Are Two Better than One? A Note

by

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Abstract

This note examines the possibility of extending the Condorcet Jury Theorem (CJT) by relaxing the assumption of homogeneous individual decision-making skills. Our main result provides two sufficient conditions for advantageous extension of a group. These conditions are referred to as (Weak) "Two Are Better than One" (WTBO) and TBO. The latter requires that the average decision-making competence of the two added members exceeds those of any existing group member. The weaker condition WTBO requires that the sum of the optimal weights of the two added members is larger than the optimal weight of any existing group member. Immediate special cases of our result include CJT settings wherein decision-making skills are assumed to be identical as well as situations wherein such skills are of two types: low and high.

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1. Introduction

According to the Condorcet Jury Theorem (CJT) (Condorcet 1785), a group of decision makers applying a simple majority rule is more likely to make a correct collective decision than any one of its members alone, and the likelihood of a correct decision becomes certain as group size tends to infinity. The theorem justifies broad democratic participation in collective decision making when all voters share the same objective, but differ in their judgments regarding the correct decision. In the simplest form of the CJT, a group of decision makers votes independently on a binary choice when each voter has the same probability $p > 1/2$ of choosing the correct alternative. By CJT, the larger the group, the better the group performs in terms of the likelihood of making the correct decision by majority voting.[1]

Various attempts have been made to generalize the theorem to heterogeneous groups assuming that the group applies the simple majority rule.[2] In particular, Shapley and Grofman (1984) and Nitzan and Paroush (1982) have shown that for a decision rule to be optimal, the weight assigned to an individual group member should be proportional to his decision-making skills, such that if p_i represents individual i's skill, then his optimal weight is $w(p_i)$ where $w(p) = \ln\frac{p}{1-p}$. As is well known, this result implies that in a three-member group, the application of simple majority rule is superior to that of rule by an expert, if the sum of the optimal weights of any two individuals exceeds the optimal weight of the third individual. In the context of the current study, we focus on the general condition, "Two are Better than One" (TBO), which requires that the two members added to a group are better at taking the correct decision than any existing group member. This condition is satisfied when the average decision-making competencies of the added group members are larger than the skill of any existing group member. We also study the weaker sufficient condition (WTBO) requiring that the sum of the optimal weights of the added members be

[1] For a recent survey of the literature inspired by CJT, see Nitzan and Paroush (2017).

[2] Owen et al. (1989) relaxed the assumption of identical competencies assuming, however, that the average competence of the team is a constant larger than one-half. Berend and Paroush (1998) formulated necessary and sufficient conditions for superior outcomes in heterogeneous groups. Kanazawa (1998) studied situations wherein heterogeneous groups perform better than homogeneous groups. Dietrich and List (2013) also discuss the case of heterogeneous judgments aggregation.

larger than the optimal weight of any existing member. The objective of our note is to show that these conditions are sufficient for justifying the extension of a group.

These sufficient conditions ensure that the incremental extension of a group is always desirable.[3] TBO is satisfied in the special case when decision-making skills are homogeneous, which is the basic result of Condorcet. It is also satisfied when the group comprises members of heterogeneous skills; however, the added individuals must be more competent in taking decisions than the existing group members.

In general, TBO is not necessarily satisfied, but when WTBO is satisfied the desirability of group extension also is justified. Assuming that the group's skills are heterogeneous, adding two members with possibly differentially poor skills can still be desirable, provided that this weaker condition is satisfied. In particular, when just two possible decision-making skills are possible, group extension is always advantageous, provided that extensions take the form of heterogeneous pairs. [4] This result is independent of the numbers of the two types in the existing group. The two-type case is especially relevant to a group consisting of females and males, old and young, or experienced and inexperienced members in the context of organizational design, e.g., designing a judicial system that consists of several courts. [5]

Note that in this study the extension of a group of decision makers is carried out by adding two members to an existing odd-numbered group. The reason for this kind of extension is that we wish to enable the use of simple majority rule, which requires that the minimal number of added group members equal two. [6]

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[3] Feld and Grofman (1984) examined whether the probability of reaching the correct decision by an enlarged group is a function of the average, median, or majority competencies of the original group for cases in which individual competence can be greater than or less than one-half. They state only a sufficient condition. Grofman Owen and Feld (1983) investigate the conditions under which expanding the group's size or adding certain individuals to it enhances the group's performance. Karotkin and Paroush (2003) studied the tradeoff between quality and quantity when members are added to the group.

[4] An extreme two-type case has been studied by Ben-Yashar and Zehavi (2011), assuming that one unskilled type is equivalent to the tossing of a fair coin.

[5] An extensive analysis of committee design in the context of two-type decisional skills appears in Ben-Yashar and Danziger (2011).

[6] To prevent tied votes, simple majority rule requires odd numbers of decision makers.

2. The model

Consider a group N_n with an odd number of members n. The group confronts two alternatives, 1 and -1, one of which is correct and therefore better for all voters.[7] As is common in decision problems, the identity of the better alternative is unknown. Every voter selects one of the two alternatives, and an aggregation mechanism is applied to select the collectively preferred alternative. Voter *i* chooses the correct alternative with probability p_i, which reflects his competence. We assume that $\forall\ j \neq i$, p_i is independent of p_j, $\forall i$, ½< p_i <1 and, with no loss of generality, $\forall\ j \neq i$, $p_i \geq p_j$ if $i < j$.[8] The vector $p^n = (p_1,..., p_n)$ is referred to as the n voters' competence structure or profile. Let $\pi_{SMR}\left(p^n\right)$ denote the probability that a group consisting of n voters associated with a competence vector p^n chooses the correct alternative under simple majority rule (SMR).[9] Since we focus on the effect of adding two members to a group, let us denote by $\pi_{SMR}\left(p^{n-2}, p_i, p_j\right)$ the probability of making a correct choice, where the vector p^{n-2} relates to the competencies of the (n-2) members of the existing group N_{n-2}, and p_i , p_j denotes the decision-making competence of the added members *i* and *j*.

Let $S_a^T = \left\{S \middle| S \subseteq T, \middle| S = a \middle|\right\}$ and $S_{a\uparrow}^T = \left\{S \middle| S \subseteq T, \middle| S > a \middle|\right\}$. Following Ben Yashar and Paroush (2000), $\pi_{SMR}\left(p^{n-2}, p_i, p_j\right)$ can be presented as:

$\pi_{SMR}\left(p^{n-2}, p_i, p_j\right) = p_i\ p_j$ A+ (1-(1- p_i)(1- p_j)) B+ C,

where A is the sum of all products of $\frac{n-3}{2}$ probabilities p_t (t$\neq i, j$) and the remaining (1- p_l), $l \neq i, j, t$. That is,

$$A = \sum_{S \in S^{N_{n-2}}_{\frac{n-3}{2}}} \prod_{t \in S} p_t \prod_{l \in N_{n-2} \setminus S} (1 - p_l)$$

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7 Earlier studies of two alternative models include Hoeffding (1956) and Young (1988). Also see Ben-Yashar (2014) and Ben-Yashar and Nitzan (2014).

8 As most common in the literature dealing with the main question of concern herein (i.e., is an increase in the size of the group desirable?), we assume independent voting. Nevertheless, Ladha (1995) examines the consequences of relaxing the independence assumption.

9 Earlier studies of simple majority rule include Baharad and Ben-Yashar (2009), Ben-Yashar and Danziger (2011), Berend and Sapir (2005), Grofman et al. (1983), Ladha (1995), Nitzan and Paroush (1982) and Shapley and Grofman (1984).

B is the sum of all products of $\frac{n-1}{2}$ probabilities p_t (t≠ i, j) and the remaining

(1- p_l), $l \neq i, j, t$. That is,

$$B=\sum_{S\in S^{N_{n-2}}_{\frac{n-1}{2}}} \prod_{t\in S} p_t \prod_{l\in N_{n-2}\setminus S} (1-p_l).$$

C is the sum of all products of more than $\frac{n-1}{2}$ up to (n-2) probabilities p_t (t≠ i, j) and the remaining (1- p_l) terms, $l \neq i, j, t$. That is,

$$C=\sum_{S\in S^{N_{n-2}}_{\frac{n-1}{2}\uparrow}} \prod_{t\in S} p_t \prod_{l\in N_{n-2}\setminus S} (1-p_l).$$

Note that the probability of making a correct collective decision, given the decisional structure of the existing group N_{n-2}, is

$$\pi_{SMR}\left(p^{n-2}\right)=B+C$$

and, therefore,

$$\pi_{SMR}\left(p^{n-2},p_i,p_j\right)-\pi_{SMR}\left(p^{n-2}\right) = p_i\ p_j \mathrm{A} - (1-p_i)(1-p_j)\mathrm{B}.$$

The probability that the two-member added group of individuals *i* and *j* will reach a correct decision is $\frac{\mathrm{p_i}+p_j}{2}$, which is the sum of the probabilities that both are correct and that in half of the cases one is correct while the other is not. That is,

$$\pi_{SMR}\left(p_i,p_j\right)= \mathrm{p_i\ p_i} + \frac{1}{2}(1-p_i)p_j + \frac{1}{2}p_i(1-p_j) = \frac{\mathrm{p_i}+p_j}{2}.$$

The condition TBO requires that the two added individuals jointly are more competent than any of the existing group members. This condition therefore requires that $\frac{p_i + p_j}{2} > p_1$. The condition WTBO requires that the sum of the optimal weights of the added members is larger than the optimal weight of any existing member. This condition therefore requires that $w(p_i) + w(p_j) > w(p_1), \quad where \quad w(p) = \ln \frac{p}{1-p}$.

3. Result

Our main result is the claim that TBO and its weaker version WTBO are sufficient conditions for advantageous group extension. That is, if the two individuals added to an existing group jointly are more competent than any member of the existing group, which implies that the sum of the optimal weights of the added members is larger than the optimal weight of any existing member, then adding them to the group improves the probability that the group majority is correct.

Theorem:

$$\frac{p_i + p_j}{2} \geq p_1 \Rightarrow w(p_i) + w(p_j) > w(p_1) \Rightarrow \pi_{SMR}\left(p^{n-2}, p_i, p_j\right) > \pi_{SMR}\left(p^{n-2}\right)$$

Proof:

$$w(p_i) + w(p_j) > w(p_1) \equiv p_i p_j p_1 + (1-p_1) p_i p_j + p_1(1-p_i) p_j + p_1 p_i (1-p_j) > p_1$$

(see Nitzan and Paroush 1982). This condition is equivalent to:

$$-2 p_i p_j p_1 + p_i p_j + p_1 p_j + p_1 p_i > p_1 .$$

The expression on the left-hand side of the above inequality is larger than $\frac{p_i + p_j}{2}$. [10]

Hence, if $\frac{p_i + p_j}{2} \geq p_1$, then $w(p_i) + w(p_j) > w(p_1)$.

To conclude the proof, we have to show that

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[10] To prove this claim, notice that

$$-2 p_i p_j p_1 + p_i p_j + p_1 p_j + p_1 p_i > (p_1 + p_j)/2 \Leftrightarrow -4 p_i p_j p_1 + 2 p_i p_j + 2 p_1 p_j + 2 p_1 p_i > (p_i + p_j) \Leftrightarrow$$
$$2 p_i p_j (1 - 2p_1) + 2 p_1 (p_i + p_j) - (p_i + p_j) > 0 \Leftrightarrow -2 p_i p_j (2p_1 - 1) + (p_i + p_j)(2p_1 - 1) > 0 \Leftrightarrow$$
$$(2p_1 - 1)(p_i(1-p_i) + p_j(1-p_j)) > 0 \quad \textit{which is true.}$$

$$w(p_i)+w(p_j)>w(p_1)\Rightarrow \pi_{SMR}\left(p^{n-2},p_i,p_j\right)- \pi_{SMR}\left(p^{n-2}\right)>0 .$$

Notice that

$$w(p_i)+w(p_j)>w(p_1)\Leftrightarrow \ln\frac{p_i}{1-p_i}+\ln\frac{p_j}{1-p_j}>\ln\frac{p_1}{1-p_1}\Leftrightarrow \frac{p_i}{1-p_i}\frac{p_j}{1-p_j}>\frac{p_1}{1-p_1}.$$

Recall that $\pi_{SMR}\left(p^{n-2},p_i,p_j\right)-\pi_{SMR}\left(p^{n-2}\right) = p_i\, p_j A - (1-p_l)(1-p_j)B$. The latter expression is positive if and only if $\frac{p_i p_j}{(1-p_i)(1-p_j)}>\frac{B}{A}$. Since $\frac{B}{A}\le\frac{p_1}{(1-p_1)}$ the proof is completed.

Q.E.D

When individual decision-making skills are identical, TBO is satisfied. Group extension therefore always is advantageous. This special case of the theorem is CJT.

As time elapses, decision makers tend to become equally skilled owing to accumulated common experience and exposure to effective learning processes. Skill uniformity can be distorted, at least temporarily, by injecting "new blood" into the group, namely by adding members with heterogeneous competencies. When the additional members are more skilled than the existing members, the enlargement of the group obviously improves the group's performance. Also, if one new member is more skilled and the other is less skilled than the existing group's average, such that the average skill in the extended group increases or remains unchanged, then the extension is advantageous.[11] But the improvement in group performance can also happen when the individual quality of its reinforcements is lower . Specifically, group extension is warranted when the competencies of the added individuals are even lower than those of the existing members, such that TBO is not satisfied, but WTBO is. For example, suppose that the skills of an existing group members are equal to 0.7 and those of the two new members are 0.6 and 0.65. Although the group's average decision-making skill is reduced, the collective probability of reaching the correct decision under simple majority rule rises, independent of the number of individuals in the existing group.[12]

[11] Ben-Yashar and Paroush (2000) show this result when the average skill remains unchanged.

[12] Miller (1983) has noted that adding members may increase the collective competence even if it

The reason behind the condition WTBO is the following: the addition of the two new members to N_{n-2} is meaningful only when their decisions are identical to *(n-3)/2* members of the original group who share different judgments relative to the remaining *(n-1)/2* members. In other words, the influence of two new members hinges on being pivotal in the collective decision.

Assume that the individuals are not homogeneous, but of just two types (female and male, old and young, experienced and inexperienced). Such a two-type setting is relevant and applicable both in small committee settings, such as courts, where the types typically comprise senior and less experienced judges as well as in voting on referendums, where the types are male and female voters. Under such minimal skill variability, extending the decision-making group with a mixed-type pair group is desirable independent of the number of the two types in the original group. Although TBO is not satisfied because the group's average skill can be reduced, the weak condition WTBO is satisfied. Denoting the two types of competence by p_1 and p_2, $w(p_1)+w(p_2)>w(p_1)$, which is WTBO.

Following Feld and Grofman (1984), the intuition behind this result can be explained by relating it to group extension in a two-stage process. In the first stage, a highly competent individual is added to the odd-numbered two-type group. This obviously increases the group's competence. In the second stage, a less skilled individual is added to the even numbered *n-1* group. This added member is pivotal only when the judgments of the other members are split. Since his probability of making the correct decision is larger than one-half, taking his judgement into account is advantageous to a random decision, that is, his addition enhances the group's competence.

In general, when individual decisional skills in the existing group are heterogeneous, TBO is satisfied, if the average skill of the added individuals exceeds that of the existing group members.

A special case is obtained when one of the added members is more competent and the other is less competent than the existing group members, such that the two added individuals are jointly more competent than any of the existing members.

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reduces the average individual competence. Our result provides a general sufficient condition, WTBO, for this possibility.

4. Conclusion

Our main result extends Condorcet's Jury Theorem (CJT). It establishes that adding two members to a group improves the probability that the group's majority is correct, if the two added members jointly are better than any single existing group member. This property, which is called "Two Better than One" (TBO), implies that the average decision-making competence of the two added individuals exceeds that of the most competent member of the existing group. CJT is obtained as a special case of our result when we assume that individuals' decision-making skills are homogeneous. Another trivial special case is obtained when the added group members are more competent than the most competent member of the existing group. Interestingly, in the special case where decisional skills are of two types and the two added group members differ in those skills, group extension is desirable. This result is valid even though the sufficient condition TBO is not satisfied, but the weaker sufficient condition WTBO is.

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