Symmetry, equilibria, and approximate equilibria in games with countably many players

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Abstract

I consider games with finite pure-strategy sets and countably many players. I present a "simple" example of such a game for which an ϵ-equilibrium exists for all $\epsilon > 0$, but for which a Nash equilibrium does not exist. This game is not symmetric, which is inevitable in the following sense: under a mild condition on the utility function—the *co-finiteness condition*—existence of an ϵ-equilibrium for all $\epsilon > 0$ in a symmetric game implies the existence of a Nash equilibrium in that game. The co-finiteness condition is logically unrelated to continuity.

Keywords: ϵ equilibrium, equilibrium non-existence, infinite games, symmetry, tail events.

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1 Introduction

Peleg (1969) proved that a Nash equilibrium exists in a game whose player-set is of an arbitrary cardinality, provided that all pure-strategy sets are finite and all utility functions are continuous in a suitable sense. He also demonstrated the existence of a game with infinitely many players, each of whom has finitely many pure strategies (and discontinuous utility) in which a Nash equilibrium does not exist. Here is his example:

Example (Peleg, 1969): Let $\mathbb{N}$ be the set of players, each player i has the set of pure strategies $\{0,1\}$, and each player i's preferences over pure profiles $a \in \{0,1\}^{\mathbb{N}}$ are given by the following utility function:

$$u_i(a) = \begin{cases} a_i & \text{if } \sum_{j=1}^{\infty} a_j < \infty \\ -a_i & \text{otherwise} \end{cases}$$

The sum $\sum_{j=1}^{\infty} a_j$ is finite if and only if there is a finite number of 1's. Since each a_i is an independent random variable, the occurrence of the event $\{\sum_{j=1}^{\infty} a_j < \infty\}$ depends on a countable sequence of independent random variables; since it is invariant to the realization of any finite number of them, it is a *tail event*. Kolmogorov's 0-1 Law (henceforth, the 0-1 Law) states that the probability of a tail event is either zero or one.[1] It follows from the 0-1 Law that this game does not have a Nash equilibrium. To see this, let p denote the probability of $\{\sum_{j=1}^{\infty} a_j < \infty\}$ in a putative equilibrium. If $p = 1$ then the unique best-response of each player is to play 1, which implies $p = 0$. If, on the other hand, $p = 0$, then each player's unique best-response is to play 0, which implies $p = 1$.[2]

Peleg's example is strong, in the following senses. First, in the game it describes,

[1] See Billingsley (1995).

[2] The connection between tail events and equilibrium non-existence is profound, and goes beyond the scope of this particular example. See Voorneveld (2010).

not only a Nash equilibrium fails to exist, even an ϵ-equilibrium (Radner (1980)) does not exist, for all sufficiently small $\epsilon > 0$. Secondly, the associated non-existence proof is not elementary, in the sense that it relies on a "high power" mathematical tool. This gives rise to the following questions:

1. Is there an example of a game with finite pure-strategy sets and no equilibrium, for which the non-existence proof is elementary?

2. What is the relation between the existence/non-existence of ϵ-equilibrium and Nash equilibrium in such games?

I answer the first question positively by describing a game with countably many players and finite pure-strategy sets, in which no equilibrium exists, and where non-existence follows from direct inspection of the players' strategic considerations. In this game, precisely one player has a discontinuous utility function; for any other player, the utility function is not only continuous, but, moreover, depends only on the actions of finitely many other players. In this game an ϵ-equilibrium exists for any $\epsilon > 0$, which addresses the second question: existence of an ϵ-equilibrium for all $\epsilon > 0$ does not guarantee the existence of a Nash equilibrium in such games.

In my game, just as in Peleg's, all pure strategy sets are $\{0, 1\}$. However, an important difference between the two games is that in Peleg's game the utility of each player i is invariant to any permutation on the actions of players $j \neq i$, while in my game this is not the case. Call a game with a common strategy set and the aforementioned invariance property a *symmetric game*. The lack of symmetry in my game is "almost" inevitable: I prove that if a symmetric game satisfies a mild condition called the *co-finiteness condition*, and if this game has an ϵ-equilibrium for all $\epsilon > 0$, then it also has a Nash equilibrium. This result does not rely on continuity, as the co-finiteness condition is logically unrelated to continuity.

Section 2 describes the model and Section 3 contains the results.

2 Model

A game in normal-form is a tuple $G = [N, (A_i)_{i \in N}, (u_i)_{i \in N}]$, where $N \neq \emptyset$ is the set of players, A_i is the set of player i's pure strategies (or actions), and $u_i \colon \Pi_{i \in N} A_i \to \mathbb{R}$ is i's utility function, defined on pure action profiles. A mixed strategy for i, generically denoted by α_i, is a probability distribution over A_i.

In this paper I consider games such that:

1. N is infinite and countable,

2. A_i is finite for all $i \in N$.

In the sequel, a *game* means a tuple $G = [N, (A_i)_{i \in N}, (u_i)_{i \in N}]$ that respects these two restrictions.

A profile of mixed strategies is denoted by α and player i's expected utility under α is denoted by $U_i(\alpha)$. A *Nash equilibrium* is a profile α such that the following holds for each i: $U_i(\alpha) \geq U_i(\alpha')$, where α' is any alternative profile that satisfies $\alpha'_j = \alpha_j$ for all $j \in N \setminus \{i\}$. A Nash equilibrium α is *pure* if for each i there is an $a_i \in A_i$ such that $\alpha_i(a_i) = 1$. A profile α is an *ϵ-equilibrium* if the following holds for each i: $U_i(\alpha) \geq U_i(\alpha') - \epsilon$, where α' is any alternative profile that satisfies $\alpha'_j = \alpha_j$ for all $j \in N \setminus \{i\}$. If α is an ϵ-equilibrium, say that each of its components α_i is an *ϵ-maximizer* of i's payoff (given $(\alpha_j)_{j \neq i}$).

A game is *symmetric* if there is a set $A \neq \emptyset$ such that $A_i = A$ for all $i \in N$ and the preferences of each player i over elements of A^N are given by a two-argument function, $u(x, a)$, that satisfies the following: its first argument is i's own-action, its second argument is the profile describing the actions of players $j \neq i$, and, finally, $u(x, a) = u(x, b)$ for any own-action x and any a and b for which there is a permutation on $N \setminus \{i\}$, π, such that $b_j = a_{\pi(j)}$ for all $j \neq i$. That is, each player i cares about what actions his opponents are playing, but not about who is playing what action. A Nash equilibrium α of a symmetric game is *symmetric* if $\alpha_i = \alpha_j$ for all $i, j \in N$.

3 Results

Consider the following game, G^*: the player set is $\mathbb{N}$, each player i has the set of pure strategies $\{0, 1\}$, and utilities, defined on pure profiles, are as follows. For player 1 the utility is:

$$u_1(a) = \begin{cases} -a_1 & \text{if } a_n = 1 \text{ for all } n > 1 \\ a_1 & \text{otherwise} \end{cases}$$

The utility for any other player n is as follows:

$$u_n(a) = \begin{cases} a_n & \text{if } a_l = 1 \text{ for all } l < n \\ -a_n & \text{otherwise} \end{cases}$$

Proposition 1. *G^* does not have a Nash equilibrium.*

Proof. Assume by contradiction that $\alpha = (\alpha_i)_{i \in N}$ is a Nash equilibrium. Look at player 1. If he plays the pure action 1 (namely, if $\alpha_1(1) = 1$), then player 2 necessarily plays his unique best response—the pure action 1; subsequently, it is easy to see that every player n plays the action 1 with certainty. But in this case player 1 is not playing a best response, in contradiction to equilibrium. If player 1 plays the pure action 0 then player 2 necessarily plays the action 0 as well, with certainty. But this means that player 1 is not playing a best-response. Therefore player 1 strictly mixes.

Let I be the set of players $i > 1$ who do not play the action 1 with certainty; that is, $I \equiv \{i \in N : i > 1, \alpha_i(1) < 1\}$. Obviously $I \neq \emptyset$; otherwise, player 1 would not mix, but play the pure action 0. Let $i^* \equiv \min I$. Since $i^* \in I$, $\alpha_{i^*}(1) < 1$, and it therefore follows that $\alpha_1(1) \leq \frac{1}{2}$ ($\alpha_1(1) > \frac{1}{2}$ implies that i^*'s unique best-response is the action 1).

Case 1: $\alpha_1(1) < \frac{1}{2}$. Here i^*'s unique best-response is the action 0. This means that player 1 is not playing a best-response: he should switch to the pure action 1.

Case 2: $\alpha_1(1) = \frac{1}{2}$. Since i^* plays the action 1 with probability smaller than 1 (by the definition of the set I), it follows that the unique best-response of player $j = i^* + 1$

is to play the pure action 0. But this means (by the same argument from Case 1) that player 1 is not playing a best-response: he should switch to the pure action 1. $\square$

Despite not having a Nash equilibrium, G^* has an ϵ-equilibrium, for any $\epsilon > 0$.

Proposition 2. *G^* has an ϵ-equilibrium, for any $\epsilon > 0$.*

Proof. Let $\epsilon > 0$. Let $N^* \equiv \lceil \frac{\log 0.5}{\log(1-\epsilon)} \rceil$. Since we are interested in small ϵ's, we can assume that $N^* > 1$. Let s^* be the strategy that plays 1 with probability $1 - \epsilon$ and play 0 with probability ϵ. Let each player n play strategy s^* if $n \leq N^*$ and play the pure action 0 otherwise. It is easy to verify that the resulting profile is an ϵ-equilibrium. $\square$

Since G^* is not a symmetric game, one may wonder whether the pathology it exhibits—existence of approximate equilibria and non-existence of exact equilibria—is possible if one restricts attention to symmetric games. The following result shows that under a certain mild condition, the answer is negative: existence of ϵ-equilibrium for all $\epsilon > 0$ implies the existence of Nash equilibrium.

The condition is the following. Say that a symmetric game satisfies the *co-finiteness condition* if for any own-action x and any two action profiles of the other players, a and b, the following is true: if $|\{j : a_j \neq b_j\}| < \infty$ then $u(x, a) = u(x, b)$. Note that Peleg's game satisfies this condition.

Theorem 1. *Let G be a symmetric game that satisfies the co-finiteness condition. If G has an ϵ-equilibrium for all $\epsilon > 0$, then it also has a Nash equilibrium. Moreover, this Nash equilibrium is pure.*

Proof. Let G be a game as above. Let A be its (finite) set of pure strategies. Given $\epsilon > 0$, let $\alpha = \alpha(\epsilon)$ be an ϵ-equilibrium. Let $\Pr_\alpha$ denote the probability measure that α induces on A^N, where N is the set of players. Given a non-empty $S \subset A$, let $E(S)$ be the event "each element of S is realized infinitely many times." Since $E(S)$ is a tail event, its $\Pr_\alpha$-measure is either zero or one.[3] That is, $\emptyset \neq S \subset A \Rightarrow \Pr_\alpha(E(S)) \in \{0, 1\}$.

Claim: There is an $S \subset A$ such that $\Pr_\alpha(E(S)) = 1$.

Proof of the Claim: Let $\{S_1, \cdots, S_K\}$ be the non-empty subsets of A ($K = 2^{|A|} - 1$). Since $A^N = \cup_{k=1}^K E(S_k)$, the falseness of the Claim implies $1 \leq \sum_{\cup_{k=1}^K E(S_k)} \Pr_\alpha(E(S)) = 0$, a contradiction.

Let $X \equiv \cup_{\{\Pr_\alpha(E(S))=1\}} S$. By the Claim, $X \neq \emptyset$. The set X consists of all pure actions that occur infinitely many times in the α-equilibrium with probability one. Suppose that $X = \{x_1, \cdots, x_L\}$. Let a be the following profile:

$$a = (x_1, \cdots, x_L, x_1, \cdots, x_L, \cdots, x_1, \cdots, x_L, \cdots).$$

Look at a particular player i. With probability one the behavior of the others is given by a profile, b, that satisfies one of the following: (1) b is obtained from a by a permutation, or (2) there is a finite set of coordinates, J, such that the sub-profile $(b_j)_{j \notin J}$ is obtained from a by a permutation.[4] By symmetry and co-finiteness, every x in the support of i's strategy is an ϵ-maximizer of $u(., a)$. Since this is true for every player i, it follows that every $x \in X$ is an ϵ-maximizer of $u(., a)$. It therefore follows that a is a pure ϵ-equilibrium.

Both X and a depend on ϵ: $X = X(\epsilon)$ and $a = a(\epsilon)$. Since A is finite there is a sequence $\{\epsilon\} \downarrow 0$ such that $a(\epsilon) = a^*$ for all ϵ in the sequence. It is easy to see that a^* is a pure Nash equilibrium. $\square$

Note that the existence proof does not rely on continuity. Moreover, continuity and co-finiteness are logically unrelated. To see that co-finiteness does not imply continuity,

[3] Peleg's example is built with reference to the set $S = \{1\}$.

[4] Note that (1) is a particular manifestation of (2)—the one corresponding to $J = \emptyset$.

consider the following game, in which the set of players is $\mathbb{N}$ and each pure-strategies set is $\{0,1\}$. In this game, a player's utility from the profile a is one if a contains infinitely many 0's and infinitely many 1's, and otherwise his utility is zero. This is a symmetric game with a discontinuous utility, in which every strategy profile is a Nash equilibrium; it is easy to see that it satisfies the co-finiteness condition. To see that continuity does not imply co-finiteness, consider the following game, in which the player set is $\mathbb{N}$, the pure-strategies set is $\{0,1\}$, and the utility from the profile $a \in \{0,1\}^N$ equals $\frac{1}{1+|\{i:a_i=1\}|}$.[5]

Since Theorem 1 considers equilibria of symmetric games, it is natural to ask whether the pure equilibrium whose existence it guarantees is also a symmetric equilibrium. As the following example shows, the answer is negative: the fact that a symmetric game that satisfies the co-finiteness condition has a pure equilibrium does not imply that it has a pure symmetric equilibrium.

Consider the following symmetric game, G^{**}. The player set is $\mathbb{N}$, the set of pure strategies is $\{0,1\}$, and utility from pure profiles is as follows:

- If there are infinitely many 0's and infinitely many 1's, then a player's utility is one.
- If everybody play the same action, then a player's utility is zero.
- If there are exactly k appearances of some action $x \in \{0,1\}$, where $0 < k < \infty$, then the utility of a player whose action is x is one, and otherwise it is zero.

G^{**} satisfies the co-finiteness condition[6] and it has infinitely many non-symmetric pure Nash equilibria—every profile with infinitely many occurrences of each action is

[5] For example, let a and b two profiles such that $a_j = b_j = 0$ for all $j \notin \{2,3\}$. If $(a_2,a_3) = (1,0)$ and $(b_2,b_3) = (0,1)$ then player 1 obtains the same utility under either a or b, while if $(a_2,a_3) = (0,0)$ and $(b_2,b_3) = (1,1)$ then the utility from a is higher.

[6] This is proved in the Appendix.

an equilibrium. Obviously, it does not have a pure symmetric equilibrium.[7]

The co-finiteness condition in Theorem 1 is important: the fact that a symmetric game (that does not satisfy the co-finiteness condition) has an ϵ equilibrium for all $\epsilon > 0$ does not imply that it has a Nash equilibrium. The following game, G^{***}, exemplifies this: its player and action sets, as in the previous examples, are $\mathbb{N}$ and $\{0, 1\}$ respectively, and the utility function is as follows:

$$u_i(a) = \begin{cases} \frac{a_i}{1+|\{k:a_k=1\}|} & \text{if } \sum_{j=1}^{\infty} a_j < \infty \\ -a_i & \text{otherwise} \end{cases}$$

Note that this utility function is obtained from that of Peleg's game by a relatively minor change: replacing a_i by $\frac{a_i}{1+|\{k:a_k=1\}|}$. That this game does not have a Nash equilibrium follows from precisely the same arguments as the ones from Peleg's game. Nevertheless, G^{***} has an ϵ equilibrium for all $\epsilon > 0$.

Proposition 3. *The game G^{***} has an ϵ equilibrium for all $\epsilon > 0$.*

Proof. Let $\epsilon > 0$. Let m be such that $\frac{1}{1+m} < \epsilon$. Consider the following (pure) strategy profile: each player in $\{1, \cdots, m\}$ plays the action 1, and every other player plays the action 0. Obviously, each player $i \leq m$ is playing a best-response; each $i > m$ can only improve his payoff by $\frac{1}{2+m} < \epsilon$ via a unilateral deviation; hence, this is an ϵ equilibrium. □

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[7] It does, however, have a non-pure symmetric equilibrium: if each player plays each action with equal probability, a symmetric equilibrium obtains, because with probability one the realized profile has infinitely many occurrences of either action.

Appendix

Claim: The game G^{**} satisfies the co-finiteness condition.

Proof. Wlog, look at player 1. Wlog, suppose that he play the action 0. Let a be a pure profile describing the behavior of all players $i > n$, for some $n > 1$. We will verify that knowledge of a implies the knowledge of player 1's payoff. If a contains infinitely many 1's and infinitely many 0's, the claim is obvious. Suppose then that there is only a single action, x, that occurs infinitely many times in a.

Case 1: $x = 0$. If all the coordinates of a are 0, then player 1 obtain payoff zero, no matter what the players in $K \equiv \{2, \cdots, n\}$ play. He also receives zero (independent of play in K) if a contains a finite number of 1's.

Case 2: $x = 1$. If all the coordinates of a are 1, then player 1 obtains the utility one independent of play in K. Similarly, if a contains finitely many 0's player 1 also receives one. □

4 References

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