A Model of Human Capital Accumulation and Occupational Choices

A simplified version of Keane and Wolpin (JPE, 1997)

- We have here three, mutually exclusive decisions in each period:

  1. Attend school.

  2. Work in a white-collar occupation.

  3. Work in a blue-collar occupation.

- Individuals differ in their endowments of skills among the various possible activities, i.e., there is an *activity-specific set of individual endowments*.

- We will keep the structure simple and allow for some individual-specific endowments, but will also allow for some of the model's parameters to be *common* to all individuals.

The Model:

- Finite horizon model where individuals start at age 18, after they completed their high school education, and end their lives five periods later, that is $T = 5$.

- At each time $t = 1, ..., 5$, an individual chooses among the three alternatives.

- Let,

$$d_{mt} = \begin{cases} 1 & \text{if the } m\text{th choice is chosen} \\ 0 & \text{otherwise,} \end{cases}$$

for $m = 1, 2, 3$.

- Note that since the choices are mutually exclusive, we always have

$$\sum_{m=1}^{3} d_{mt} = 1.$$

The reward:

- The per period reward at time t is given by

$$R_t = \sum_{m=1}^{3} R_{mt} d_{mt},$$

where R_{mt} is the per-period reward, associated with the mth choice.

- Before defining the individuals' reward functions some notation is in order:

  1. Let E_t denote the number of completed years of schooling at time t, and note that $E_t = 12, ..., 16$.

  2. Let x_{mt} denote the number of years of experience in the mth occupation, $m = 2, 3$.

     We assume that the experience is occupation specific, and one is rewarded only for the experience one has in the occupation he/she currently works in.

- Notation (continued):

  3. Let $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \alpha_{i3})'$ denote the activity specific endowment of the individual at $t = 1$.

  4. Let δ denote the discount factor.

  5. Denote the full parameter vector of the entire model by θ.

The reward function:

1. **The schooling alternative ($m = 1$):**

$$R_{i1t} = \exp\left\{\alpha_{i1} + \varepsilon_{i1t}\right\} - tc \cdot I\left(E_{it} \geq 12\right),$$

where the endowment α_{i1} represents the indirect cost of effort, or enjoyment (and can be positive), while tc represents the direct costs of education, and $I(\cdot)$ denotes the usual indicator function.

2. **White-collar alternative ($m = 2$):**

$$R_{i2t} = \exp\left\{\alpha_{i2} + \beta_1 E_{it} + \beta_2 x_{i2t} + \varepsilon_{i2t}\right\}.$$

3. **Blue-collar alternative ($m = 3$):**

$$R_{i3t} = \exp\left\{\alpha_{i3} + \gamma_1 E_{it} + \gamma_2 x_{i3t} + \varepsilon_{i3t}\right\}.$$

- Note that the parameters β_1, β_2, γ_1, and γ_2, are common parameters to all individuals.

- The vector of residuals $\varepsilon_{it} = (\varepsilon_{i1t}, \varepsilon_{i2t}, \varepsilon_{i3t})'$ represents idiosyncratic shock, and it is assumed that

$$\varepsilon_{it} \sim N(0, \Omega).$$

- In this model the *state vector* at time t, z_t, is given by

$$\begin{aligned} z_{it} &= \left(\underbrace{\underbrace{\alpha_{i1}, \alpha_{i2}, \alpha_{i3}}_{\alpha_i}, E_{it}, x_{i2t}, x_{i3t}}_{s_{it}}, \underbrace{\varepsilon_{i1t}, \varepsilon_{i2t}, \varepsilon_{i3t}}_{\varepsilon_{it}} \right) \\ &= \left(s_{it}, \varepsilon_{it} \right), \end{aligned}$$

where the first part of the state vector,

$$s_{it} = \{\alpha_i, E_{it}, x_{i2t}, x_{i3t}\},$$

is *deterministic*, while ε_{it} is the stochastic part of the state vector.

- The value function (dropping the ith subscript) at time t:

$$V\left(z_t, t\right) = \max_{\{d_{m\tau}\}_{\tau=t}^{T}} E\left[\left.\sum_{\tau=t}^{T} \delta^{\tau-t} \sum_{m=1}^{3} R_{mt} d_{mt} \right| z_t\right]. \tag{1}$$

- Maximization of (1) is achieved by choosing the optimal sequence of control variables:

$$\{d_{mt}, \quad m = 1, 2, 3\} \quad \text{for } t = 1, ..., T\,(= 5)\,.$$

- Hence, we can write

$$V\left(s_t, t\right) = \max_{m \in \{1,2,3\}} \left\{V_m\left(z_t, t\right)\right\},$$

where

$$V_m\left(z_t, t\right) = R_{mt}\left(z_{it}\right) + \delta E\left[V\left(z_{t+1}, t+1\right) \mid z_t, d_{mt} = 1\right], \qquad (2)$$

for all $t < T$.

- For $t = T$ we have

$$V_m (z_T, T) = R_{mT} (z_t) .$$

- Note that the expectation is taken over the random components of z_{t+1}, namely ε_t, conditional on s_t.

- That is, there is an implicit assumption about the Markovian nature of the problem, namely, the distribution of z_{t+1}, depends only on z_t.

- The state variable of schooling evolves deterministically according to

$$E_{t+1} = E_t + d_{1t},$$

for $E_t \leq \overline{E}$, the highest attainable level of education. In our case $\overline{E} = 16$.

- Similarly, for occupation specific work experience we have

$$x_{m,t+1} = x_{mt} + d_{mt}, \quad \text{for } m = 2, 3.$$

Remarks:

1. The solution to the optimization gives five mutually exclusive regions on the domain of ε_t, for which each of the five alternatives is the optimal choice.

2. Note that from the point of view of the individual agents the solution is *deterministic*, because ε_{it} (but not $\varepsilon_{i,t+1}$) is observed when making the optimal decision. However, from the econometrician standpoint the solution is stochastic, specifically, because ε_{it} is not observed by the econometrician.

Backward induction for Serially Uncorrelated ε_t:

- Here the joint distribution of the ε_t, $t = 1, ..., T$, is given by

$$\prod_{t=1}^{T} f(\varepsilon_{1t}, \varepsilon_{2t}, \varepsilon_{3t}; \eta) = \prod_{t=1}^{T} f(\varepsilon_t; \eta),$$

  where η is a vector of parameters that correspond to the distribution $f(\cdot)$.

- Note that in our case η is simply

$$\eta = \text{vec}(\Omega).$$

- The individual knows θ and has to solve for the sequence $\{d_m(t)\}$ for $t = 1, ..., T$, as in (1).

Note also that

$$f\left(\varepsilon_{t+1}|z_t, d_{mt}\right) = f\left(\varepsilon_{t+1}|s_t, d_{mt}\right).$$

The value function at T:

- Consider now an individual who enters the last time period T with the state space vector z_T.

- The individual receives a draw from the distribution of ε_T, conditional on s_T (here ε_T is actually independent of s_T). The individual will then choose the alternative with the highest reward function.

The value function at $T-1$:

- Now consider the individual at time $T-1$, with the deterministic part of the state space vector s_{T-1}.

- In order to calculate the alternative-specific value function, the individual must first calculate

$$\begin{aligned} & E\left[\max\{R_{1T}, R_{2T}, R_{3T}\}\mid s_{T-1}, d_{m,T-1}\right] \\ = & \int_{\varepsilon_3}\int_{\varepsilon_2}\int_{\varepsilon_1} \max\left\{R_{1T}, R_{2T}, R_{3T}\mid s_{T-1}, d_{m,T-1}\right\} \\ & \times f\left(\varepsilon_{1T}, \varepsilon_{2T}, \varepsilon_{3T}\right) d\varepsilon_{1T} d\varepsilon_{2T} d\varepsilon_{3T}, \end{aligned} \tag{3}$$

for each of the possible choices, $d_{m,T-1}$, $m = 1, 2, 3$.

- The individual needs to compute (3) three times for $d_{m,T-1} = 1, 2, 3$.

- Note that the $E\,\max(\cdot)$ function is a multivariate integral, even though the ε's are independent over time.

- The individual knows the value functions at period $T-1$, $V_m(s_{T-1}, T-1)$, up to the random draw of ε_{T-1}. The individual receives the random draw ε_{T-1} and then chooses the alternative with the highest value.

The value function at t:

- Moving back in time to a general period t the individual has to solve a problem, analogous to the problem in (3), which takes the form

$$E\left[\max\left\{R_{1,t+1}, R_{2,t+1}, R_{3,t+1}\right\}\middle| s_t, d_{mt}\right]. \tag{4}$$

- Note that in order to compute (4) the alternative-specific value functions must be calculated for all of the state vector values s_{t+1} that may arise, conditional on s_t and d_{mt}.

- That is, it is necessary to compute the alternative-specific value functions at each future date, for all possible realizations of the state vector.

Estimation:

- There is a significant complexity in computing the value function. Any erroneous approximation would transmit itself into bias in the parameter of interest θ.

- Consider an homogeneous set of individuals from the same birth cohort who are observed for all periods $t = 1, ..., T$.

- Note that in any period t wages are observed if and only if a work alternative was chosen, and only for that work alternative.

- Hence, the joint probability of choosing (for example) occupation 2 at time t, *and* its corresponding (accepted) wage, $w_{2t} = R_{2t}$, is

$$
\begin{aligned}
&\Pr\left(d_{2t} = 1, w_{2t} | s_t\right) \\
&\quad = \Pr\left(w_{2t} | s_t\right) \Pr\left(d_{2t} = 1 | w_{2t}, s_t\right) \\
&\quad = \Pr\left(w_{2t} | s_t\right) \Pr\{w_{2t} + \delta E \max\left[V\left(z_{t+1}, t+1\right) | s_t, d_{2t} = 1\right] \\
&\qquad \geq \widetilde{w}_{3t} e^{\varepsilon_{3t}} + \delta E \max\left[V\left(z_{t+1}, t+1\right) | s_t, d_{3t} = 1\right], \\
&\qquad w_{2t} + \delta E \max\left[V\left(z_{t+1}, t+1\right) | s_t, d_{2t} = 1\right] \\
&\qquad \geq \exp\{\alpha_1\} e^{\varepsilon_{1t}} - tcI\left(E_t \geq 12\right) + \delta E \max\left[V\left(z_{t+1}, t+1\right) | s_t, d_{1t} = 1
\end{aligned}
$$

where

$$
\widetilde{w}_{3t}\left(s_t\right) = \exp\{\alpha_3 + \gamma_1 E_t + \gamma_2 x_{3t}\}.
$$

- In words, this is the probability that alternative 2 exceed all the others, and that the accepted wage is the observed wage.

- Similar joint probabilities can be written for the other choices.

- The likelihood function for the sample is the product of all these probabilities over *time* and *individuals*.

- Maximizing the likelihood would yield a consistent estimator for θ, which is also asymptotically normal.

- If the $E\,\mathsf{max}(\cdot)$ function could be calculated with no error, then one can also use the method of simulated moments (MSM) to obtain an estimate for θ.

- However, the MSM estimator combined with simulated estimates for $E\,\mathsf{max}(\cdot)$ is not consistent for a fixed number of simulation, R.

Important remark: There are no conceptual problems in the implementation of models with large choice sets, large state space, and serial dependence in the unobserved elements. *The problems are merely computational*.

Existing Solutions and Estimation Methods:

Full-Solution Methods:

- These methods involve computational simplifications and involve finding convenient forms for the *reward functions* and *error distributions*.

- Three crucial assumptions make the computation relatively easy:

  1. The reward functions are additively separable in the unobservable, with each unobservable associated with a mutually exclusive choice. That is

  $$u^*(s_t, t, \varepsilon_t) = u(s, t) + \varepsilon_t,$$

  where ε_t is choice specific.

2. Conditional on the observed state variables, the unobservables are independent.

3. The unobservables are distributed as multivariate extreme values.

- These three assumptions lead to two very appealing consequences for solution and estimation:

  1. The $E\,\mathsf{max}(\cdot)$ function has the closed form solution given by

  $$E\left[V\left(s\left(t\right),t\right)\right]=\gamma+\tau\,\mathsf{ln}\left\{\sum_{m=1}^{M}\frac{\mathsf{exp}\left\{V_m\left(s_t,t\right)\right\}}{\tau}\right\}, \tag{5}$$

     where $V_m\left(s_t,t\right)$ is the expectation of the alternative-specific value functions, such as the one in (2), and γ is an Euler's constant. The importance of this is that multivariate numerical integrations are avoided in solving the DP problem.

- Consequences (continued):

   2. The choice probabilities are multinomial logit and are given by (with the normalization $\tau = 1$) by:

$$\Pr\left(d_{mt} = 1 | s_t\right) = \frac{\exp\left\{V_m\left(s_t, t\right)\right\}}{\sum_{j=1}^{M} \exp\left\{V_j\left(s_t, t\right)\right\}}. \tag{6}$$

   That is, the multivariate numerical integrations are avoided in the likelihood estimation. The drawback of this is that only a very limited form of correlation across alternatives can be accommodated.

- **Note:** To get these last two results, we need *all* the assumptions above. An assumption about the extreme distribution is not enough, we need to have the assumption about separability as well!

Non-Full-Solution Methods:

- One can use an alternative representation of the future components of the choice-specific valuations that do not depend explicitly on the structural parameters of the model.

- The $E\,\max(\cdot)$ function is estimated then using data on future choice probabilities.

- The intuition for this method (Hotz and Miller (1993)) is that, using (5) and (6), one can show that

$$E\left[V\left(z_t, t\right)\right] = \sum_{m=1}^{M} \Pr\left(d_{mt} = 1|s_t\right)\left[\gamma + V_m\left(s_t, t\right) - \ln\left(\Pr\left(d_{mt} = 1|s_t\right)\right)\right]. \tag{7}$$

- Now, successive forward substitutions for $V_m\left(s_t, t\right)$, recognizing that it contains future expected maximum functions, implies that the expected maximum function at any t can be written as a function of the future conditional choice probabilities.

- Although this method is a lot more tractable than the full solution methods there are few disadvantages that are associated with this method:

  1. One needs a lot of data to be able to estimate all the transition probabilities, which are essentially estimated non-parametrically.

  2. This approach cannot admit individual-specific heterogeneity, at least in its current form. This, generally, rules out any form of serial correlation, including that implied by permanent unobserved heterogeneity.

  3. The $E \max(\cdot)$ values, which are calculated from the data, are not policy invariant, because they depend on the parameters of the model. Hence, a full solution is necessary after a solution for θ has been obtained in order to carry out policy experiments.