Fair Bias*

Uzi Segal†

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Abstract

This paper takes a simple, informal suggestion by Broome and another more explicit suggestion by Kamm for how to deal with asymmetric claims and shows how they can be interpreted to be consistent with two different social welfare functions: Sum-of-square-roots of individual utilities, and product of utilities. These functions are then used to analyze more complicated situations but I show that the first yields more intuitive results, and a better compromise of efficiency and justice, than the other.

*I wish to thank John Broome, David Heyd, Joe Ostroy, Martin Quaas, and Joel Sobel for their helpful comments. Reading the paper it should become clear that although the paper's main idea stems from an informal suggestion by Broome [1, 2], he himself clearly disagrees with some of the conclusions presented below.

†Dept. of Economics, Boston College, Chestnut Hill MA 02467. E-mail: segalu@bc.edu

1 Introduction

One of the major problems in social choice theory is the opposing appeals of the two extreme solutions — utilitarianism (that is, maximizing the sum of individual utilities) and egalitarianism. In cases of symmetric claims, it seems optimal to use a symmetric allocation mechanism. And when the good to be allocated in indivisible, randomization seems natural (see Diamond [4] and Broome [2]). Giving each claimant the same probability creates an egalitarian mechanism which under some assumption does also maximize the sum of individual utilities. But such situations are rare and little insight can be exacted from them. What is a just randomization in asymmetric situations is less transparent and may depend on the circumstances. In this paper I take a simple, informal suggestion by Broome [1, 2] and another more explicit suggestion by Kamm [9] for how to deal with asymmetric claims and show how they can be interpreted to be consistent with two different social welfare functions: The one suggests to maximize the sum of the square roots of the utilities of the social members, the other to maximize the product of these utilities. These functions are then used to analyze more complicated situations but I show that the first yields more intuitive results, and a better compromise of efficiency and justice, than the other.

The analysis of this paper should be contrasted with two of the major issues of just distributions of scarce resources: The actual distribution of utility and the mechanism used to determine this distribution. Following Harsanyi [8], there are many axiomatizations of such allocations. This literature assumes that individuals receive utility from outcomes and the question is what is the best utility allocation from a social perspective. Harsanyi [7] offered axioms leading to a utilitarian approach—society should choose the option that maximizes the sum of the personal utilities. Other utility-oriented approaches analyze no-envy environments, that is, identify fair allocations as those where no individual would like to change position with anyone else, or drop Harsanyi's linearity assumption (see Epstein and Segal [5]).

The philosophical literature on the other hand seems to pay more attention to the *procedures* used by society to determine utility allocations. Such procedures may be equal division of the goods, randomization between individuals, or affirmative action policies. One problem with this approach is that different situations call for different procedures and it is typically hard to tell whether these procedures are consistent with each other or not. This paper transforms a philosophical suggestion of a procedure into a social welfare function, thus offering a link between the two approaches.

The arguments of the paper are presented in an informal manner. All the formal claims are given in the Appendix.

2 Proportional Claims

Suppose we have one unit of an indivisible good (or bad) we wish to allocate to one of two individuals. This good can be a donation of an organ, military service, or an airline ticket to Hawaii. Both individuals have the same utility function and both have the same claim for the good. Diamond [4] and others suggest that it is best to randomize between the two, selecting each person with probability one half. For a comprehensive analysis of different possible justifications for randomization, see Broome [2].

But what if the two don't have the same claim for the good? For example, what if one individual has a better chance to survive an operation, is better qualified to become a soldier, or derives higher utility from a trip to Hawaii? In case none of the two claimants owns the good and society wants to give it to one of them, utilitarianism requires giving the good to the person with the strongest claim for it. Observing the unfairness of such a division, Broome [1, 2] suggests an extension of the Aristotelian rule of proportionality, and states that "fairness seems to require … that the people's treatment should be in proportion to their claims" and "claims should be satisfied in proportion to their strength." And since the good cannot be divided, Broome offers "to give each person a chance proportional to his claim."[1]

This suggestion has a lot of appeal. Unlike utilitarianism, it is sensitive to the needs of both individuals. Taking utilitarianism to the extreme suggests that even the smallest difference in claims will determine the recipient of the good.[2] On the other hand, proportional randomization does not ignore the fact that a person with a higher claim is going to create a larger contribution to social well-being when receiving the good. It seems to be a reasonable compromise between efficiency and egalitarianism.

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[1] In [2], Broome explicitly says that he does "not mean 'proportion' to be taken too precisely. But… equal claims require equal satisfaction, stronger claims require more satisfaction… and weaker claims require some satisfaction." This paper takes a precise notion of 'proportion,' therefore Broome's three requirements are naturally satisfied.

[2] See more on this in Section 5 below, where I discuss Kamm's [9] analysis of the Flower Case.

A claim must be made with respect to some alternatives, and 'proportional claims' inevitably involve interpersonal comparisons of well-being. Circumscribing these difficulties, restrict the proportionality rule to situations where all individuals are interested in the expected level of utility they'll receive from a social randomization. Assume for simplicity that there are two individuals, that the utility level of each of them from *not* receiving the good is zero, and that their utilities from receiving the good are u for person 1 and v for person 2. Giving the good to person 1 with probability p and to person 2 with probability $1-p$ thus means that the expected (or average) utility of person 1 is pu and the expected utility of person 2 is $(1-p)v$. I now discuss several possible interpretations of the idea that probabilities should be proportional to claims and show their formal implications. A more general analysis, for n individuals, appears in the Appendix.

2.1 Proportional utilities

One possible interpretation of the proportionality rule is that if the utility person 1 receives from the good is twice the utility person 2 receives from it, then the probability of giving the good to person 1 should be twice the probability of giving it to person 2. In general, as the utility person 1 receives from the good is u and the utility person 2 receives from it is v, the rule suggests giving the good to person 1 with probability p and to person 2 with probability $1-p$, where the ratio between the probabilities equals the ratio between the utilities, that is, $p : 1-p = u : v$. The solution of this equation is $p = \frac{u}{u+v}$ and $1-p = \frac{v}{u+v}$.

The expected (or average) utility person 1 receives from this randomization is pu, while the expected utility of person 2 is $(1-p)v$. Using the above solution we obtain that the optimal allocation of *utility* between the two individuals is

$$\frac{u^2}{u+v} \text{ to person 1 and } \frac{v^2}{u+v} \text{ to person 2} \tag{1}$$

For example, if the utilities the two individuals receive from the good are 30 and 15, then the probabilities should be $\frac{2}{3}$ and $\frac{1}{3}$ and the utilities they receive are 20 and 5, respectively.

So far the focus of attention has been the mechanism used by society to determine the recipient of the good. But as such decisions also imply an allocation of expected levels of utility, one can evaluate possible randomizations from this perspective as well. Suppose that society is using a social welfare function of the utilities the individuals receive. The utilitarian social welfare function, that is, the one that evaluates social situations by the sum of these utilities, implies giving the good to the person whose utility from this good is highest, without using any randomization. Randomizing with probabilities p and $1-p$ over the two individuals will generate the (expected) utilities pu and $(1-p)v$. Each probability thus generates a social welfare value. Following the utility allocation (1) above, the question is what, if any, social welfare function will imply that this allocation of utilities is optimal. Denote by U and V possible utility levels for the two individuals (as before, u and v represent their actual utility levels from receiving the good).

Claim 1 *The social welfare function $W(U,V)=\sqrt{U}+\sqrt{V}$ implies the optimal allocation* (1).

In other words, the requirement for the optimal ratio $p:1-p$ to be equal to $u:v$ is obtained when society is maximizing the sum $\sqrt{pu}+\sqrt{(1-p)v}$. For a proof and a generalization of Claim 1, see App. 1, where it is also proved that the optimal allocation (1) actually characterizes the sum-of-square-roots social welfare function.

The meaning of the claim is simple. Suppose society is employing a social welfare function that is the sum of the square roots of individual utilities. Then it will choose the same utility allocation as the one obtained from randomizing over the two individuals with probabilities that are proportional to their utility from the good. Moreover, this is the *only* social welfare function to have this property.

2.2 The limited capacity case

In the above analysis, the rule "people's treatment should be in proportion to their claims" is interpreted to mean "randomization probabilities should be proportional to the utilities from the good." But other interpretations are also possible. Suppose three people, weighting 100, 100, and 200*lb*, need to be saved.[3] A boat can carry only 200*lb* and can be used only once. Since we can save either the first two or the third person, it is meaningful to say that each of the first two individuals has twice the claim for the boat as the

[3]This is similar in some aspects to Taurek's [11, p. 310] volcanic island example. The difference between the two stories is discussed in Section 2.3 below.

third person and therefore should be given twice the probability of person 3. In other words, we should save the first group of two individuals with probability $\frac{2}{3}$ and the other person with probability $\frac{1}{3}$. This is one of the solutions suggested by Kamm [9, vol. I, pp. 128–134] to a similar problem, where we can save either five or one person (in this case, Kamm suggests the probabilities should be $\frac{5}{6}$ and $\frac{1}{6}$). This solution is rejected by Broome [3], claiming that one should save the five with probability one, and certainly by Taurek [11] who does not see any merit in saving more life than less. (See Section 3 below for a further discussion).

To generalize the boat example, suppose that there are two groups, one of n individuals, each weighting a pounds and the other of m individuals, weighting b pounds each. Assume further that the total weight of the first group equals the total weight of the second group and both are equal to the capacity of the boat. The above analysis suggests that we should randomize, picking the first group with probability p and the second group with probability $1-p$, where $p : 1-p = n : m$. As before, the solution to this equation is $p = \frac{n}{n+m}$ and $1-p = \frac{m}{n+m}$. If the utility of each person (of both types) from being rescued is 1 while his utility from being left behind is zero, then the expected utility of individuals of type 1 is p and the expected utility of each person of type 2 is $1-p$.

First, I show that the probabilities $\frac{2}{3}$ and $\frac{1}{3}$ are inconsistent with the results of the previous case ("proportional utilities"). When the weights of the three individuals are 100, 100, and 200, we have $n = 2$, $m = 1$, and the probabilities are $p = \frac{2}{3}$ for the first two and $1-p = \frac{1}{3}$ for the third person. Using the sum-of-square-roots social welfare function we get $\sqrt{2/3} + \sqrt{2/3} + \sqrt{1/3} = 2.21$. But if we use the probabilities $p = 0.8$ and $1-p = 0.2$ we get that the value of this social welfare function is $\sqrt{0.8} + \sqrt{0.8} + \sqrt{0.2} = 2.24$. In fact, these are the optimal probabilities according to this social welfare function (App. 2).

It turns out however that there is a social welfare function for which the probabilities $\frac{n}{n+m}$ and $\frac{m}{n+m}$ are optimal. According to this function, society should maximize the *product* of individual utilities. It is called the Nash social welfare function, after Nash's solution to the bargaining problem.[4] Formally (see App. 3):

[4]Observe that the Nash, the sum-of-square-roots, the utilitarian, and the maximin functions all belong to the CES class.

Claim 2 *The Nash social welfare function implies that the optimal allocation of expected utilities is* $\frac{n}{n+m}$ *to all individuals of type 1 and* $\frac{m}{n+m}$ *to all individuals of type 2.*

In other words, the Nash social welfare function implies that society should randomize between the two groups, where the probabilities are proportional to the sizes of the group. As mentioned before, this is one of the solutions suggested by Kamm [9].

Apply now this social welfare function to the first problem, where one unit of an indivisible good needs to be allocated to one of two individuals. Their utilities from this good are u and v and randomizing with probabilities p and $1-p$ we obtain the utility distribution pu and $(1-p)v$. The product of these two utilities is $p(1-p)uv$ and it is easy to verify that the highest value is obtained when $p = \frac{1}{2}$, *regardless of the values of* u *and* v. The utilities received by the two individuals are $\frac{1}{2}u$ and $\frac{1}{2}v$, that is, the ratio between the *utilities* (not the probabilities) equals the ratio between the original claims.

But why do the two interpretations lead to different results? After all, isn't it true that in both cases the probabilities are proportional to the trade-offs in terms of utility? Here is a possible explanation. In the first interpretation, where one unit is to be given to one of two individuals whose utilities from the good are 2 and 1, the trade-off is between two units of utility of one person and one unit of utility of another person. In the second interpretation, where we can save either two individuals or a third person, the trade-off is between two units of utility, one from each of two individuals and one unit of utility from another person. Only in a utilitarian framework is there no difference between one person loosing two units of utility and two individuals, each loosing one unit of utility. Once we are out of the realm of utilitarianism, and as argued above, the proportionality rule must be inconsistent with utilitarianism, we cannot assume that two units from one person are the same as the one unit from each of two persons.

2.3 Why use a social welfare function?

There seems to be no difference between the boat example of Section 2.2 and any other story where the choice is between saving one group of people or another. But there is another way in which the boat can be used. One can save half the people of type 1 and half the people of type 2, or in general, $a\%$ of the people of type 1 and $(100-a)\%$ of the people of type 2. Moreover, the total weights of the two groups may not be the same, and the capacity of the boat can be more or less than the total weight of each of the groups. (And of course, there may be more than two groups). What should we do in such cases? For example, what should we do if the boat's capacity is 14,000*lb.*, there are 50, 80, and 70 people weighting 100, 150, and 200*lb.* respectively? Intuitive arguments are not straightforward. Taurek's [11] analysis suggests giving everyone the same probability, while other would probably suggest maximizing the total number of survivors (presumably by saving all 40 people of the first type, none of the third type, and assigning $\frac{3}{4}$ probability to each person of the second group). But the intuitions of Sections 2.1 and 2.2 are best extended via the derived social welfare functions.

In general, each person is assigned a certain probability of being rescued. App. 4 explains the nature of the constraint and shows how probabilities should be assigned to satisfy it. One possible solution is to allocate probabilities that are proportional to the inverse of the share claimed by individuals. As explained in Section 2.2, this solution is consistent with one of Kamm's [9] suggestions. App. 4 shows that under the Nash social welfare function $u_1 \cdot \ldots \cdot u_n$, the probabilities assigned to individuals are indeed proportional to the inverse of the shares they claim. Moreover, the Nash social welfare function is the *only* function to have this property.

Alternatively, one can allocate probabilities that are proportional to the inverse of the claims squared. This solution is implied by the sum-of-square-roots function $\sqrt{u_1} + \ldots + \sqrt{u_n}$ (see App. 5). Needless to say, other social welfare function will lead to other optimal allocations.

3 Which of the Two?

The two interpretations of the proportional rule lead to two different social welfare functions. There are two ways in which this contradiction can be handled. We can use other examples to better understand the implications of the two social welfare functions and reject one of them, or we can accept the contradiction and reject the consistency requirement. In this section I try the first approach, suggesting to accept the sum-of-square-roots function over the Nash social welfare function. In particular, I will try to explain why we get in the boat case that the ratio of the probabilities should be 4:1 and not 2:1.

3.1 Survival rates

Suppose we have one unit of a medication (the minimum required to be of any help), and two patients in a life-threatening situation in need of it. The probability it will help person 1 is $\frac{1}{2}$ and the probability it will help person 2 is $\frac{1}{4}$. As the claim of the first person is twice that of the second person, proportionality implies that we should administer the medicine to the first person with twice the probability assigned to the second person, that is, $\frac{2}{3} : \frac{1}{3}$. It is easy to verify that this interpretation leads to the same conclusions as that of Claim 1. Assuming both patients to have the same utility from survival (and from death), we obtain that the ratio between their utilities from receiving the medication equals the ratio between their probabilities of survival.

Using this policy, the first person will survive with probability $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$, while the second person's chances are only $\frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$. The ratio between these probabilities is $4 : 1$, the same ratio we got in the boat case $(.8 : .2)$. I try to explain this result is Section 3.3 below.

3.2 The volcanic island case

Consider the following extension of a problem discussed by Taurek [11, p. 310] (see also Broome [1, p. 54] and Kamm [9, vol. I, ch. 5–7]). Fleeing an upcoming volcanic eruption, 100 people are trapped at the north end of an island and 50 people are at the south end. We can make only one trip to the island, either to the north or the south end, and our boat can carry t people. Each person receives utility 1 from being rescued and zero from not being rescued. What should we do?

Consider the following two-stage policy. First, randomize between North and South, going North with probability p and South with probability $1-p$. Then, if the number of people at the chosen end is less than or equal to t, save all of them. If it is more than t, rescue t people at random, giving everyone at that end an equal chance. What should be the value of p?

There are three cases deserving attention. *1.* The number of seats t is 50 or less;[5] *2.* The number of seats is between 51 and 99; and *3.* The number of seats is 100 or more.

When the number of seats is less than or equal to 50, all seats will be used, regardless of where the boat goes. All individuals have the same claim for a

[5]This is the case discussed in Broome [1].

seat and should therefore receive the same probability of rescue. When we go North with probability p, each person there survives with probability $p \cdot t/100$ and each person at the south end survives with probability $(1 - p) \cdot t/50$. Equating these two expressions we get $p = \frac{2}{3}$. Both social welfare functions, the sum-of-square-roots and Nash, imply this value (App. 6). This is also the solution obtained from Taurek's [11] implied principle of equating the probabilities, and as all seats will be used, it is consistent with the principle of maximizing the number or survivors (see Broome [3]).

If the number of seats is 100, then we are back at the situation described in the "limited capacity case" (see Section 2.2 above) and the two social welfare functions give values for the probability of going North that are similar to those obtained before. Under the Nash function $p = \frac{2}{3}$ (which is Kamm's [9] "proportional chances" solution), while under the sum-of-square-roots function, $p = \frac{4}{5}$. Of course, utilitarianism implies $p = 1$ (which is also the solution supported by Broome [3]).

Unlike the case of Section 2.2, where higher capacity could be used, here increasing the capacity of the boat beyond 100 seats makes no difference. We should therefore expect the optimal probability, whatever its value, not to change when the capacity goes beyond 100. Both functions agree with this prediction (App. 6).

The most interesting and difficult to judge case is when the number of seats is between 51 and 99. On the one hand, people at the north end seem to have a higher claim for the boat, but what exactly is the ratio between the claims of the north- and the south-enders? Although intuition is not clear here, one thing seems certain. As the number of seats in the boat increases from 50 to 100, the cost of going South in terms of alternative life-saving opportunities at North is increasing. And therefore, the higher is the number of seats (between 50 and 100), the stronger is the claim for it by each of the northern survivors. We should thus expect that within this range, the optimal probability of going North will increase with t. This is indeed the case with the sum-of-square-roots social welfare function, but not with the Nash function (see App. 6). This latter function assigns always the same probability to the two ends — the probability should be $\frac{2}{3}$ regardless of the number of seats in the boat. And in general, this probability depends on the number of people at each point, but not the capacity of the boat. I believe that this result proves the analysis of Section 2.2 to be wrong — it does not capture the true intuition of the rule that "people's treatment should be in proportion to their claims."

Another unsatisfactory result of the Nash function is obtained in the following variation of the volcanic island problem. Suppose there are 1001 islands, one with 1000 survivors and 1000 islands with one survivor each. The capacity of the boat is 1000 and it can go to one island only. The Nash function is maximized when we decide to go to the populated island with probability $\frac{1}{2}$ and to each of the other islands with probability 1/2000 (see App. 7). In other words, with probability $\frac{1}{2}$ the boat will return virtually empty. The sum-of-square-roots function suggests the much more reasonable probability of 1000/1001 for the populated island.

3.3 Where did intuition go wrong?

In Section 2.2 I discussed the limited capacity case where a boat can carry 200*lb* while the weights of the three people in need of it are 100, 100, and 200. Even if one rejects the intuition of that section, it still seems puzzling that the optimal rule obtained from the sum-of-square-roots social welfare functions is to randomize with probabilities 0.8:0.2 over the two groups of individuals ("thin" and "heavy"). But here is another representation of this mechanism that seems to be in total agreement with the proportionality rule. Assign each of the two thin individuals probability 0.4 and assign the heavy person probability 0.2. If one of the first two is selected, then we still have room in the boat for another (thin) person. Each of them will survive if he or the other one is selected, hence the 0.8 probability. But this probability is composed of two parts. The mechanism assignment of 0.4 and Nature's assignment of another 0.4. This argument is clearly the correct interpretation of Section 3.1. In that story, the person with the higher survival rate receives the medication with twice the probability of the other person ($\frac{2}{3}$ and $\frac{1}{3}$), but his chances to survive are *four* times that of the other person. Half of it is due to the mechanism and half of it is due to Nature giving him twice the survival rate.

4 Longevity and Relative Claims

So far I dealt with situations where final allocations give everything to some individuals and nothing to others. Randomization seems therefore a useful tool in solving such conflicts and Broome's suggestion leads to a specific social welfare function. We can now apply this function to situations where randomization is not essential and check its predictions.

Suppose two people need a daily ration of a medication to survive. One patient needs one unit per day, the other two units. The utility of each of them from surviving n days is n. We have 180 units of the medication. How should we allocate these units in the following three scenarios?

1. Units are delivered daily (for example, the two are hospitalized).

2. All units must be allocated right away (the two must be quarantined to avoid the disease from spreading).

3. One patient must receive all units (we have only one syringe and sharing it will kill both patients).

The third situation fits well into the above analysis as it naturally calls for some randomization. As the claim of the first person in terms of utility is twice that of the other person, he should receive all the quantity of the medication with probability $\frac{2}{3}$, leading to the utility allocation of 120:30. This is also the solution implied by the sum-of-square-roots social welfare function.

We can get the same utility allocation in the other two cases. In the first case, for thirty days person 1 receives one unit daily while the other person receives two units. From the 31st day on, and for another ninety days, only person 1 receives the medication (and person 2 dies right away). In the second case, we can give the first person 120 units of the medication and the second person sixty units of it. But are these the right solutions?

The first problem is clearly different, as it leads to a sequence of decisions rather than one decision only. Every day (the 31st day included) we face the decision "should we give person 1 one unit, person 2 two units, or both?" Whatever the answer, it is reasonable to require that every day it will be the same. Letting both live for sixty days is therefore the most natural solution.

But what should be the solution to the second situation? The Nash social welfare function implies giving each person 90 units (see App. 8). But is the equal split of the good the right allocation? In this case person 1 lives for ninety days and person 2 for 45. I find the egalitarian aspect of this solution to fall under the criticism expressed by Anatol France in *The Red Lily*: "They [the poor] have to labor in the face of the majestic equality of the law, which forbids the rich as well as the poor to sleep under bridges, to beg in the streets, and to steal bread." If an egalitarian distribution is our goal, then we should give the first person sixty and the second person 120 units so that both will survive for sixty days, which is the intuitive solution to the first situation.

But suppose that the second person needs nine units per day (while the first person still needs only one). Will it be right to give the second person 162 units and person 1 only 18 units so that both survive for 18 days? Applying the sum-of-square-roots social welfare function, we get that if the second person needs two units per day then the optimal allocation of the medication is 120:60 so that the first lives for 120 and the second for 30 days. If the second person needs nine units per day then the optimal allocation is 162:18 (see App. 8). Person 1 lives for 162 days while person 2 lives for 2 days. I find this allocation to be a lot more reasonable then 18:162, where both live for 18 days. The higher is the cost of egalitarianism, the more should we move in the direction of utilitarianism.

To sum: I believe that the examples of this paper are consistent with my interpretation of Broome's rule and that the analysis of these examples by the sum-of-square-roots social welfare function leads to reasonable results.

5 The Trolley and the Flower

In what is known as the "Trolley Case," Foot [6] raises the following question: Suppose a runaway trolley is headed towards killing five people. A conductor can change the course of the trolley to another track, where only one person, who is not one of the original five, will be killed. Should the conductor change the course of the trolley?

Montmarquet [10] seems to claim that in deciding this issue, one should consider maximizing the sum of overall good. In her criticism of this doctrine, Kamm [9, vol. II, pp. 157–158] raises the following question:

> Suppose we may choose whether to send the trolley toward one person on the left track or toward one person on the right track. Either way one person will die, but if we send the trolley to the right track some beautiful flowers that give many people pleasure will also be destroyed.

Kamm claims that "increasing overall good by saving the flowers should play no role in deciding along which track to send the trolley. The extra utility is irrelevant."

Although I agree that determining the issue of where to send the trolley (only) by the fate of the flowers is wrong, ignoring them altogether is also wrong. Following the analysis of this paper, the obvious solution should be to randomize between the two sides, giving left a slightly higher probability than right. For example, suppose that in addition to the two people in danger, there are ten other people. The two individuals in danger receive utility zero if not saved. Everyone who is alive receives utility 1 if the flowers are kept and $1 - \varepsilon$ if they are destroyed. It is easy to verify (App. 9) that for both of the social welfare functions discussed above, it holds that for sufficiently small value of ε, the optimal probability of going left is only slightly above $\frac{1}{2}$.[6] In other words, one can be sensitive to the existence of the flowers without letting their existence decide whose life we should save.

6 Summary

Following Harsanyi [8], there are many axiomatizations of social welfare functions where the moral insights of the axioms relate to the eventual allocation of utilities and goods. My interpretation of Broome's proportional rule refers to the allocation *mechanism*, and as argued above, can lead to a specific social welfare function, the one that evaluates social situations by the sum of the square roots of individual utilities.

Randomization seems natural in cases of equal claims. If these are the only situations where this mechanism is to be used, then it becomes virtually meaningless, as exact equal claims are very rare. Kamm's flower case clarifies why it would be wrong to restrict randomizations only to such cases. Either we let insignificant factors (like the flowers) become decisive, or we ignore them (which is what Kamm suggests). Both are unsatisfactory. It would be morally wrong to let the existence of the flowers decide who will live and who will die. On the other hand, there are factors we would like to take into consideration, like one of the two people being able to save other people in the future or social services one of them performed in the past.[7] Moving continuously from an insignificant factor to a significant one, there will be a

[6] For example, for $\varepsilon = 0.001$ the sum-of-square-roots social welfare function obtains that the optimal probability of going left is 0.504. In general, all strictly quasi concave social welfare functions lead to an optimal value greater than half.

[7] Kamm [9, vol. I, p. 107] would probably disagree that the first of these factors is relevant, but see Broome [3].

point where we will switch from an even chance randomization to saving one of the two with no randomization. But at this point a very minor change becomes decisive. Biased randomization is the natural solution. It gives both parties an even chance if their claims are equal and favors the one with the stronger claim by giving him a higher probability in cases of asymmetric claims. For other arguments in favor of randomization, see Diamond [4], Broome [1, 2], Epstein and Segal [5], and Kamm [9].

Once we accept the rule of biased randomization, the idea that probabilities should be proportional to claims seems a natural. The social welfare function it implies is simple and, at least in some situations, compelling. I cannot claim that it covers all social situations. As argued above, there are situations where an egalitarian allocation seems right even if people have different claims. Moreover, using always the same social welfare function requires the independence of irrelevant alternatives axiom, which is known to be behaviorally questionable. But I believe that the analysis of this paper does capture the idea that in many situations people with higher claims deserve a more favorable randomization.

Appendix

App. 1 Suppose there are n individuals and their utilities from receiving the one unit of the indivisible good are $v_1, \ldots, v_n$, respectively. Section 2.1 suggests that society should randomize over the n individuals, using the probability vector p that is proportional to the utilities vector $(v_1, \ldots, v_n)$, that is, $p_i = v_i / \sum_j v_j$. With these probabilities, person i will receive the expected utility

$$p_i v_i = \frac{v_i^2}{\sum_j v_j} \tag{2}$$

Proposition 1 *The social welfare function $W(U_1, \ldots, U_n)$ is maximized at* (2) *if, and only if, W is given by $\sum \sqrt{U_i}$, where U_i is person i's utility index.*

Proof: The possibility of randomization leads to the utility-opportunity set $\sum U_i / v_i = 1$. The utility allocation (2) is the solution to

$$\begin{aligned} \max \quad & \sum \sqrt{U_i} \\ \text{s.t.} \quad & \sum \frac{U_i}{v_i} = 1 \end{aligned}$$

If $W \neq h(\sum \sqrt{U_i})$, then there is a point where $\nabla W \neq \nabla(\sum \sqrt{U_i})$ and the two functions lead to different optimal points. ■

App. 2 The maximum of the function $2\sqrt{p} + \sqrt{1-p}$ is obtained at $p = 0.8$.

App. 3 Saving the first group with probability p and the second group with probability $1-p$ yields the Nash value of $p^n(1-p)^m$. This function is maximized at $p = \frac{n}{n+m}$.

App. 4 Suppose that there are n individuals and person i requires a_i fraction of a certain good to survive, where $\sum a_i > 1$. (In the first boat example of the text, $n = 3$, $a_1 = a_2 = \frac{1}{2}$ while $a_3 = 1$). The utility of person i is 1 if he receives his share a_i and zero if not. Each person is assigned a probability p_i of being rescued and the social constraint over the values of these probabilities is given by

$$\sum p_i a_i = 1 \tag{3}$$

For example, suppose that the boat's capacity is 10,000, there are 200 people whose weight is 100 each, numbered 1 to 200, and 100 people whose weight is 200 each, numbered 201 to 300. Suppose further that we would like to assign each person of the first group three time the probability assigned to each person of the second. Denote the latter p, and obtain that the constraint 3 becomes

$$200 \cdot 3p \cdot \frac{1}{100} + 100 \cdot p \cdot \frac{1}{50} = 1 \Longrightarrow$$
$$6p + 2p = 1 \Longrightarrow p = \frac{1}{8}$$

Each person of the first group is selected with probability $\frac{3}{8}$, and each person of the second group is selected with probability $\frac{1}{8}$. Here is a simple mechanism that implies these probabilities.

Put in an urn 700 balls, three balls with each of the numbers 1 to 200 and one ball with each of the numbers 201 to 300. Draw balls at random and send the individuals whose numbers were picked to the boat until it is full. If a selected person is one of the first 200, also remove the other two balls with his number. It is true that there is some friction at the end — what if at a certain point there are only $100lb$ of the boat's capacity left — but if the population and the boat are sufficiently large this friction is negligible. As desired, the probability of selection for each of the first 200 people is three times that of the last 100.

The second interpretation of the proportionality rule is that the probability p_i assigned to person i should be proportional to the inverse of a_i. For example, if person i claims $\frac{1}{3}$ of the good and person j claims $\frac{1}{6}$ or it, then person j's probability of receiving his claim should be twice that of person i's. Formally, if we denote $b_i = \frac{1}{a_i}$, then we want the probability vector $(p_1, \ldots, p_n)$ to be proportional to the vector $(b_1, \ldots, b_n)$. Together with the constraint (3) and the observation $U_i = p_i$, this implies that

$$U_i = p_i = \frac{b_i}{n}, \quad i = 1, \ldots, n \tag{4}$$

Proposition 2 *Suppose that for every i, $a_i \geqslant \frac{1}{n}$. That is, everyone needs more than the average room available. Given the constraint* (3)*, the social welfare function W is maximized at* (4) *if, and only if, W is (an ordinal transformation of) the Nash social welfare function $U_1 \times \ldots \times U_n$.*

Proof: From eq. (3), the utility-opportunity set is $\{(u_1, \ldots, u_n) \in [0,1]^n\}$ such that $\sum a_i u_i = 1$. Clearly, the utility allocation

$$\left(\frac{b_1}{n}, \ldots, \frac{b_n}{n}\right) = \left(\frac{1}{na_1}, \ldots, \frac{1}{na_n}\right)$$

is the solution to

$$\begin{array}{rl} \max & u_1 \times \cdots \times u_n \\ \text{s.t.} & \sum a_i u_i = 1 \\ & (u_1, \ldots, u_n) \in [0,1]^n \end{array}$$

The assumption $a_i \geqslant \frac{1}{n}$ is needed to ensure that $1/na_i \leqslant 1$. ■

App. 5 Given the claims $a_1, \ldots, a_n$ for shares of the boat, the sum-of-square-roots function implies probabilities $p_1, \ldots, p_n$ that solve the maximization problem

$$\begin{array}{rl} \max & \sum \sqrt{p_i} \\ \text{s.t.} & \sum p_i a_i = 1 \end{array}$$

It follows that

$$\frac{p_i}{p_j} = \frac{a_j^2}{a_i^2}$$

App. 6 The three cases are $t \leqslant 50$; $50 < t < 100$; and $t \geqslant 100$. Going North with probability p yields the individual probabilities of rescue (and hence utilities) of $pt/100$; $pt/100$; and p at the north end and $(1-p)t/50$; $1-p$; and $1-p$ at the south end. The Nash social welfare function yields the values $(pt/100)^{100}([1-p]t/50)^{50}$; $(pt/100)^{100}(1-p)^{50}$; and $p^{100}(1-p)^{50}$. Clearly, the optimal p is the same in all three cases and is equal to $\frac{2}{3}$.

The alternative, sum-of-square-roots social welfare function yields the values $100\sqrt{pt/100}+50\sqrt{(1-p)t/50}$; $100\sqrt{pt/100}+50\sqrt{1-p}$; and $100\sqrt{p}+50\sqrt{1-p}$. The optimal values of p are $\frac{2}{3}$, $100t/(100t+2500)$; and $\frac{4}{5}$.

App. 7 The maximum of the constrained optimization

$$\begin{array}{rl} \max & p^{1000} \times q_1 \times \ldots \times q_{1000} \\ \text{s.t.} & p + q_1 + \ldots + q_{1000} = 1 \end{array}$$

is obtained at $p = \frac{1}{2}$ and $q_1 = \ldots = q_{1000} = 1/2000$. Under the same constraint, the maximum of the function $1000\sqrt{p} + \sqrt{q_1} + \ldots + \sqrt{q_{1000}}$ is obtained at $p = 1000/1001$ and $q_1 = \ldots = q_{1000} = 10^{-6}$.

App. 8 If we have a units of the medication, person 1 needs one unit while person 2 needs k per day, then giving x units to person 1 and $a-x$ to person 2 yields the social value of

$$\sqrt{x} + \sqrt{\frac{a-x}{k}} \tag{5}$$

for the sum-of-square-roots social welfare function, and

$$x \cdot \frac{a-x}{k} \tag{6}$$

for the Nash function. The optimal values for eq. (5) are given by $x = ak/(1+k)$ and $a-x = a/(1+k)$. The first person survives for $ak/(1+k)$ days and the second person for $a/k(1+k)$ days. Eq. (6) implies $x = a-x = \frac{a}{2}$. The first person survives for $\frac{a}{2}$ days and the second person for $\frac{a}{2k}$ days.

App. 9 If there are n other people beyond the two in danger (in the text, $n = 10$), then the optimal value of p for the sum-of-square-roots social welfare function solves

$$\frac{1}{\sqrt{p}} - \frac{\sqrt{1-\varepsilon}}{\sqrt{1-p}} + \frac{n\varepsilon}{\sqrt{1-\varepsilon+p\varepsilon}} = 0 \quad (7)$$

while the optimal value of p for the Nash function is the solution to the quadratic equation

$$(n+2)\varepsilon p^2 + [2-(n+3)\varepsilon]p + \varepsilon - 1 = 0 \quad (8)$$

As ε approaches zero, the optimal values of p in (7) and (8) approach $\frac{1}{2}$.

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