indeed these graphs are ZV-line graphs w.r.t. $Z = \{\widehat{r}_j\}_{j=1}^{\ell}$ being the pre-roots with any order over them and $V_i = \mathcal{V}_i$, by noticing that the recursive step of the definition satisfies the recursive connectivity constraint of Definition 4.

Our mechanism for these graphs:

- Find the subgraph G' of lowest complexity s.t. all ballots belong to G'.
- If there exists a ballot for a pre-root of G', the mechanism returns the leftmost pre-root of G' that was voted for.
- Otherwise, the mechanism returns the leftmost pre-root of G'.

Notice that the order over the pre-roots is arbitrary, so different mechanisms could arise from different choices of orders (For instance, the order might represent the social norm of the society).[11]

Example (iii): Block graphs

Our last example is *connected block graphs* [13].[12]

Definition 7 (Connected block graphs)
A connected graph $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ is a block graph if the following equivalent conditions hold:
- Every biconnected subgraph of $\mathcal{G}$ is a clique.[13]
- The intersection of any two connected subgraphs of $\mathcal{G}$ is either empty or connected.
- For every four vertices $u, v, w, x \in \mathcal{V}$, the largest two of the three distance sums

$$d(u,v) + d(w,x), \quad d(u,w) + d(v,x), \text{ and } \quad d(u,x) + d(v,w)$$

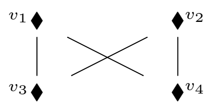
are equal.

In general, any connected graph $\mathcal{G}$ decomposes into a tree of biconnected components called the *block-cut tree* of the graph. The block-cut tree of a graph $\mathcal{G}$ is a tree $\mathcal{T}(\mathcal{G})$ which is defined in the following way. In $\mathcal{T}(\mathcal{G})$ there is a vertex (*component-vertex*) for each maximal biconnected component of $\mathcal{G}$ and a vertex (*intersection-vertex*) for each vertex in $\mathcal{G}$ which belongs to more than one maximal biconnected component. There is an edge in $\mathcal{T}(\mathcal{G})$ between each component-vertex and the intersection-vertices belonging to this component.

Hence, since for connected block graphs all maximal biconnected components are cliques, a connected block graph $\mathcal{G}$ can be represented by its block-cut tree $\mathcal{T}(\mathcal{G})$ s.t. each *component-vertex* is labeled by the size of the represented clique and each *intersection-vertex* is labeled by the indices of the represented vertex in the respective cliques. Moreover, any such labeled tree defines a (unique) block graph.[14]

Given a block graph $\mathcal{G}$, its block-cut tree $\mathcal{T}(\mathcal{G})$ induces a recursive structure decomposing $\mathcal{G}$ to smaller block graphs. Our mechanism is defined w.r.t. a choice

[11] In the version of this work which appeared on AAMAS [21, Claim 3.16] we erroneously claimed that the mechanism $F(\boldsymbol{x}) = \operatorname{argmin}_{v \in PO(\boldsymbol{x})} d(v,r)$, that returns the Pareto-optimal location closest to the root and breaks ties according to some predefined order is a group-manipulation-resistant mechanism. A counter-example for this claim is the following $(2,2)$-biclique. Assume F returns the Pareto-optimal location closest to v_3 and consider the profile $\langle v_1, v_2, v_4 \rangle$. Then the Pareto-optimal locations are $\{v_1, v_2, v_4\}$ and the Pareto-optimal locations closest to v_3 are v_1 and v_2. Assume that the mechanism returns v_1 (and the case of v_2 is symmetric). Then the agent located on v_2 can manipulate by changing her vote to v_3 and changing the outcome to be v_3 which is closer to her.



[12] We thank Ayumi Igarashi for suggesting us this family as an example.

[13] A graph is biconnected if it is a connected graph that is not broken into disconnected pieces by deleting any single vertex (and its incident edges). An equivalent definition is that a graph is biconnected if, for every pair of its vertices, it is possible to find two vertex-independent paths connecting these two vertices.

[14] This is the reason connected block graphs are also called *clique trees.*

of an arbitrary root of $\mathcal{T}(\mathcal{G})$ and arbitrary orders for every clique over its vertices. We can see to see that indeed these graphs are ZV-line graphs w.r.t. Z being the vertices of the clique which is represented by the root of $\mathcal{T}(\mathcal{G})$ and the order over them, by noticing hat the recursive step of the definition satisfies the recursive connectivity constraint of Definition 4 (In each V_i-subgraph there is exactly one vertex which is a Z-vertex of the original graph).

Our mechanism for a connected block graph $\mathcal{G}$ is:

- Find the component G' which is represented by the lowest common ancestor of the ballots.
- If one of the locations of G' was voted for, the mechanism returns the first location of G' (according to the order) that was voted for.
- Otherwise, the mechanism returns the first location of G'.

An equivalent definition of this family of mechanisms is returning the closest Pareto-optimal location to some arbitrary location v, breaking ties according to an arbitrary fixed order.

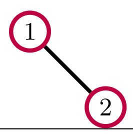
5.3 Small graphs (at most five vertices)

In this section, we show that all connected graphs with at most five vertices except C_5, the cycle of five vertices, are ZV-line graphs. As we saw in Section 5.2 there is an anonymous, Pareto-optimal, group-manipulation-resistant mechanism for C_5. Therefore, we get that for all connected graphs with at most five vertices there is an anonymous, Pareto-optimal, group-manipulation-resistant mechanism. In Section 6.1 we show this gives us such mechanisms for unconnected graphs with at most five vertices as well. Note this is no longer true for larger graphs since we showed no such mechanism exists for C_6, the cycle of six vertices.

The rest of this section is a list of all connected graphs with at most five vertices with a schematic graph showing a respective ZV-ordered partition. We use the figures ○ for V-vertices and ⓘ for Z-vertices with indexes noting the order over the Z-vertices.[15] For some of the graphs, there are also other mechanisms arising from other ZV-ordered partitions, but we chose to show only one or two ZV-ordered partitions to prove the graph is a ZV-line graph. For any of these graphs (except C_5) we are unaware of anonymous, Pareto-optimal, group-manipulation-resistant mechanisms that are not based on a representation of the graph as a ZV-line graph.

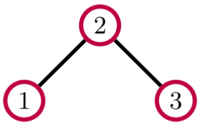
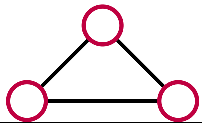
Connected graphs with two vertices:

P_2 **(line)**:

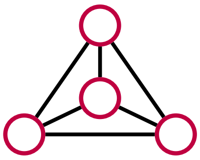
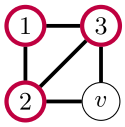
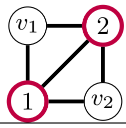
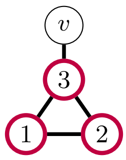
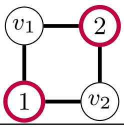
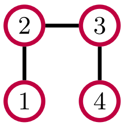
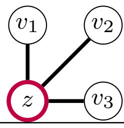


Connected graphs with three vertices (2 graphs):

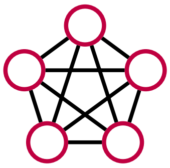
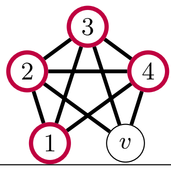
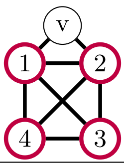
[15] We base the naming of the graphs on `https://www.graphclasses.org/smallgraphs.html`.

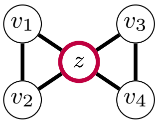
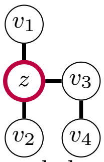
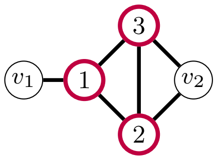
P_3 **(line)**: 	C_3 **(3-cycle)** K_3 **(3-Clique)**: 

Connected graphs with four vertices (6 graphs):

1) K_4 **(4-Clique)**/W_3: 	2) **Diamond**/$K_4 \backslash e$: (Prop. 4) $V_1 = \{v\}$ or (Prop. 3) $V_1 = \{v_1\}$ $V_2 = \{v_2\}$	3) **Paw/3-pan**: (Prop. 3) $V_1 = \{v\}$
4) $K_{2,2}$ **((2, 2)-biclique)** C_4 **(4-cycle)**: $V_1 = \{v_1\}$ $V_2 = \{v_2\}$	5) P_4 **(line)**: 	6) **Claw** $K_{1,3}$ **((1, 3)-biclique)**: $V_1 = \{v_1\}$ $V_2 = \{v_2\}$ $V_3 = \{v_3\}$

Connected graphs with five vertices (21 graphs):

1) K_5 **(5-Clique)** : 	2) $K_5 \backslash e$: (Prop. 4) $V_1 = \{v\}$	3) $\overline{P_3 \cup 2K_1}$: (Prop. 3) $V_1 = \{v\}$

<table>
<tr>
<td>4) Butterfly/Hourglass:

$V_1 = \{v_1, v_2, z\}$ and $V_2 = \{v_3, v_4, z\}$ both induced sub-graphs are K_3.</td>
<td>5) Fork/Chair:

$V_1 = \{v_1, z\}$
$V_2 = \{v_2, z\}$
$V_3 = \{v_3, v_4\}$
and the three induced sub-graphs are lines.</td>
<td>6) Co-fork/Co-chair/Kite:
(Prop. 3)

$V_1 = \{v_1\}$
$V_2 = \{v_2\}$</td>
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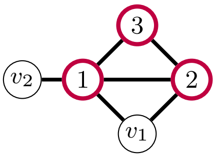
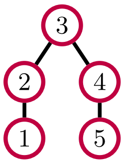
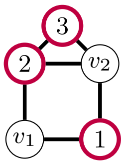
<td>7) Dart:

$V_1 = \{v_1\}$
$V_2 = \{v_2\}$</td>
<td>8) P_5 (line):
</td>
<td>9) House/$\overline{P_5}$:

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$V_2 = \{v_2\}$</td>
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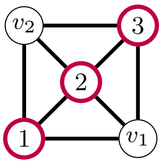
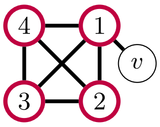
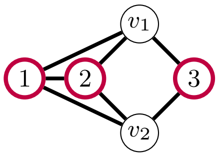
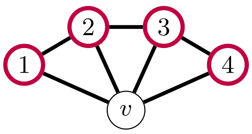
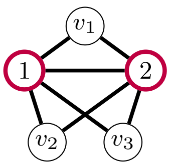
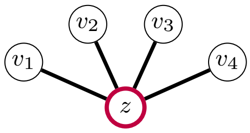
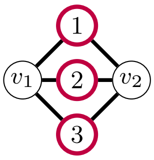
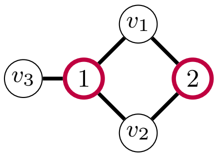
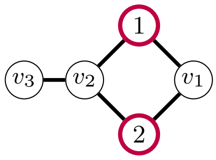
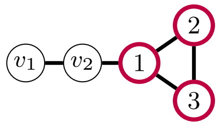
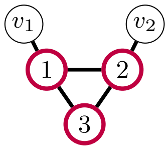
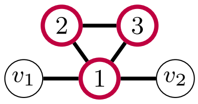
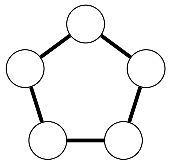
<td>10) W_4:

$V_1 = \{v_1\}$
$V_2 = \{v_2\}$</td>
<td>11) $\overline{\textbf{claw} \cup K_1}$:
(Prop. 3)

$V_1 = \{v\}$</td>
<td>12) $\overline{P_2 \cup P_3}$:

$V_1 = \{v_1\}$
$V_2 = \{v_2\}$</td>
</tr>
<tr>
<td>13) Gem/3-fan:

$V_1 = \{v\}$</td>
<td>14) $\overline{K_3 \cup 2K_1}$:
(Prop. 3)

$V_1 = \{v_1\}$
$V_2 = \{v_2\}$
$V_3 = \{v_3\}$</td>
<td>15) $K_{1,4}$ ((1, 4))-biclique:

$V_1 = \{v_1\}$ $V_3 = \{v_3\}$
$V_2 = \{v_2\}$ $V_4 = \{v_4\}$</td>
</tr>
</table>

16) $K_{2,3}$ **((2, 3))-biclique**):  $V_1 = \{v_1\}$ $V_2 = \{v_2\}$	17) **4-pan/banner/**P:  $V_1 = \{v_1\}$ $V_2 = \{v_2\}$ $V_3 = \{v_3\}$ or  $V_1 = \{v_1\}$ $V_2 = \{v_2, v_3\}$	18) $\overline{\textbf{4-pan}}$: (Prop. 3)  $V_1 = \{v_{1,}, v_2\}$ and the induced graph is a line.
19) **Bull**: (Prop. 3)  $V_1 = \{v_1\}$ $V_2 = \{v_2\}$	20) **Cricket/**$K_{1,4} + e$: (Prop. 3)  $V_1 = \{v_1\}$ $V_2 = \{v_2\}$	21) C_5 **(5-Cycle)**:  C_5 is not a ZV-line graph (See Section. 5.1)

6 Summary & Future Work

In this work, we presented a new family of graphs, ZV-line graphs, and a generic, anonymous, Pareto-optimal, group-manipulation-resistant mechanism for the facility location problem on these graphs (Thm. 1). To the best of our knowledge, the (very few) false-name-proof mechanisms which were previously known, were tailored for specific graphs and this work is the first to show a generic false-name-proof mechanism for a large family, utilizing a broad graph property and unifying all existence results which we are aware of. The construction of the mechanism is recursive (Thm 2): We derive a mechanism for a given ZV-line graph from mechanisms for its subgraphs (which might not be ZV-line graphs). Hence, it is straightforward to derive from our construction general mechanisms for recursive graph families (as exemplified in Section 5.2).

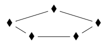
The mechanism $\boldsymbol{\mathcal{F}}^{\star}$ we presented is not the only mechanism satisfying the desired properties. Also taking any other order over the Z-locations s.t. the constraints of Def. 4 hold and defining $\boldsymbol{\mathcal{F}}^{\star}$ accordingly will satisfy them. For instance, a mechanism that outputs at the second stage of Def. 3 the rightmost Pareto-optimal Z-location instead of the leftmost, would also satisfy the desiderata. We did not find any mechanism satisfying the desiderata which is not of this template. Furthermore, unifying non-existence results we've found, we think that the ZV-ordered partition to Z-locations and V-locations is a fundamental property of a false-name-proof mechanism.

Conjecture 1

Let $\mathcal{G} = \langle \mathcal{V}, \mathcal{E} \rangle$ be a graph and let $F : \mathcal{V}^\star \to \mathcal{V}$ be an anonymous, Pareto-optimal, group-manipulation-resistant mechanism for $\mathcal{G}$. Then there exists a sequence of non-empty sets of vertices $Z, V_1, \ldots, V_k \subseteq \mathcal{V}$ s.t.

- $\mathcal{V} = Z \cup (V_1 \dot\cup \cdots \dot\cup V_k)$ is a ZV-ordered partition of $\mathcal{G}$;
- For $i = 1, \ldots, k$: Whenever $\boldsymbol{x} \in (V_i)^n$, i.e., all locations are in V_i, also $F(\boldsymbol{x}) \in V_i$;
- F is the outcome of applying Def. 3 for the mechanisms F_i which are defined by $\boldsymbol{x} \in (V_i)^n \overset{F_i}{\mapsto} F(\boldsymbol{x})$; and
- Either $\mathcal{G}$ is a ZV-line graph w.r.t. $\mathcal{V} = Z \cup (V_1 \dot\cup \cdots \dot\cup V_k)$ or the induced graph on Z is C_5 and for $i = 1, \ldots, k \quad \mathcal{R}(V_i) \in Z$.

Consequentially, showing that a given graph does not have such ZV-ordered partition-structure could be an easy and efficient way to prove non-existence of an anonymous, Pareto-optimal, group-manipulation-resistant mechanism.

The only non ZV-line graphs for which we found an anonymous, Pareto-optimal, group-manipulation-resistant mechanism are the cycle of size 5  (See Section 5.1) and graphs derived from it by the second part of Thm. 2, e.g.,

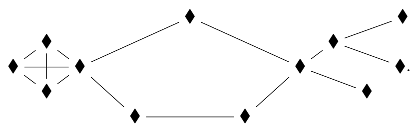


Fig. 11

We conjecture that the cycle of size 5 is a representative extreme exception and that except for very few small graphs, there are anonymous, Pareto-optimal, group-manipulation-resistant graphs only for ZV-line graphs.

6.1 Relaxing the assumptions of the model

Three assumptions we had are connectivity of the graph, a finite number of agents, and a finite number of locations. One could define mechanisms for unconnected graphs, infinite graphs, or an infinite number of agents (both countable and uncountable). The definitions of the desiderata are extended naturally to deal with these scenarios (while constraining the profiles, manipulations, coalitions to be measurable functions or sets).

For **unconnected graphs**, if each connected component is a ZV-line graph, the following mechanism generalizes $\boldsymbol{\mathcal{F}}^\star$ and it satisfies the desiderata:

- ▶ At the first stage, choose the first connected component according to some predefined order s.t. at least one agent voted for a location in this component.
- ▶ At the second stage, run $\boldsymbol{\mathcal{F}}^\star$ taking into account only ballots in the chosen component.

Note that, just like the mechanism for the connected case, also this mechanism can be equivalently defined as the first Pareto-optimal location according to some order over the locations: The concatenation of the respective orders of the different components.

Recall that in Section 5.3 we saw that there exist anonymous, Pareto-optimal, group-manipulation-resistant mechanisms for all connected graphs of up-to five vertices. Hence, we get by the above insight anonymous, Pareto-optimal, group-manipulation-resistant mechanisms for all graphs of up-to five vertices (both connected and unconnected). This is no longer true for larger graphs since we showed no such mechanism exists for the cycle of six vertices.

For the scenario of an **infinite number of agents**, we get that $\boldsymbol{\mathcal{F}}^{\star}$ still satisfies the desiderata (using the same proof).

For dealing with **infinite graphs**, we need to extend Def. 2 of ZV-ordered partitions and add a requirement that the linear order over the Z-locations is a well-order, that is, that the leftmost location is defined for any (measurable) subset of Z-locations. Adding this assumption, our results are extended (using the same proof) to show that $\boldsymbol{\mathcal{F}}^{\star}$ satisfies the desiderata for infinite graphs as well (with either finite or infinite number of agents). Note that without the well-order assumption, $\boldsymbol{\mathcal{F}}^{\star}$ might not be well-defined even when the number of agents is finite. For instance, consider the following ZV-line graph

$$V = \{v_1, v_2\} \quad ; \quad Z = \mathbb{Z} \quad ; \quad \mathcal{E} = V \times Z$$

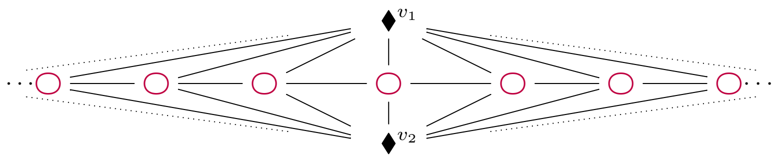


Fig. 12

Then for the profile $\{v_1, v_2\}$ the leftmost Pareto-optimal Z-location is not defined.

6.2 Approximate Mechanism Design without Money

Last, an important continuation of this work is analyzing the implications for *approximate mechanism design without money* [23]. That is, assuming the agents are accurately represented by a cost function (e.g., the distance to the facility or a monotone function of the distance) and analyzing the implications of manipulation-resistance on the approximability of the minimization problem of natural social cost functions, e.g., the average cost (Harsanyi's social welfare), the geometric mean of the costs (Nash's social welfare), or the maximal cost (Rawls' criterion). For instance, assuming the conjecture above, one gets that whenever there is a large disagreement in the population (i.e., the agents are dispersed over many V_i-subgraphs) an extreme status-quo alternative must be chosen by the mechanism, which results in a bad *price of false-name-proofness* (roughly, the number of agents times the diameter of the graph). Nowadays, many aggregation mechanisms are highly susceptible to double-voting and to false-name voting in general (e.g., mechanisms over huge anonymous networks like the internet, but also other scenarios in which vote frauds are known to be easy). We think that such results should open a discussion on the costs of these protocols (since the benefits are clear).

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