## Lecture Note 2

# Numerical Dynamic Programming in Economics 

From Handbook of Computational Economics
by Amman, H., Kendrick, D., and J. Rust (editors)

## 1 Plan of lecture note

1. Introduction
2. Review of MDP's and the Theory of Dynamic Programming

- Definitions of MDP's, DDP's and CDP's
- Bellman's Equation, Contractions, Blackwell's Theorem
- Analytic Solutions to "Test Problems"
- Euler Equations and Euler Operators

3. Computational Complexity and Optimal Algorithms

- Discrete Computational Complexity
- Continuous Computational Complexity

4. Numerical Methods for MDP's

- Discrete Finite Horizon
- Discrete Infinite Horizon
- Continuous Finite Horizon
- Continuous Infinite Horizon

5. Conclusions

## 2 Review of MDP's and the Theory of Dynamic Programming

Definition 2.1: A (discrete-time) Markovian Decision Process (MDP) consists of the following objects:

- A state space $S$
- An action space $A$
- A family of constraint sets $\{A(s) \subseteq A\}$
- A transition probability $p\left(d s^{\prime} \mid s, a\right)$
- A discount factor $\beta \in(0,1)$
- A single period utility function $u(s, a)$

The individual's problem at time $t=0$ is then

$$
\begin{equation*}
V_{0}(s)=\max _{\alpha=\left\{\alpha_{0}, \ldots, \alpha_{T}\right\}} E_{a}\left\{\sum_{t=0}^{T} \beta^{t} u\left(s_{t}, \alpha_{t}\right) \mid s_{0}=s\right\} . \tag{1}
\end{equation*}
$$

## Discrete MDP's versus Continuous MDP's

Definition 2.2: A Discrete Decision Process (DDP) is an MDP with the following property:

- There is a finite set $A$ such that $A(s) \subset A$ for each $s \in S$.

Definition 2.3: A Continuous Decision Process (CDP) is an MDP with the following property:

- For each $s \in S$, the action set $A(s)$ is a compact subset of $R^{d_{a}}$, with non-empty interior.

Remark 1: Important distinction between CDP's and DDP's:

- Solutions to DDP's can be computed exactly (provided state space is finite), CDP's can only be approximated.
- CDP's are subject to an inherent curse of dimensionality, but can break the curse of dimensionality for DDP's.
- Approximate solution methods might also be attractive for very large DDP problems even though exact solution methods exist.

Remark 2: The CDP/DDP dichotomy is mirrored by two fundamental approaches to approximation of solutions to MDP's:

- Discrete Approximation Methods
- Continuous Approximation Methods

Remark 3: There is substantial controversy in the economics literature over the "best" approximation method:

- The proliferation of different algorithms and controversies over which approach is "best" is similar to controversies that have arise in statistics and econometrics about which estimations methods are "best". The computational problem is akin to nonparametric econometrics: which is "best" way to "estimate" (i.e., compute) unknown functions $V, \alpha$ ?
- This chapter appeals to computational complexity theory to help shed new light on these controversies.
- Complexity theory is in principle capable of characterizing optimal algorithms for solving MDP's.


## 3 Computational Complexity and Optimal Algorithms

Optimal strategies for finding optimal strategies.

- Discrete Computational Complexity
- Turing model of computation, exact solutions to discrete, finite-dimensional problems (matrix multiplication, LP, etc.).
$-\operatorname{comp}(n)$ minimal computer time to solve a problem of size $n$. Example, matrix multiplication $\operatorname{comp}(n)=O\left(n^{2.376}\right)$.
- Outstanding problems: $P=N P$ and $P=N C$ problems.
- Continuous Computational Complexity
- Real model of computation, $\varepsilon$-approximations to continuous infinite-dimensional problems (e.g. PDE's, integration, optimization, etc.).
$-\operatorname{comp}(\varepsilon, d)$ minimal time to compute an $\varepsilon$-approximation of a $d$-dimensional problem.
- Tight upper and lower bounds on complexity of many problems have been established.
- Enables us to formalize the curse of dimensionality:

Definition: A class of discrete MDP problems with $d_{s}$ state variables and $d_{a}$ control variables is subject to the curse of dimensionality if $\operatorname{comp}\left(d_{s}, d_{a}\right)=\Omega\left(2^{d_{s}+d_{a}}\right)$. A class of continuous MDP problems is subject to the curse of dimensionality if $\operatorname{comp}\left(\varepsilon, d_{s}, d_{a}\right)=\Omega\left(1 / \varepsilon^{d_{s}+d_{a}}\right)$.

## 4 Numerical Methods for MDP's

### 4.1 Dynamic Programming Solution, Finite Horizon Problems

$$
\begin{align*}
& \alpha_{t}\left(s_{t}\right)=\arg \max _{a_{t} \in A\left(s_{t}\right)}\left\{u\left(s_{t}, a_{t}\right)+\beta \int V_{t+1}\left(s_{t+1}\right) p\left(d s_{t+1} \mid s_{t}, a_{t}\right)\right\},  \tag{2}\\
& V_{t}\left(s_{t}\right)=\max _{a_{t} \in A\left(s_{t}\right)}\left\{u\left(s_{t}, a_{t}\right)+\beta \int V_{t+1}\left(s_{t+1}\right) p\left(d s_{t+1} \mid s_{t}, a_{t}\right)\right\} . \tag{3}
\end{align*}
$$

### 4.2 Dynamic Programming Solution, Infinite Horizon Problems

$$
\begin{align*}
& \alpha(s)=\arg \max _{a \in A(s)}\left\{u(s, a)+\beta \int V\left(s^{\prime}\right) p\left(d s^{\prime} \mid s, a\right)\right\},  \tag{4}\\
& V(s)=\max _{a \in A(s)}\left\{u(s, a)+\beta \int V\left(s^{\prime}\right) p\left(d s^{\prime} \mid s, a\right)\right\} . \tag{5}
\end{align*}
$$

Bellman's equation in operator form:

$$
V=\Gamma(V),
$$

where $\Gamma: B \rightarrow B$ is the Bellman operator defined by

$$
\begin{equation*}
\Gamma(V)(s)=\max _{a \in A(s)}\left\{u(s, a)+\beta \int V\left(s^{\prime}\right) p\left(d s^{\prime} \mid s, a\right)\right\} \tag{6}
\end{equation*}
$$

## Contraction Mapping Theorem:

$\Gamma$ is a contraction mapping with unique fixed point $V \in B$.

$$
\begin{equation*}
\|\Gamma(V)-\Gamma(W)\| \leq \beta\|V-W\| . \tag{7}
\end{equation*}
$$

Lemma 2.1: Suppose $\left\{\Gamma_{N}\right\}$ converge pointwise to $\Gamma$, i.e., for all $W \in B$ we have

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \Gamma_{N}(W)=\Gamma(W) \tag{8}
\end{equation*}
$$

Then the approximate fixed point $V_{N}=\Gamma_{N}\left(V_{N}\right)$ satisfies the bound:

$$
\begin{equation*}
\left\|V_{N}-V\right\| \leq \frac{\left\|\Gamma_{N}(V)-\Gamma(V)\right\|}{1-\beta} \tag{9}
\end{equation*}
$$

Lemma 2.2: Let $\Gamma$ be a contraction mapping with fixed point $V=\Gamma(V)$. For each $W$ the following inequalities hold:

$$
\begin{align*}
& \|W-\Gamma(W)\| \leq(1+\beta)\|V-W\|  \tag{10}\\
& \left\|\Gamma^{t}(W)-V\right\| \leq \beta^{t} \frac{\|\Gamma(W)-W\|}{1-\beta} \tag{11}
\end{align*}
$$

and

$$
\left\|\Gamma^{t+1}(W)-V\right\| \leq \beta\left\|\Gamma^{t}(W)-V\right\| .
$$

Lemma 2.3: Let $\Gamma$ be a Bellman operator. Then we have:

$$
\Gamma: B\left(0, \frac{K}{1-\beta}\right) \rightarrow B\left(0, \frac{K}{1-\beta}\right)
$$

where $B(0, r)=\{V \in B:\|V\| \leq r\}$ and $K$ is given by:

$$
K=\sup _{s \in S} \sup _{a \in A(s)}|u(s, a)| .
$$

## Infinite Horizon MDP's as Generalized Geometric Series:

$$
\begin{align*}
V_{\alpha} & =\left[I-\beta M_{a}\right]^{-1} u_{a}  \tag{12}\\
& =u_{\alpha}+\beta M_{\alpha} u_{\alpha}+\beta^{2} M_{\alpha}^{2} u_{\alpha}+\beta M_{\alpha}^{3} u_{\alpha}^{3}+\cdots,
\end{align*}
$$

where $M_{a}$ is the Markov operator defined by:

$$
\begin{equation*}
M_{\alpha}(V)(s)=\int V\left(s^{\prime}\right) p\left(d s^{\prime} \mid s, \alpha(s)\right) . \tag{13}
\end{equation*}
$$

## Solution Algorithm for Contraction Mappings:

- Successive Approximations

$$
V_{k}=\Gamma\left(V_{k-1}\right)=\Gamma^{k}\left(V_{0}\right) .
$$

- Newton-Kantorovich iterations

$$
\begin{equation*}
V_{k+1}=V_{k}-\left[I-\Gamma^{\prime}\left(V_{k}\right)\right]^{-1}(I-\Gamma)\left(V_{k}\right) . \tag{14}
\end{equation*}
$$

## Fundamental Trade-offs:

1. SA is linearly convergent, NK is quadratically convergent.
2. SA is cheap per iteration, NK is expensive per iteration.
3. SA is globally convergent, NK is locally convergent.

Example: Smooth Bellman operator, $\Gamma_{\sigma}$

$$
\begin{aligned}
\Gamma_{\sigma}(V)(s) & =\sigma \log \left[\sum_{a \in A(s)} \exp \left\{\frac{1}{\sigma}\left[u(s, a)+\beta \int V\left(s^{\prime}\right) p\left(d s^{\prime} \mid s, a\right)\right]\right\}\right] \\
\Gamma_{\sigma}^{\prime}(V)(W)(s) & =\beta \sum_{a \in A(s)} \int W\left(s^{\prime}\right) p\left(d s^{\prime} \mid s, a\right) p(a \mid s)
\end{aligned}
$$

where $p(a \mid s)$ is the conditional choice probability given by

$$
p(a \mid s)=\frac{\exp \left\{\frac{1}{\sigma}\left[u(s, a)+\beta \int V\left(s^{\prime}\right) p\left(d s^{\prime} \mid s, a\right)\right]\right\}}{\sum_{a^{\prime} \in A(s)} \exp \left\{\frac{1}{\sigma}\left[u\left(s, a^{\prime}\right)+\beta \int V\left(s^{\prime}\right) p\left(d s^{\prime} \mid s, a^{\prime}\right)\right]\right\}} .
$$

## Solution Algorithm for Discrete MDP's:

- General iterative methods Gauss-Seidel, Successive Over-relaxation, etc. from literature on nonlinear equations.
- Successive Approximations.
- Accelerated Successive Approximations

$$
\begin{equation*}
\Gamma^{k}(V)+\underline{b}_{k} e \leq V \leq \Gamma^{k}(V)+\bar{b}_{k} e, \tag{15}
\end{equation*}
$$

where

$$
\begin{align*}
\underline{b}_{k} & =\frac{1}{1-\beta} \min \left[\Gamma^{k}(V)-\Gamma^{k-1}(V)\right], \quad \text { and } \\
\bar{b}_{k} & =\frac{1}{1-\beta} \max \left[\Gamma^{k}(V)-\Gamma^{k-1}(V)\right] . \tag{16}
\end{align*}
$$

- Linear Programming (with Constrain Generation)

$$
\min _{V \in R^{|S|}} \sum_{s=1}^{|S|} V(s),
$$

subject to:

$$
V(s) \geq u(s, a)+\beta \sum_{s^{\prime}=1}^{|S|} V\left(s^{\prime}\right) p\left(d s^{\prime} \mid s, a\right) .
$$

- Policy iteration

$$
\begin{align*}
& \alpha_{t}(s)=\arg \max _{a \in A(s)}\left\{u(s, a)+\beta \sum_{s^{\prime}=1}^{|S|} V_{\alpha_{t-1}}\left(s^{\prime}\right) p\left(d s^{\prime} \mid s, a\right)\right\},  \tag{17}\\
& V_{\alpha_{t-1}}=\left[I-M_{\alpha_{t}}\right]^{-1} u_{\alpha_{t}} . \tag{18}
\end{align*}
$$

- Modified Policy Iteration solve (5.7) approximately using successive approximations:

$$
V_{\alpha_{t-1}} \simeq\left[\sum_{i=0}^{k} \beta^{i} M_{\alpha_{t}}^{i}\right] u_{\alpha_{t}} .
$$

- Policy Iteration Using State Aggregation Methods.
- Massively Parallel Policy Iteration Pan/Reif idea: use Newton's method to invert $L=\left[I-M_{\alpha}\right]$ :

$$
\begin{aligned}
F(X) & =[I-L X]=0 \\
L_{k} & =\left(2 I-L_{k-1} M\right) L_{k-1}
\end{aligned}
$$

Theorem: Suppose that the initial estimate $L_{0}$ of $L^{-1}$ satisfies $\left\|I-L_{0} L\right\|<1$. Then a massive parallel processor with $O\left(|S|^{\omega}\right)$ processors $(\omega \leq 2.376)$ can find an approximate solution $L_{k}$ satisfying the convergence criterion $\left\|L_{k}-L^{-1}\right\| \leq 2^{-n^{c}}\left\|L_{0}\right\|$ in $O\left(\log \left(|S|^{2}\right)\right)$ time.

Conjecture: Let $\varepsilon>0$ be an arbitrary solution tolerance between an approximate solution $\widehat{V}$ and the true solution $V=\Gamma(V)$ to an $|S|$-state infinite horizon MDP problem. Then the massively parallel policy iteration algorithm using the Pan-Reif algorithm to approximately solve each policy valuation step generates an approximate solution $\widehat{V}$ satisfying $\|\widehat{V}-V\|<\varepsilon$ in at most in $O\left(\log (\log (1 / \varepsilon)) \log (|S|)^{2}\right)$ time steps using a massively parallel processor with $O\left(|S|^{\omega}\right)$ processors.

- Policy Iteration using Minimum Residual Algorithm

$$
\begin{align*}
u & =L V, \\
L & =[I-\beta M], \tag{19}
\end{align*}
$$

with

$$
\widehat{V}_{k}=\sum_{i=1}^{k} c_{i}^{*} L^{i-1} u
$$

and

$$
\left(c_{1}^{*}, \ldots, c_{k}^{*}\right)=\arg \min _{\left(c_{1}, \ldots, c_{k}\right) \in R^{k}}\left\|u-\sum_{i=1}^{k} c_{i} L^{i-1} u\right\| .
$$

Note that MR algorithm uses Krylov information:

$$
I_{k}(L, u)=\left(u, L u, L^{2} u, \ldots, L^{k} u\right)
$$

Theorem: The MR algorithm using Krylov information is almost optimal for solving system (5.32) for the class $F=\{L: L=I-\beta M,\|M\| \leq 1\}$ in the sense that the value $k_{M R}$ is nearly equal to the minimal level needed to get an $\varepsilon$-approximation to the solution $V$.

$$
k_{M R}(\varepsilon, F)=\min \left(|S|,\left\lfloor\frac{\log (\varepsilon)}{\log (\beta)}(1-\delta)\right\rfloor+1\right) .
$$

Observation: If we do not need extremely high precision, then if $k_{M R}(\varepsilon, F) \leq|S|$, and if $L$ is sparse with $O(|S|)$ non-zeros, then MR algorithm will find approximate solution $\widehat{V}$ in $O(|S|)$ versus $O\left(|S|^{3}\right)$ steps used by standard Gaussian elimination to solve linear system (5.32).

Fundamental trade-off: Use tailor-made algorithms that exploit special structure of a given problem, or general purpose algorithms that do not take advantage of special structure of problem?

### 4.3 Approximate Solution Methods for Continuous MDP's

- Discrete Approximation Methods: Approximate $V$ via an expanding sequence of approximate finite-dimensional fixed points

$$
V \simeq V_{N}=\Gamma_{N}\left(V_{N}\right)
$$

where $\Gamma_{N}: R^{N} \rightarrow R^{N}$ is defined over a finite grid $\left\{s_{1}, \ldots, s_{N}\right\}$ of points in $S$ :

$$
\Gamma_{N}(V)\left(s_{j}\right)=\max _{a \in A\left(s_{j}\right)}\left\{u\left(s_{j}, a\right)+\frac{\beta}{N} \sum_{i=1}^{N} V\left(s_{i}\right) \widehat{p}\left(s_{i} \mid s_{j}, a\right)\right\} .
$$

- Smooth Approximation Methods: Approximate $V$ via an expanding sequence of finite dimensional manifolds in $B$, i.e.,

$$
V \simeq V_{\theta}=g_{k}(\theta)
$$

for some smooth mapping $g_{k}: R^{k} \rightarrow B$. Choose $\widehat{\theta}$ so that $V_{\widehat{\theta}}$ is "as close as possible" to the true fixed point $V$.

$$
\begin{align*}
V_{\theta}(s) & =\sum_{i=1}^{k} \theta_{i} p_{i}(s)  \tag{20}\\
\sigma(\theta) & =\left\|V_{\theta}-\Gamma\left(V_{\theta}\right)\right\| \equiv \sqrt{\int\left|V_{\theta}(s)-\Gamma\left(V_{\theta}\right)(s)\right|^{2} \mu(d s)}  \tag{21}\\
\sigma_{N}(\theta) & \equiv \sqrt{\sum_{i=1}^{N}\left|V_{\theta}\left(s_{i}\right)-\widehat{\Gamma}\left(V_{\theta}\right)\left(s_{i}\right)\right|^{2}} \tag{22}
\end{align*}
$$

## How to Choose Grid Points for DA methods?

- Uniform Grid Points (Chow and Tsitsiklis, 1991) Assume $S=[0,1]^{d_{s}}$ and $A(s)=[0,1]^{d_{a}}=A$. Partition $S$ and $A$ into equal sub-cubes of length $h$ on each side. Then have $N_{s}=(1 / h)^{d_{s}}$ discrete states and $N_{a}=(1 / h)^{d_{a}}$ discrete actions. Define discretized utility and transition probability by:

$$
\begin{aligned}
u_{h}(s, a) & =u\left(s_{k(s)}, a\right) \\
p_{h}\left(s^{\prime} \mid s, a\right) & =\frac{p\left(s_{k\left(s^{\prime}\right)} \mid s_{k(s)}, a\right)}{\int p\left(s_{k\left(s^{\prime}\right)} \mid s_{k(s)}, a\right) d s^{\prime}},
\end{aligned}
$$

where $s_{k(s)}$ is an arbitrary "representative" from sub cube $k(s)$ containing point $s \in S$. Define
approximate Bellman operator $\Gamma_{h}$ by:

$$
\begin{aligned}
\widehat{\Gamma}_{h}(V)(s) & =\max _{a_{k}}\left\{u_{h}\left(s, a_{k}\right)+\beta \int V\left(s^{\prime}\right) p_{h}\left(s^{\prime} \mid s, a_{k}\right) d s^{\prime}\right\} \\
& =\max _{a_{k}}\left\{u_{h}\left(s, a_{k}\right)+\beta \sum_{k=1}^{N} \int V\left(s_{k}\right) p_{h}\left(s_{k} \mid s_{k(s)}, a_{k}\right) d s^{\prime}\right\} .
\end{aligned}
$$

Theorem: There exist constants $K_{1}$ and $K_{2}$ such that for all $h$ sufficiently small and all $V \in B$ we have:

$$
\left\|\Gamma(V)-\widehat{\Gamma}_{h}(V)\right\| \leq\left(K_{1}+\beta K_{2}\|V\|\right) h .
$$

Note this method is subject to the "curse of dimensionality": The number of operations to find $\varepsilon$-approximation to $V$ is $O\left(T /\left((1-\beta)^{2} \varepsilon\right)^{\left(2 d_{s}+d_{a}\right)}\right)$.

- Quadrature Grids (Tauchen and Hussey, 1991)

$$
\begin{aligned}
\int V\left(s^{\prime}\right) p\left(s^{\prime} \mid s, a\right) d s^{\prime} & =\int V\left(s^{\prime}\right) \frac{p\left(s^{\prime} \mid s, a\right)}{p\left(s^{\prime}\right)} p\left(s^{\prime}\right) d s^{\prime} \\
& \simeq \sum_{k=1}^{N} V\left(s_{k}\right) \frac{p\left(s_{k} \mid s, a\right)}{p\left(s_{k}\right)} p_{k}
\end{aligned}
$$

The quadrature abscissa $\left\{s_{1}, \ldots, s_{N}\right\}$ then yield our grid, and the weights $\left\{p_{1}, \ldots, p_{N}\right\}$ can be used to define an $N$-state Markov chain with transition probability $p_{N}$ given by:

$$
p_{N}\left(s_{k} \mid s_{j}, a\right)=\frac{p\left(s_{k} \mid s_{j}, a\right) p_{k} / p\left(s_{k}\right)}{\sum_{i=1}^{N} p\left(s_{i} \mid s_{j}, a\right) p_{i} / p\left(s_{i}\right)},
$$

for $j, k=1, \ldots, N$.
Then the approximate Bellman operator is given by

$$
\widehat{\Gamma}_{N}(V)(s)=\max _{a_{k} \in A_{N}(s)}\left\{u(s, a)+\beta \sum_{k=1}^{N} \int V\left(s_{k}\right) p_{N}\left(s_{k} \mid s, a_{k}\right)\right\} .
$$

Theorem: Suppose that $S \in[0,1]$. There exist constants $K_{1}$ and $K_{2}$ such that for all $V \in B=C(S)$ and all $N$ sufficiently large we have:

$$
\left\|\Gamma(V)-\widehat{\Gamma}_{N}(V)\right\| \leq \frac{\left(K_{1}+\beta K_{2}\right)}{N}
$$

Remark: Quadrature grids are also subject to the curse of dimensionality.

- A solution for this is the random grids (Rust, 1994): Choose a random grid $\left\{\widetilde{s}_{1}, \ldots, \widetilde{s}_{N}\right\}$
via i.i.d. draws from $S=[0,1]^{d_{s}}$. Form random Bellman operator $\widetilde{\Gamma}_{N}$ defined by:

$$
\widetilde{\Gamma}_{N}(V)(s)=\max _{a \in A(s)}\left\{u(s, a)+\frac{\beta}{N} \sum_{i=1}^{N} V\left(\widetilde{s}_{i}\right) p\left(\widetilde{s}_{i} \mid s, a\right)\right\} .
$$

Theorem: Suppose $S=[0,1]^{d}, A$ is a finite set, and ( $u, p$ ) satisfy the Lipschitz conditions. Then for each $N>1$ the expected error in the random Bellman operator satisfies the following uniform bound:

$$
\sup _{p} \sup _{\|V\| \leq K /(1-\beta)} E\left[\left\|\widetilde{\Gamma}_{N}(V)-\Gamma(V)\right\|\right] \leq \frac{\gamma(d)|A| K_{p} K}{(1-\beta) \sqrt{N}}
$$

where $K_{p}$ is the Lipschitz bound on $p$ and $\gamma(d)$ is a bounding constant given by:

$$
\gamma(d)=\sqrt{\frac{\pi}{2}} \beta[1+d \sqrt{\pi} C]
$$

and $C$ is an absolute constant, independent of $p, V, \beta$, or $d$.

- Remark: This result says that randomization succeeds in breaking the curse ofdimensionality for DDP's. However, randomization cannot break the curse of dimensionality of CDP's as is stated formally in the next theorem.
- Theorem: Randomization cannot succeed in breaking the curse of dimensionality of the class of all continuous MDP problems, i.e., a lower bound on the computational complexity is given by:

$$
\operatorname{comp}^{w o r-r a n}\left(\varepsilon, \beta, d_{a}, d_{s}\right)=\Omega\left(\frac{1}{\left[(1-\beta)^{2} \varepsilon\right]^{d_{a}}}\right)
$$

## - Other methods:

- "Low Discrepancy" Grid Points (Traub and Paskov, 1994, Judd, 1994). Use analog of random Bellman operator, but use deterministically chosen Hammersley, Halton or Sobol' points instead of random grid points from $S$.
- Hybrid methods (Keane and Wolpin, 1994). Based on altemative-specific value functions:

$$
\begin{aligned}
V_{t}(s, a) & =u(s, a)+\beta \int \max _{a^{\prime} \in A\left(s^{\prime}\right)}\left[V_{t+1}\left(s^{\prime}, a^{\prime}\right)\right] p\left(d s^{\prime} \mid s, a\right), \\
V_{t}(s) & =\max _{a \in A(s)}\left[V_{t}(s, a)\right], \\
\widehat{V}_{t}\left(s_{j}, a\right) & =u\left(s_{j}, a\right)+\frac{\beta}{N} \sum_{i=1}^{N} \max _{a^{\prime} \in A\left(\widetilde{s}_{i j a}\right)}\left[V_{t+1}\left(\widetilde{s}_{i j a}, a\right)\right] p\left(d s^{\prime} \mid s, a\right) .
\end{aligned}
$$

- Remark: Note that unlike previous methods, Keane- Wolpin method is not self-interpolating,
i.e., $\widehat{V}_{t}$ is only defined at the pre-assigned grid points $\left\{s_{1}, \ldots, s_{N}\right\}$ and not at other points $s \in S$. Since the realizations $\left\{\widetilde{s}_{1 j a}, \ldots, \widetilde{s}_{N j a}\right\}$ will generally not fall on the pre-assigned grid points $\left\{s_{1}, \ldots, s_{N}\right\}$, Keane and Wolpin fit an interpolating regression using the $N$ simulated values $\left\{\widehat{V}_{t}\left(s_{1}\right), \ldots, \widehat{V}_{t}\left(s_{N}\right)\right\}$ as dependent variables in order to evaluate $\widehat{V}_{t}(s)$ at arbitrary points $s \in S$. In principle many other interpolation procedures could be used such as simple linear interpolation or any of the multivariate interpolation procedures. Our formula assumes that this procedure has already been carried out on $\widehat{V}_{t+1}$, so that it can be evaluated at all the random points $\widetilde{s}_{i j a}, i=1, \ldots, N$.


### 4.4 Numerical Solution for the Auto Replacement Problem

- Consider the problem of optimal replacement of durable assets analyzed in Howard 1960, for the discrete state, and Rust $(1985,1986)$, for the continuous state.
- State space $S=R_{+}$, where $s_{t}$ is interpreted as a measure or the accumulated utilization of the durable (such as the odometer reading on a car). Thus $s_{t}=0$ denotes a brand new durable good.
- There are two possible decisions \{keep, replace\}, corresponding to the binary constraint set $A(s)=\{0,1\}$, where $a_{t}=1$ corresponds to selling the existing durable for scrap price $\underline{P}$ and replacing it with a new durable at cost $\bar{P}$.
- Utilization of the asset each period has an exogenous exponential distribution. This corresponds to a transition probability $p$ is given by:

$$
p\left(d s_{t+1} \mid s_{t}, a_{t}\right)= \begin{cases}\lambda \exp \left\{-\lambda\left(s_{t+1}-s_{t}\right)\right\} & \text { if } a_{t}=0 \text { and } s_{t+1} \geq s_{t} \\ \lambda \exp \left\{-\lambda\left(s_{t+1}-0\right)\right\} & \text { if } a_{t}=1 \text { and } s_{t+1} \geq s_{t} \\ 0 & \text { otherwise }\end{cases}
$$

- The per-period cost of operating the asset in state $s$ is given by a function $c(s)$. This implies the following per period utility function is:

$$
U\left(s_{t}, a_{t}\right)= \begin{cases}-c\left(s_{t}\right) & \text { if } a_{t}=0 \\ -[\bar{P}-P]-c(0) & \text { if } a_{t}=1\end{cases}
$$

- The Bellman's equation is given by:

$$
\begin{align*}
V(s)= & \max \left\{-c(s)+\beta \int_{s}^{\infty} V\left(s^{\prime}\right) \lambda \exp \left\{-\lambda\left(s^{\prime}-s\right)\right\} d s^{\prime}\right.  \tag{23}\\
& \left.-[\bar{P}-P]-c(0)+\beta \int_{0}^{\infty} V\left(s^{\prime}\right) \lambda \exp \left\{-\lambda s^{\prime}\right\} d s^{\prime}\right\}
\end{align*}
$$

- Let $\gamma$ be the smallest value of $s$ such that the agent is indifferent between keeping and replacing. Differentiating Bellman's equation (0.6), it follows that on the continuation region, $[0, \gamma]$,
$\alpha(s)=0$ (i.e., keep the current durable) and $V$ satisfies the differential equation:

$$
\begin{equation*}
V^{\prime}(s)=c^{\prime}(s)+\gamma c(s)+\lambda(1-\beta) V(s) . \tag{24}
\end{equation*}
$$

- This is known as a free boundary value problem, since the boundary condition is determined endogenously:

$$
\begin{equation*}
V(\gamma)=[\bar{P}-P]+V(0)=-c(\gamma)+\beta V(\gamma)=\frac{-c(\gamma)}{1-\beta} \tag{25}
\end{equation*}
$$

- Integrating the ODE, we obtain:

$$
\begin{equation*}
V(s)=\max \left\{\frac{-c(\gamma)}{1-\beta}, \frac{-c(\gamma)}{1-\beta}+\int_{s}^{\gamma} \frac{c^{\prime}(\gamma)}{1-\beta}[1-\beta \exp \{-\lambda(1-\beta)(y-s)\}] d y\right\} \tag{26}
\end{equation*}
$$

where $\gamma$ is the unique solution to:

$$
\begin{equation*}
[\bar{P}-P]=\int_{0}^{\gamma} \frac{c^{\prime}(\gamma)}{1-\beta}[1-\beta \exp \{-\lambda(1-\beta) y\}] d y \tag{27}
\end{equation*}
$$

### 4.5 Solving the Auto Replacement Problem by Discrete Approximation

- Formulate discrete-state version of the replacement problem using $|S|=100$ states, and the same discount factor, cost function, and replacement costs.
- Approximate the exponential transition density $p\left(\cdot \mid s_{t}, d_{t}\right)$ by a twelve-point pprobability distribution using a simple continuity correction so that each of the 12 mass points equal the probability that an exponential random variable falls within plus or minus .5 of the integer value $j$ that $s_{t}$ assumes in the discrete case:

$$
p\left(s_{t+1}=s_{t}+j \mid s_{t}, d_{t}\right)= \begin{cases}\int_{0}^{.5} \lambda \exp \{-\lambda y\} d y & j=0  \tag{28}\\ \int_{j-.5}^{j+.5} \lambda \exp \{-\lambda y\} d y & j=1, \ldots, 10 \\ \int_{10.5}^{\infty} \lambda \exp \{-\lambda y\} d y & j=11\end{cases}
$$

