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Lecture Note 1

Dynamic Programming I

1 A Heuristic Approach

1.1 Neoclassical Growth Model

Consider the following optimization problem:

$$\max_{\{c_t,k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(c_t) \tag{1}$$
$$\text{subject to: } c_t + k_{t+1} = F(k_t),$$

where

$$\begin{aligned} U &: R_+ \to R, \\ F &: R_+ \to R_+. \end{aligned}$$

The above problem in (1) can be reformulated as

$$\max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(F(k_t) - k_{t+1}).$$

Note that one need to choose a sequence of k_t for all periods from $t = 1$ to infinity. How can such a problem be solved?

A finite horizon problem (with finite T):

Consider first the problem with a finite horizon, that is,

$$\max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{T} \beta^{t-1} U(F(k_t) - k_{t+1}).$$

Period T problem:
In period T we have the solution is

$$\begin{aligned} V^1(k_T) &= \max_{k_{T+1}} U(F(k_T) - k_{T+1}), \\ &= U\left(F(k_T) - k^*_{T+1}\right), \end{aligned} \tag{2}$$

where

$$k_{T+1}^* \equiv 0,$$

is the optimal solution for the problem (2). In words, $V^1(k_T)$ is the value of having k_T units of capital when entering the last period, and choosing k_{T+1} optimally. Note that the superscript 1 refers to the number of periods remaining in the planning problem.

Here, the function $V^1(k_T)$ is referred to as the *value function*, while the variable k_T is called the *state variable*.

Period $T-1$ problem:

$$\begin{aligned} V^2(k_{T-1}) &= \max_{k_{T+1}} \left\{ U(F(k_{T-1}) - k_T) + \beta V^1(k_T) \right\}, \\ &= U(F(k_{T-1}) - k_T^*) + \beta V^1(k_T^*), \end{aligned} \tag{3}$$

where

$$k_T^* = G^2(k_{T-1}),$$

solves the problem (3). The function $G^2(\cdot)$ is called the *decision rule*, or alternatively, the *policy function*. Here, the optimal k_T, denote by k_T^*, simply solves the first-order condition:

$$-U_1(F(k_{T-1}) - k_T^*) + \beta V_1^1(k_T^*) = 0. \tag{4}$$

Question: What can be said about the functions $G^2(\cdot)$ and $V^2(\cdot)$?

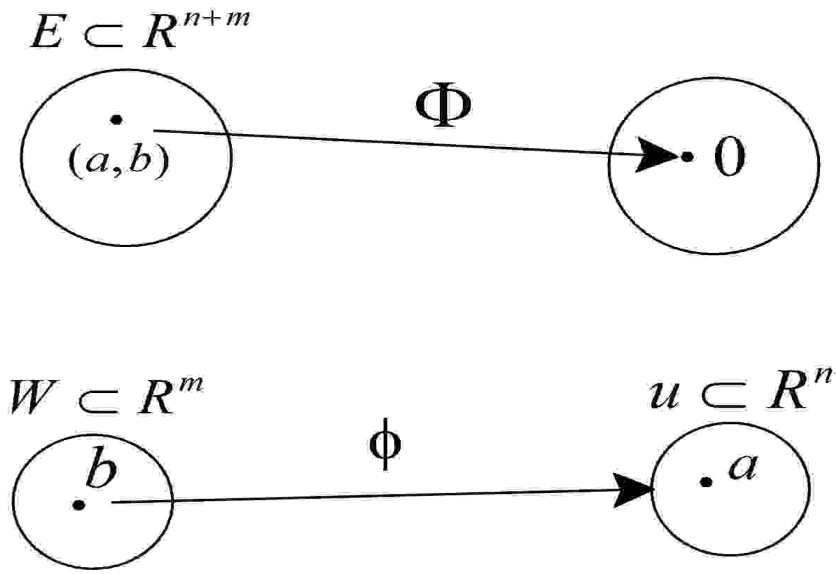
The first-order condition (4) defines an implicit function determining k_T as a function of k_{T-1}. More generally in economics one often comes across equation systems of the form

$$\begin{aligned} \Phi(x,y) &= 0, \quad \text{where} \\ x &\in R^n, \quad y \in R^m, \quad \text{and} \\ \Phi &: R^{n+m} \to R^n. \end{aligned}$$

Can we find a function ϕ that solves for x in terms of y, so that $x = \phi(y)$? The answer is given in the following theorem (see also Figure 1 below):

Theorem 1 (implicit function theorem): *Let Φ be a C^q mapping from an open set $E \subset R^{n+m}$* into R^n *such that $\Phi(a,b) = 0$ for some point $(a,b) \in E$. Suppose that the Jacobian determinant $|J| = |\partial\Phi(a,b)/\partial x| \neq 0$. then there exists a neighborhood $U \subset R^n$ around a and a neighborhood $W \subset R^m$ around b, and a unique function $\phi : W \to U$ such that: (i) $a = \phi(b)$; (ii) ϕ is class C^q on W; and (iii) for all $y \in W$, $(\phi(y), y) \in E$ and $\Phi((\phi(y), y)) = 0$.*

Figure 1: Implicit Function Theorem



Applying the implicit function theorem to the first-order condition in (4) it is straight forward to show that (under some standard regularity conditions) $k_T^* = G^2(k_{T-1})$ will be a C^1 function, which, in turn, implies that $V^2(k_{T-1})$, will be one as well.
Period t problem:

$$\begin{aligned} V^{T+1-t}(k_t) &= \max_{k_{t+1}} \left\{ U(F(k_t) - k_{t+1}) + \beta V^{T-t}(k_{t+1}) \right\}, \\ &= U\left(F(k_t) - k_{t+1}^*\right) + \beta V^{T-t}\left(k_{t+1}^*\right), \end{aligned} \tag{5}$$

where $k_{t+1}^* = G^{T+1-t}(k_t)$ is the optimal k_{t+1} that solves the problem (5).

Note now that the dynamic programming effectively collapsed the single large problem, involving $T+1-t$ choice variables, into $T+1-t$ smaller problems, each of which involving only one choice variable. To see this solve out for $V^{T-t}(k_{t+1})$ in (5) to get

$$\begin{aligned} V^{T+1-t}(k_t) &= \max_{k_{t+1}} \left\{ U(F(k_t) - k_{t+1}) + \beta \max_{k_{t+2}} \left\{ U(F(k_{t+1}) - k_{t+2}) + \beta V^{T-t-1}(k_{t+2}) \right\} \right\}, \\ &= \max_{k_{t+1}, k_{t+2}} \left\{ U(F(k_t) - k_{t+1}) + \beta U(F(k_{t+1}) - k_{t+2}) + \beta^2 V^{T-t-1}(k_{t+2}) \right\}. \end{aligned}$$

Now, solving out recursively for $V^{T-t-1}(k_{t+2})$, $V^{T-t-2}(k_{t+3})$,...., yields

$$\max_{\{k_{t+j+1}\}_{j=0}^{T-t}} \sum_{j=0}^{T-t} \beta^j U(F(k_{t+j}) - k_{t+j+1}).$$

An infinite horizon problem:

As $T \to \infty$ we would expect to have for *any* time t:

$$\begin{aligned} V^{T+1-t}(k_t) &\to V(k_t), \quad \text{and} \\ G^{T+1-t}(k_t) &\to G(k_t). \end{aligned}$$

That is, when we have an infinite horizon problem, we should have the same value function at any period in which we have the same value for the state variable k_t.

While this is difficult to show formally, it is a true statement. Hence, for the infinite horizon problem the value function takes the form

$$\begin{aligned} V(k_t) &= \max_{k_{t+1}} \{U(F(k_t) - k_{t+1}) + \beta V(k_{t+1})\}, \qquad (6) \\ &= U\left(F(k_t) - k^*_{t+1}\right) + \beta V\left(k^*_{t+1}\right), \end{aligned}$$

where

$$k^*_{t+1} = G(k_t).$$

1.2 The Envelope Theorem

Assumption: The function V is continuously differentiable.

First we characterize the solution for the infinite horizon problem. The first-order condition for this problem is

$$-U_1(F(k_t) - k_{t+1}) + \beta V_1(k_{t+1}) = 0,$$

or alternatively

$$U_1(F(k_t) - k_{t+1}) = \beta V_1(k_{t+1}). \qquad (7)$$

However, we have a problem because the first-order condition in (7) includes the unknown function V. What should we do about it?

Differentiate both sides of (6) with respect to k_t, to get

$$\begin{aligned} V_1(k_t) &= U_1(F(k_t) - k_{t+1}) F_1(k_t) - U_1(F(k_t) - k_{t+1}) \frac{\partial k_{t+1}}{\partial k_t} + \beta V_1(k_{t+1}) \frac{\partial k_{t+1}}{\partial k_t} \\ &= U_1(F(k_t) - k_{t+1}) F_1(k_t) + [-U_1(F(k_t) - k_{t+1}) + \beta V_1(k_{t+1})] \frac{\partial k_{t+1}}{\partial k_t} \qquad (8) \\ &= U_1(F(k_t) - k_{t+1}) F_1(k_t), \qquad (9) \end{aligned}$$

because the term in square bracket in (8) is zero by the first-order condition (7).

In period $t+1$ this equation will be then

$$V_1(k_{t+1}) = U_1(F(k_{t+1}) - k_{t+2}) F_1(k_{t+1}). \qquad (10)$$

Therefore, we can rewrite the first-order condition in (7), by substitution of from (10) into (7), as

$$U_1\left(F\left(k_t\right)-k_{t+1}\right)=\beta U_1\left(F\left(k_{t+1}\right)-k_{t+2}\right)F_1\left(k_{t+1}\right). \tag{11}$$

2 A More Formal Analysis

2.1 Neoclassical Growth Model

Dynamic programming representation:

$$V\left(k\right)=\max_{k'}\left\{U\left(F(k)-k'\right)+\beta V\left(k'\right)\right\}.$$

There are a number of issues that we need to investigate:

1. Does V exist?
2. Is V unique?
3. Is V continuous?
4. Is V continuously differentiable?
5. Is V increasing in k?
6. Is V concave in k?

2.2 Method of Successive Approximation

Goal: Approximate the value function V by a sequence of successively better guesses, denote by V^j, at stage j.

The procedure:

Stage 0. Make an initial guess for V. Denote that function by V^0.

Stage 1. Compute

$$V^1\left(k\right)\equiv\max_{k'}\left\{U\left(F(k)-k'\right)+\beta V^0\left(k'\right)\right\}. \tag{12}$$

Stage 1. Compute V^{n+1}, given V^n, as follows:

$$V^{n+1}\left(k\right)\equiv\max_{k'}\left\{U\left(F(k)-k'\right)+\beta V^n\left(k'\right)\right\}. \tag{13}$$

In the operator notation this procedure can be denoted simply by

$$V^{n+1}\equiv TV^n.$$

The operator T is shorthand notation for the list of operations, described by (13) that are performed on the function V^n to transform it into the new one, that is, V^{n+1}. Often, the operator T maps some set of functions, say $\mathcal{F}$, into itself. That is $T : \mathcal{F} \to \mathcal{F}$. The hope is that as n gets large it transpires that $V^n \to V$, where V is such that $V = TV$.

2.3 Metric Space

Definition 1. *A metric space is a set S, together with a metric $\rho : S \times S \to R_+$, such that for all $x, y, z \in S$* (see Figure 2 below)*:*

1. $\rho(x, y) \geq 0$, *with* $\rho(x, y) = 0$ *if and only if* $x = y$.
2. $\rho(x, y) = \rho(y, x)$.
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

Figure 2: A metric $\rho(x, y)$

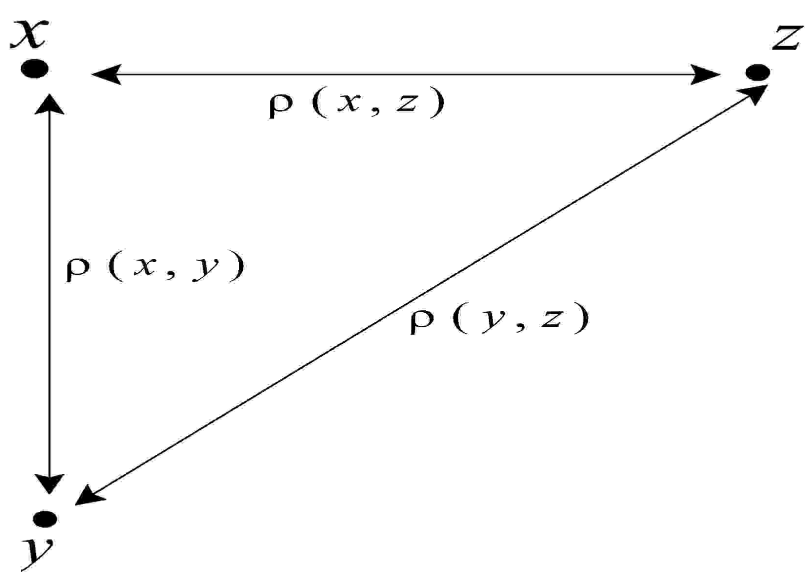
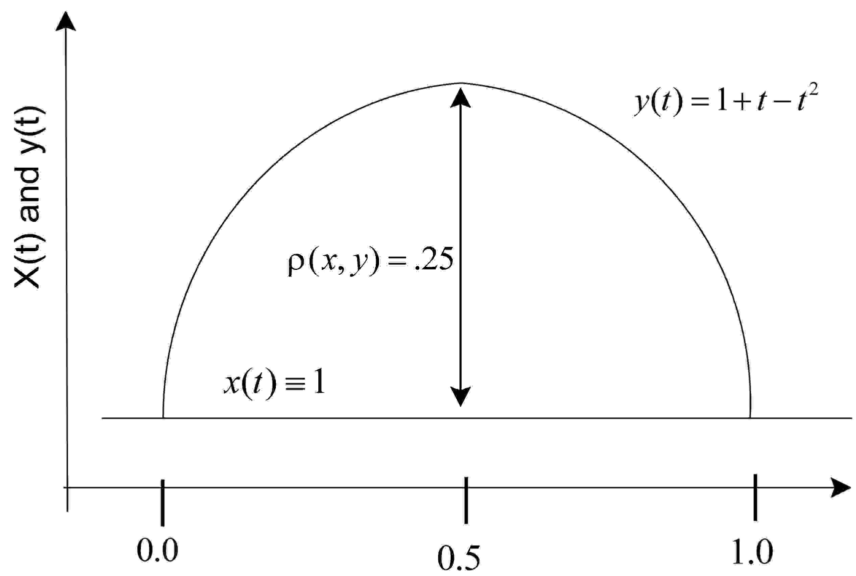


Figure 3: Uniform metric



Example 1. Space of continuous functions $C : [a, b] \to R_+$. Here, we have

$$\rho(x, y) = \max_{t \in [a,b]} |x(t) - y(t)| .$$

Definition 2. *A sequence $\{x_n\}_{n=0}^{\infty}$ in S converges to $x \in S$, if for each $\varepsilon > 0$ there exists an N_ε, such that*

$$\rho(x_n, x) < \varepsilon, \quad \textit{for all } n > N_\varepsilon.$$

Definition 3. *A sequence $\{x_n\}_{n=0}^{\infty}$ in S is a Cauchy sequence if for each $\varepsilon > 0$ there exists an N_ε, such that*

$$\rho(x_m, x_n) < \varepsilon, \quad \textit{for all } m, n > N_\varepsilon.$$

Note that a Cauchy sequence in S may not converge to a point in S.

Example 2. Let S= $(0, 1]$, $\rho(x, y) = |x - y|$, and $\{x_n\}_{n=0}^{\infty} = \{1/n\}_{n=0}^{\infty}$. Clearly, $x_n \to 0 \notin (0, 1]$. This sequence satisfies the Cauchy criteria, though,

$$\rho(x_m, x_n) = \left| \frac{1}{m} - \frac{1}{n} \right| \leq \frac{1}{m} + \frac{1}{n} < \varepsilon, \quad \text{if } m, n > \frac{2}{\varepsilon}.$$

Definition 4. *A metric space (S, ρ) is complete if every Cauchy sequence in S converges to a point in S.*

Theorem 2. *Let $X \subseteq R^l$ and $C(X)$ be the set of bounded continuous functions $V : X \to R$ with the uniform metric $\rho(V, W) = \sup_{x \in X} |V - W|$. Then $C(X)$ is a complete metric space.*

Proof 2. **Let** $\{V^n\}_{n=0}^{\infty}$ be any Cauchy sequence in $C(X)$. Now, for each $x \in X$ the sequence

$\{V^n\}_{n=0}^{\infty}$ is Cauchy, since

$$|V^n(x) - V^m(x)| \leq \sup_{y \in X} |V^n(y) - V^m(y)| = \rho\left(V^n, V^m\right).$$

By the completeness of the real line $V^n(x) \to V(x)$, as $n \to \infty$. Define the function V by $V(x)$ for each $x \in X$.

It will be shown now that $\rho\left(V^n, V\right) \to 0$ as $n \to \infty$. Choose an $\varepsilon > 0$.

Now,

$$\begin{aligned} |V^n(x) - V(x)| &\leq |V^n(x) - V^m(x)| + |V^m(x) - V(x)| \\ &\leq \underbrace{\rho\left(V^n, V^m\right)}_{\leq \varepsilon/2} + \underbrace{|V^m(x) - V(x)|}_{\leq \varepsilon/2}. \end{aligned}$$

Note that the first term of the last inequality can be made smaller than $\varepsilon/2$ by the Cauchy criteria, that is, there exists an N_ε, such that for all $m, n > N_\varepsilon$ it transpires that $\rho(V^n, V^m) < \varepsilon/2$. The second term can be made smaller than $\varepsilon/2$ by point-wise convergence of V^m to V, that is, there exists an $M_\varepsilon(x)$ such that for all $m \geq M_\varepsilon(x)$ it follows that $|V^m(x) - V(x)| \leq \varepsilon/2$. Note that here, $M_\varepsilon(x)$ depends on x, but N_ε does not. Also note that for any value of x such an $M_\varepsilon(x)$ will always exist. Hence, $|V^n(x) - V(x)| \leq \varepsilon$ for all $n > N_\varepsilon$ *independent* of the value of x. It follows then that $\rho\left(V^n, V\right) \leq \varepsilon$, the desired result.

Now, it remain to be shown that V is a continuous function. To do this, pick an $\varepsilon > 0$. It is then the question: Does there exist a δ such that whenever $\rho(x, x_0) \leq \delta$, $|V(x) - V(x_0)| \leq \varepsilon$?

Note that

$$|V(x) - V(x_0)| \leq \underbrace{|V(x) - V^n(x)|}_{\varepsilon/3} + \underbrace{|V^n(x) - V^n(x_0)|}_{\varepsilon/3} + \underbrace{|V^n(x_0) - V(x_0)|}_{\varepsilon/3}.$$

The first and the third terms can be made arbitrarily small by the uniform convergence of V^n to V. The second term can be made to vanish by the fact that V^n is a continuous function, that is, by picking a δ small enough such that the last term will be less than $\varepsilon/3$.

QED

Remark. Point-wise convergence of a sequence of continuous functions does not imply that the limiting function is continuous.

Example 3. (See Figure 4)

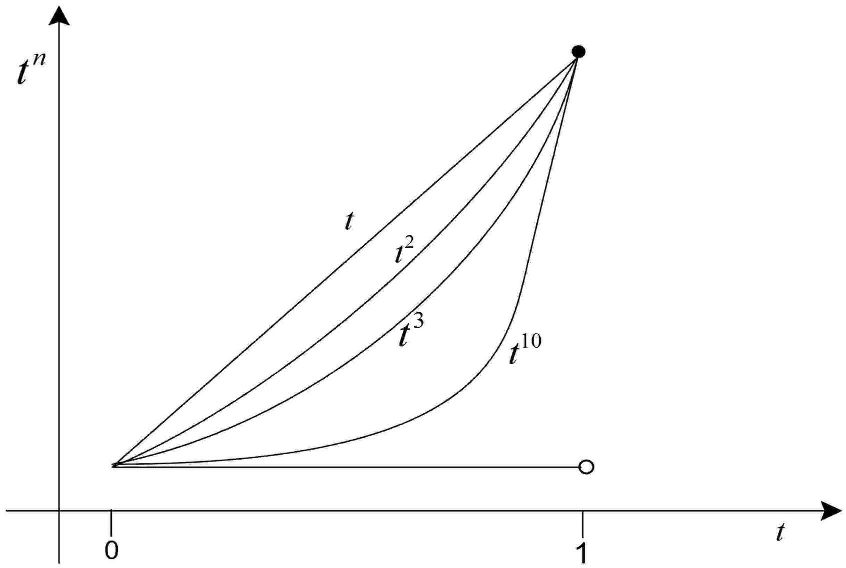
Let $\{V^n\}_{n=1}^{\infty}$ in $C[0,1]$ be defined by $V^n(t) = t^n$. As $n \to \infty$ it transpires that: (i) $V^n(t) \to 0$ for $t \in [0,1)$; and (ii) $V^n(t) \to 1$ for $t = 1$. Thus,

$$V(t) = \begin{cases} 0 & \text{for } t \in [0,1), \\ 1 & \text{for } t = 1. \end{cases}$$

Hence, $V(t)$ is a discontinuous function. Clearly, by Theorem 2 $\{V^n\}_{n=1}^{\infty}$ cannot describe a Cauchy sequence.

This, however, can also be shown directly. Note that for any given N_ε it is always possible to pick $m, n \geq N_\varepsilon$ and $t \in [0,1)$ so that $|t^n - t^m| \geq 1/2$. To see that, pick $n = N_\varepsilon$ and a $t \in (0,1)$ so that $t^n \geq 3/4$; i.e., choose $t \geq (3/4)^{1/N_\varepsilon}$. Next, pick an m large enough such that $t^m < 1/4$ or $m \geq (\ln 1/4) / \ln(t)$. Then the desired results follows.

Figure 4: Point-wise convergence to a discontinuous function



Example 4.

Consider the space of continuous functions $C[-1,1]$ with metric

$$\rho(x,y) = \int_{-1}^{+1} |x(t) - y(t)|\, dt.$$

Let $\{V^n\}_{n=1}^{\infty}$ in $C[-1,1]$ be defined by

$$V^n(t) = \begin{cases} 0 & \text{if } -1 \leq t \leq 0, \\ nt & \text{if } 0 \leq t \leq 1/n, \\ 1 & \text{if } 1/n \leq t \leq 1. \end{cases}$$

Show that $\{V^n\}_{n=1}^{\infty}$ is a Cauchy sequence. Deduce that the space of continuous functions in not complete with this metric.

2.4 The Contraction Mapping Theorem

Definition 5. Let (S, ρ) be a metric space and $T : S \to S$ be a function mapping S into itself. We say that T is a contraction mapping (with modulus β) if for $\beta \in (0,1)$,

$$\rho(Tx, Ty) \leq \beta \rho(x,y), \qquad \text{for all } x, y \in S.$$

Theorem 3. (Contraction Mapping Theorem, or Banach Fixed Point Theorem) *If (S, ρ) is a complete metric space and $T : S \to S$ is a continuous mapping with modulus β, then*

1. *T has exactly one fixed point $V \in S$ such that $V = TV$;*

2. *for any $V^0 \in S$, $\rho\left(T^n V^0, V\right) < \beta^n \rho\left(V^0, V\right)$, $n = 0, 1, 2, \ldots$.*

Proof 3. Define the sequence $\{V^n\}_{n=0}^{\infty}$ by

$$V^n = TV^{n-1} = \underbrace{TT}_{T^2} V^{n-2} = T^n V^0.$$

It will be shown that $\{V^n\}_{n=0}^{\infty}$ is a Cauchy sequence. To this end, the contraction property of T implies that

$$\rho\left(V^2, V^1\right) = \rho\left(TV^1, TV^0\right) \leq \beta\rho\left(V^1, V^0\right).$$

Hence,

$$\rho\left(V^{n+1}, V^n\right) = \rho\left(TV^n, TV^{n-1}\right) \leq \beta\rho\left(V^n, V^{n-1}\right) \leq \beta^n\rho\left(V^1, V^0\right).$$

Therefore, for any $m > n$ we have

$$\begin{aligned}
\rho\left(V^m, V^n\right) &\leq \rho\left(V^m, V^{m-1}\right) + \rho\left(V^{m-1}, V^{m-2}\right) + \cdots + \rho\left(V^{n+1}, V^n\right), \\
&\leq \left(\beta^{m-1} + \beta^{m-2} + \cdots + \beta^n\right)\rho\left(V^1, V^0\right), \\
&\leq \frac{\beta^n}{1-\beta}\rho\left(V^1, V^0\right),
\end{aligned}$$

where the first inequality follows from the triangular inequality property. Therefore, $\{V^n\}_{n=0}^{\infty}$ is a Cauchy sequence, since

$$\frac{\beta^n}{1-\beta} \to 0 \qquad \text{as } n \to \infty.$$

Since S is complete, $V^n \to V$.

Now we need to show that $V = TV$. To do that, note that for any $\varepsilon > 0$ and $V^0 \in S$, we have

$$\begin{aligned}
\rho(V, TV) &\leq \rho\left(V, T^n V^0\right) + \rho\left(T^n V^0, TV\right), \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2},
\end{aligned}$$

for large enough n since $\{V^n\}_{n=0}^{\infty}$ is a Cauchy sequence. Therefore, $V = TV$.

Finally, suppose that another function $W \in S$ satisfies $W = TW$. Then,

$$\rho(V, W) = \rho(TV, TW) \leq \beta\rho(V, W).$$

This is a contradiction, unless $V = W$.

$$\rho\left(T^n V^0, V\right) = \rho\left(T^n V^0, TV\right) \leq \beta\rho\left(T^{n-1}V^0, V\right) \leq \beta^n\rho\left(V^0, V\right).$$

QED

Corollary 1. *Let* (S, ρ) *be a complete metric space and let* $T : S \to S$ *be a contraction mapping with fixed point* $V \in S$*. If* S' *is a closed subset of* S *and* $T(S') \subseteq S'$*, then* $V \in S'$*. If in addition* $T(S') \subseteq S'' \subseteq S'$*, then* $V \in S''$*.*

Proof. Choose $V^0 \in S'$ and note that $\{T^n V^0\}$ is a sequence in S' converging to V. Since S' is closed, it follows that $V \in S'$. If $T(S') \subseteq S''$, it then follows that $V = TV \in S''$.

QED

Theorem 4. (Blackwell's Sufficiency Condition) *Let* $X \subseteq R^l$ *and* $B(X)$ *be the space of bounded functions* $V : X \to R$ *with the uniform metric* $\rho(V, W) = \sup_{x \in X} |V - W|$*. Let* $T : B(X) \to B(X)$ *be an operator satisfying*

1. *(Monotonicity)* $W, V \in B(X)$*. If* $V \leq W$ *(i.e.,* $V(x) \leq W(x)$ *for all* x*) then* $TV \leq TW$*.*

2. *(Discontinuity) There exists some constant* $\beta \in (0, 1)$ *such that* $T(V + a) \leq TV + \beta a$*, for all* $V \in B(X)$ *and* $a \geq 0$*.*

Then T *is a contraction mapping with modulus* β*.*

Proof. For every $W, V \in B(X)$, , $V \leq W + \rho(V, W)$. Thus, conditions (1) and (2) imply that

$$TV \leq T(W + \rho(V, W)) \leq TW + \beta\rho(V, W),$$

where the first inequality is by monotonicity and the second inequality is by discontinuity. Thus,

$$TV - TW \leq \beta\rho(V, W).$$

By permuting the functions it is easy to show that

$$TW - TV \leq \beta\rho(V, W).$$

Consequently,

$$|TV - TW| \leq \beta\rho(V, W),$$

so that

$$\rho(TV, TW) \leq \beta\rho(V, W).$$

Therefore T is a contraction.

QED

2.5 Neoclassical Growth Model

Consider the mapping

$$(TV)(k) = \max_{k' \in \mathcal{K}} \left\{ U\left(F(k) - k'\right) + \beta V(k') \right\}, \tag{14}$$

where $k, k' \in \mathcal{K} = \{k_1, k_2, ..., k_n\}$.

Question: Is T a contraction?

We should check monotonicity and discontinuity, so that we can use Blackwell's Theorem.

1. **Monotonicity.** Suppose $V(k) \leq W(k)$ for all k. We need to show that $(TV)(k) \leq (TW)(k)$.

$$(TV)(k) = \left\{U\left(F(k) - k'^{*}\right) + \beta V(k'^{*})\right\},$$

where k'^{*} maximizes (14). Now we can clearly see that

$$\begin{aligned} (TV)(k) &= \left\{U\left(F(k) - k'^{*}\right) + \beta V(k'^{*})\right\}, \\ &\leq \max_{k' \in \mathcal{K}} \left\{U\left(F(k) - k'\right) + \beta W(k')\right\}, \\ &= (TW)(k). \end{aligned}$$

2. **Discontinuity.**

$$\begin{aligned} (TV + a)(k) &= \max_{k' \in \mathcal{K}} \left\{U\left(F(k) - k'\right) + \beta\left(V(k') + a\right)\right\}, \\ &= \max_{k' \in \mathcal{K}} \left\{U\left(F(k) - k'\right) + \beta V(k')\right\} + \beta a, \\ &= (TV)(k) + \beta a. \end{aligned}$$

2.6 Characterizing the Value Function

What can we say about the value function V? In particular,

1. Is V continuous in k?
2. Is V strictly increasing in k?
3. Is V strictly concave in k?
4. Is V differentiable in k?

Definition 6. *A function $V : X \to R$ is strictly increasing if $x > y$ implies $V(x) > V(y)$. A function $V : X \to R$ is non-decreasing (or increasing) if $x > y$ implies $V(x) \geq V(y)$.*

Definition 6. *A function $V : X \to R$ is strictly concave if*

$$V\left(\theta x + (1-\theta)y\right) > \theta V(x) + (1-\theta)V(y),$$

for all $x, y \in X$ such that $x \neq y$ and $\theta \in (0,1)$. A function $V : X \to R$ is concave if

$$V\left(\theta x + (1-\theta)y\right) \geq \theta V(x) + (1-\theta)V(y),$$

for all $x, y \in X$ such that $x \neq y$ and $\theta \in (0,1)$.

Assumption 1. Let U and F be strictly increasing functions.

Assumption 2. Let U and F be strictly concave functions.

Theorem 5. *The function V is strictly increasing and strictly concave.*

Proof. Consider the mapping, as in (14), given by

$$(TV)(k) = \max_{k' \in \mathcal{K}} \left\{ U\left(F(k) - k'\right) + \beta V(k') \right\}.$$

It will be shown that the operator T maps concave functions into strictly concave ones. It is also true that the operator T maps increasing functions into strictly increasing ones. Let V be concave function. Take two point $k_0 \neq k_1$ and let $k_\theta = \theta k_0 + (1-\theta) k_1$. Observe that $F(k_\theta) > \theta F(k_0) + (1-\theta) F(k_1)$, since F is strictly concave. Now it need to be shown that

$$(TV)(k_\theta) > \theta (TV)(k_0) + (1-\theta)(TV)(k_1).$$

To this end, define $k_0^{'*}$ as the maximizer of $(TV)(k_0)$, $k_1^{'*}$ as the maximizer of $(TV)(k_1)$, and $k'_\theta = \theta k_0^{'*} + (1-\theta) k_1^{'*}$. Note that k'_θ is a feasible choice when $k = k_\theta$, since $k_0^{'*} \le F(k_0)$ and $k_1^{'*} \le F(k_1)$, while $\theta F(k_0) + (1-\theta) F(k_1) < F(k_\theta)$. Now,

$$\begin{aligned} (TV)(k_\theta) &\geq U\left(F(k_\theta) - k'_\theta\right) + \beta V(k'_\theta), \quad \text{since } k'_\theta \text{ is non-optimal,} \\ &> \theta \left[U\left(F(k_0) - k_0^{\prime *}\right) + \beta V(k_0^{\prime *}) \right] \\ &\quad + (1-\theta) \left[U\left(F(k_1) - k_1^{\prime *}\right) + \beta V(k_1^{\prime *}) \right], \quad \text{by strict concavity} \\ &> \theta (TV)(k_0) + (1-\theta)(TV)(k_1) \quad \text{by definition.} \end{aligned}$$

QED

Remark. *The space of strictly concave functions is not complete. Hence, to finish the argument an appeal to Corollary 1 of the contraction mapping can be made.*

Theorem 6. *The function V is continuous in k.*

Proof. It will be shown that the operator described in (14) maps strictly increasing, strictly concave C^2 functions into strictly increasing, strictly concave C^2 functions. Suppose that V^n is a continuous, strictly increasing, strictly concave C^2 function. The decision rule for k' is determine from the first-order condition

$$U_1\left(F(k) - k'\right) = \beta V^{n\prime}(k').$$

This determine k' as a continuously differentiable function of k by the implicit function theorem. Note that $0 < dk'/dk < F_1(k)$. Therefore, $V^{n+1}(k)$ is strictly increasing, strictly concave C^2 function too, since

$$V_1^{(n+1)}(k) = U_1\left(F(k) - k'\right) F_1(k).$$

The limit of such a sequence must be continuous function (although it need not be a C^2function).

QED

Differentiability:

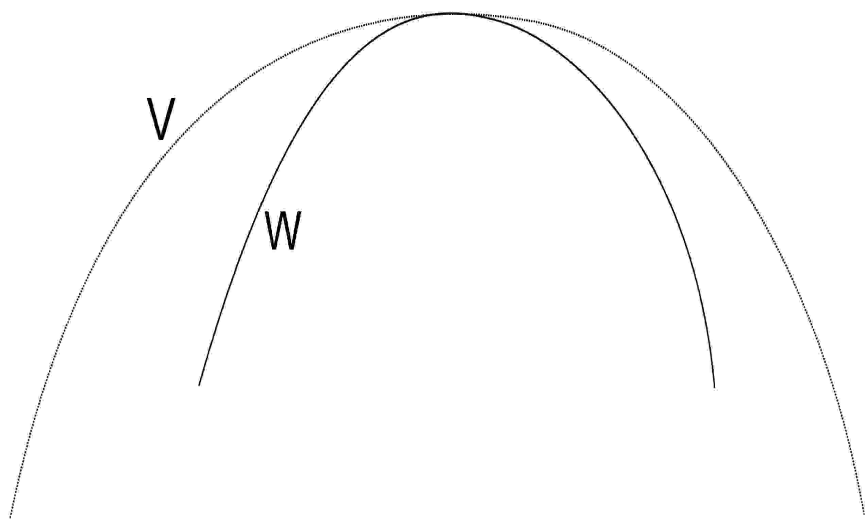
Lemma 1. *Let $X \subseteq R^l$ be a convex set, $V : X \to R$ be a concave function. Pick an $x_0 \in int(X)$*

and let D be a neighborhood of x_0. If there is a concave, differentiable function $W : D \to R$ with $W(x_0) = V(x_0)$ and $W(x) = V(x)$ for all $x \in D$, then V is differentiable at x_0 and

$$V_i(x_0) = W_i(x_0), \quad \textit{for } i = 1, 2, ..., l.$$

Proof. See Figure 5 below.

Figure 5: Differentiability of V



Theorem 7. (Benveniste and Scheinkman) *Suppose that K is a convex set and that U and F are strictly concave C^1 functions. Let $V : K \to R$ in line with (14) and denote the decision rule associated with this problem by $k' = G(k)$. Pick $k_0 \in int(K)$ and assume that $0 < G(k_0) < F(k_0)$. Then $V(k)$ is continuously differentiable at k_0 with its derivative given by*

$$V_1 = U_1 (F(k_0) - G(k_0)) F_1 (k_0).$$

Proof. Clearly, there exists some neighborhood D of $k_0$such that $0 < G(k_0) < F(k)$ for all $k \in D$. Define W on D by

$$W(k) = U(F(k) - G(k_0)) + \beta V(G(k_0)).$$

Now, note that W is concave and differentiable, since U and F are. Furthermore, it follows that

$$W(k) \leq \max_{k'} \left\{ U\left(F(k) - k'\right) + \beta V\left(k'\right) \right\} = V(k),$$

with the equality holding strictly at $k = k_0$. The result then follows immediately from Lemma 1.

QED