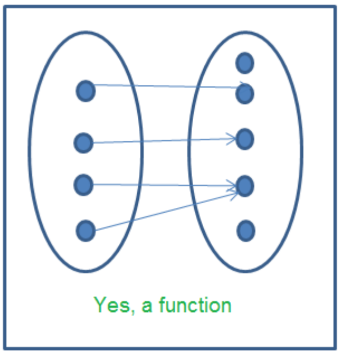
Mathematics for Economists (66110)

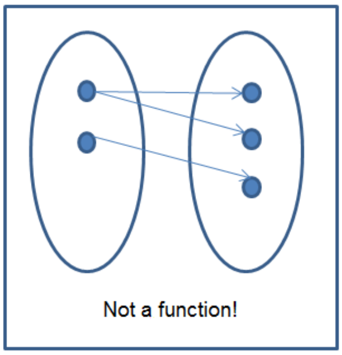
Lecture Notes 3

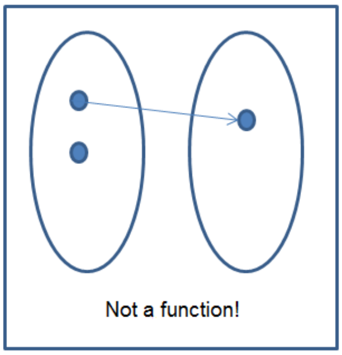
Functions

Elementary Definitions

Definition: Let A and B be two sets. f is a function from A to B, denoted $f : A \to B$, if every element of A is associated by f to one and only one element of B. One write $f(x) = y$ when f associates $\mathrm{x} \in \mathrm{A}$ with $y \in B$.





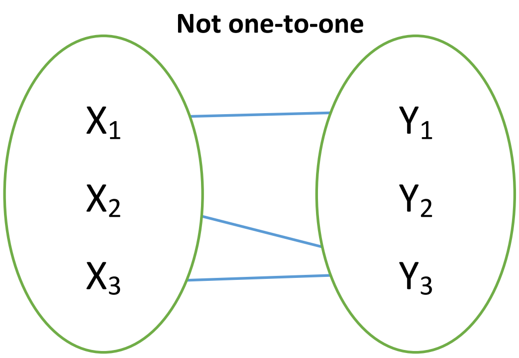


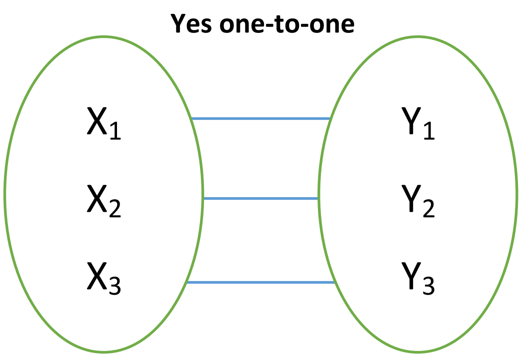
Definition: If $\mathrm{f}: \mathrm{A} \to \mathrm{B}$ is a function, then A is called the domain of the function and B is the range of the function. An element $\mathrm{x} \in \mathrm{A}$ is called a source element. If $f(x) = y$ then y is called an image element under f.

If f is a function, each element in the domain is associated with one and only one element in the range, but the converse is not true: an image element may be related by f to two or more source elements.

For example, consider the function associating each person with his or her height in centimetres. This is a well-defined function from people to $\mathbb{R}_+$, since each one of us is associated this way with one and only one number. However, it is very likely that for a given height there are several people who share that height. There are also values in the range $\mathbb{R}_+$ that are not associated with any person (e.g., 100 metres).

Definition: A function is one-to-one, or injective, if every image element is associated by the function to only one source element.





Definition: A function f is onto (surjective) if every element of the range of f is an image element.

Definition: If a function f is both one-to-one and onto then it is called bijective.

Definition: Two functions f and g are equal if

1. Both f and g have the same domain
2. For every x in this domain, $f(x) = g(x)$

Example:

$$g: \mathbb{R}_+ \to \mathbb{R}_+ \qquad f: \mathbb{R}_+ \to \mathbb{R}$$

$$g(x) = \frac{\sqrt{x+2}}{x+1}, \ f(x) = \frac{\sqrt{x+2}}{x+1}$$

$$\text{then } f(x) = g(x)$$

Examples:

1. $f(x) = ax$ is one-to-one and onto, as a function $f: \mathbb{R} \to \mathbb{R}$, if $a \neq 0$.
2. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. This function is not one-to-one. However, if we restrict the domain of f to $\mathbb{R}_+$ then this function is one-to-one.
3. $f(x) = \begin{Bmatrix} 2x-3, x<2 \\ 5x, x \geq 2 \end{Bmatrix}$. This function is one-to-one but not onto, since it fails to take values between 1 and 10.
4. $f(x) = \begin{Bmatrix} 2x-3, x<2 \\ 5x-10, x \geq 2 \end{Bmatrix}$. This function is not one-to-one but is onto.

Definition: If $f: A \to B$ is bijective then f has an inverse. The inverse of f, denoted $f^{-1}: B \to A$, is the function from B to A such that every $y \in B$ is associated by f^{-1} to the source of *y* in A, that is, $f^{-1}(y) = x \Leftrightarrow f(x) = y$.

Why is it important in this definition that f is bijective?

If f is not surjective (i.e., not onto) then there are elements of B that are not image elements of any source element in A. If $y \in B$ is such an element then $f^{-1}(y)$ will not be defined, and then f^{-1} will not be a well-defined function.

If f is not injective (i.e., not one-to-one) then there are image elements in B that are associated with two or more source elements. If $y \in B$ is such an element then which of these source elements should be $f^{-1}(y)$?

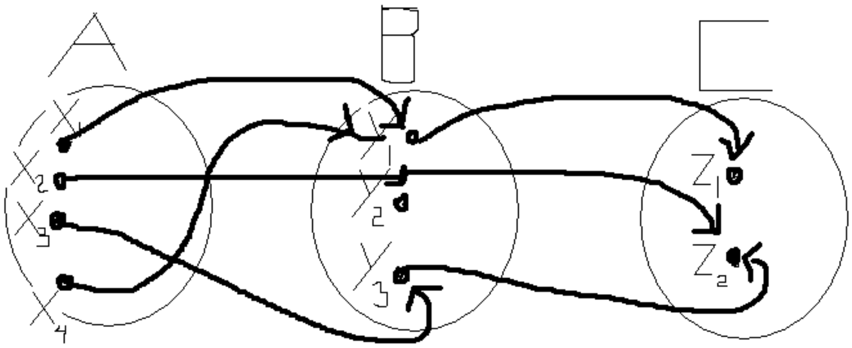
Note: the inverse of f^{-1} is f.

Examples:

1. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$ with $a \neq 0$ is bijective. If $ax + b = y$ then $x = (y-b)/a$, which gives us the inverse function $f^{-1}(y) = (y-b)/a$.

2. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. This function is surjective. Setting $y = x^3$ then one has $x = \sqrt[3]{y}$, yielding the inverse function $f^{-1}(y) = \sqrt[3]{y}$.
3. $f; \mathbb{R}+ \to \mathbb{R}+$, $f(x) = x^2$. This function is surjective (since the domain is restricted to $\mathbb{R}+$). Setting $y = x^2$ yields $x = +\sqrt{y}$, implying $f^{-1}(y) = \sqrt{y}$.
4. $f: \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$,$f(x) = \frac{1}{x}$. This function is surjective. Setting $y = \frac{1}{x}$ yields $x = \frac{1}{y}$, hence $f^{-1}(y) = \frac{1}{y}$. In other words, the function is really its own inverse.

<u>Definition</u>: Given two functions $f: A \to B$ and $g: B \to C$, the composition function g of f is the function $h: A \to C$ that is given by $h(x) = g(f(x))$ for each x in A.



In the illustration, $f(x_1) = g(f(x_1)) = g(y_1) = z_1$.

Examples:

1. $f(x) = x + 8$, $g(x) = \frac{1}{x+3}$, $g, f: \mathbb{R} \to \mathbb{R}$. Given that the domains and ranges of both functions are identical ($\mathbb{R}$), we can form both $f(g(x))$ and $g(f(x))$. This is not always true. For example, consider $f: \mathbb{R}_+ \to \mathbb{R}$, $f(x) = \ln(x)$, $g: \mathbb{R} \to \mathbb{R}$, $g(x) = x - 2$ *, in this case we cannot form the composition* *f(g(x))* because if we try to do this we get the *ln* of a negative number.
2. $f: \mathbb{R} \to \mathbb{R}_+$, $f(x) = e^x$, $g: \mathbb{R}_+ \to \mathbb{R}_+$, $g(x) = \sqrt{x}$*,* then setting $h(x) = g(f(x))$, where $h: \mathbb{R} \to \mathbb{R}_+$, yields $h(x) = g(f(x)) = g(e^x) = \sqrt{e^x}$.
3. Given

   $f(x) = \begin{Bmatrix} \frac{1}{x}, & x > 0 \\ x^2, & x \le 0 \end{Bmatrix}$ and $g(x) = \begin{Bmatrix} x + 3, x > 4 \\ 3x, x \le 4 \end{Bmatrix}$ yields the compositions:

$$f(g(x)) = \begin{Bmatrix} f(x + 3), x > 4 \\ f(3x), x \le 4 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{x+3}, x > 4 \\ \frac{1}{3x}, \quad (0,4] \\ (3x)^2 = 9x^2, x \le 0 \end{Bmatrix}$$

and

$$g(f(x)) = \begin{Bmatrix} g(1/x), x > 0 \\ g(x^2), x \le 0 \end{Bmatrix} = \begin{Bmatrix} 1/x + 3, 0 < x < 0.25 \\ 3/x, x > .25 \\ 3x^2, [-2,0] \\ x^2 + 3, x < -2 \end{Bmatrix}$$

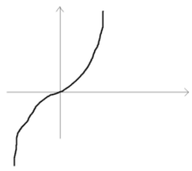
<u>Definition</u>: A function whose range is the set of real numbers is called a real-valued function.

A real-valued function f is weakly monotonically increasing (also called non-decreasing) if $x_1 < x_2$ implies that $f(x_1) \le f(x_2)$. The function f is strictly monotonically increasing if $x_1 < x_2$ implies that $f(x_2) > f(x_1)$.

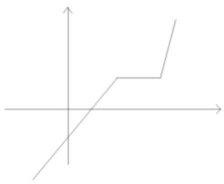
A real-valued function f is weakly monotonically decreasing (also called non-increasing) if $x_1 < x_2$ implies that $f(x_1) \ge f(x_2)$. The function f is strictly monotonically decreasing if $x_1 < x_2$ implies that $f(x_2) < f(x_1)$.

Examples:

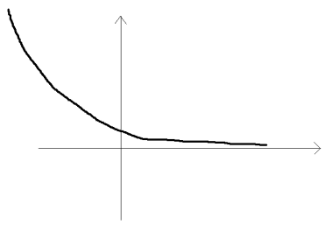
$f(x) = x^3$ is strictly increasing.



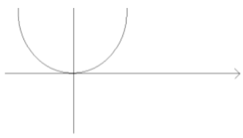
This graph is weakly monotonically increasing.



This graph is monotonically decreasing.



This graph is neither increasing nor decreasing.



Claim: If $g(x)$,$f(x)$ are both weakly monotonically increasing functions (i.e., non-decreasing) then

1. $f(\mathrm{x}) + \mathrm{g}(\mathrm{x})$ is also weakly monotonically increasing

2. $a \cdot f(x)$: if a>0 then this is a weakly monotonically increasing function, if a<0 then this function is a weakly monotonically decreasing function.
3. $f(x) - g(x)$: in this case, we cannot tell (without checking the details of the functions) whether this is an increasing or decreasing function.
4. $f(g(x))$ is monotonically increasing. Proof: let $\mathrm{x}_2 > \mathrm{x}_1$. Since g is increasing, we get $g(x_1) \leq g(x_2)$. Denote $y_2 = g(x_2)$, $y_1 = g(x_1)$. Then if $y_2 \geq y_1$, we get $f(y_2) \geq f(y_1)$, since f is monotonically increasing. We conclude that if $x_2 > x_1$ then $f\left(g(x_1)\right) \leq f(g(x_2))$, which is the same as stating that $f(g(x))$ is increasing.